UPPER BOUNDS FOR THE EXTENDED GENERALIZED ALUTHGE TRANSFORM OF BOUNDED OPERATORS IN HILBERT SPACES

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ABSTRACT. Let H be a complex Hilbert space. For a *contraction* $V \in \mathcal{B}(H)$, i.e. $0 \le V^*V \le I$, an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$ we define the operator

$$\Delta_{t,V}(T) := |T|^t V |T|^{1-t}$$

that we *call the extended generalized Aluthge transform.* In this paper we provide several upper bounds for the extended generalized Aluthge transform $\Delta_{t,V}(T)$. The cases of usual generalized Aluthge, Dougal and Aluthge transforms are also presented.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \le w(T)\|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \ge 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda|\,\omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T+V) \le \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \le \|T\| \le 2\omega(T) \tag{1.3}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [9], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T) \le \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [10] improved the inequality (1.3) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \le \omega^2(T) \le \frac{1}{2}\|T^*T + TT^*\| \tag{1.5}$$

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for any operator $T\in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [8]:

If for an operator $T\in B(H)$ we denote $|T|:=(T^{*}T)^{1/2}$, then

$$\omega^{r}(T)\leq\frac{1}{2}\left\||T|^{2\alpha r}+|T^{*}|^{2(1-\alpha)r}\right\| \tag{1.6}$$

and

$$\omega^{2r}(T)\leq\left\|\alpha|T|^{2r}+(1-\alpha)|T^{*}|^{2r}\right\|, \tag{1.7}$$

where $\alpha\in(0,1)$ and $r\geq 1$.

If we take $\alpha=\frac{1}{2}$ and $r=1$ we get from (1.6) that

$$\omega(T)\leq\frac{1}{2}\||T|+|T^{*}|\| \tag{1.8}$$

and from (1.7) that

$$\omega^{2}(T)\leq\frac{1}{2}\left\||T|^{2}+|T^{*}|^{2}\right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [7] and [4].

Let $T=U|T|$ be the *polar decomposition* of the bounded linear operator T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T}:=|T|^{1/2}U|T|^{1/2}$, see [1].

The following properties of $\widetilde{T}$ are as follows:

(i) $\left\|\widetilde{T}\right\|\leq\|T\|$,

(ii) $w\left(\widetilde{T}\right)\leq\omega(T)$,

(iii) $r\left(\widetilde{T}\right)=\omega(T)$,

(iv) $\omega\left(\widetilde{T}\right)\leq\left\|T^{2}\right\|^{1/2}(\leq\|T\|)$, [12].

Utilizing this transform T. Yamazaki, [12] obtained in 2007 the following refinement of Kittaneh's inequality (1.4):

$$\omega(T)\leq\frac{1}{2}\left(\|T\|+\omega\left(\widetilde{T}\right)\right)\leq\frac{1}{2}\left(\|T\|+\left\|T^{2}\right\|^{1/2}\right) \tag{1.10}$$

for any operator $T\in B(H)$.

We remark that if $\widetilde{T}=0$, then obviously $w(T)=\frac{1}{2}\|T\|$.

For a *contraction* $V\in\mathcal{B}(H)$, i.e. $0\leq V^{*}V\leq I$ and an operator $T\in\mathcal{B}(H)$ and $t\in[0,1]$ we define the operator

$$\Delta_{t,V}(T):=|T|^{t}V|T|^{1-t}$$

that we *call the extended generalized Aluthge transform.*

We assume in what follows that $|T|^{0}:=I$.

For $t=1$ we have

$$\widehat{T}_{V}:=\Delta_{1,V}(T)=|T|V,$$

that we call *the extended Dougal transform*, for $t=1/2$,

$$\widetilde{T}_{V}=\Delta_{1/2,V}(T):=|T|^{1/2}V|T|^{1/2},$$

that we *call the extended Aluthge transform* and for $t=0$,

$$T_{V}:=\Delta_{0,V}(T)=V|T|.$$

An operator $U \in \mathcal{B}(H)$ is called a *partial isometry* if $\|Ux\| = \|x\|$ for all $x \in \mathcal{N}^{\perp}(U)$.

Now, let $x \in H$, then there exists a unique $x_1 \in \mathcal{N}(U)$ and a unique $x_2 \in \mathcal{N}^{\perp}(U)$ such that $x = x_1 + x_2$. Then

$$0 \leq \langle U^* U x, x \rangle = \|Ux\|^2 = \|Ux_1 + Ux_2\|^2 = \|Ux_2\|^2 = \|x_2\|^2 .$$

By the fact that $x_1 \perp x_2$,

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 .$$

Therefore

$$0 \leq \langle U^* U x, x \rangle \leq \|x\|^2 ,$$

which shows that U is a contraction on H.

Let $T \in \mathcal{B}(H)$ and $T = U|T|$ the polar decomposition of T with U a partial isometry. Then

$$T_U = U|T| = T,$$
$$\widetilde{T}_U = |T|^{1/2} U |T|^{1/2} = \widetilde{T}$$

is the usual *Aluthge transform* and

$$\widehat{T}_U = |T| U = \widehat{T}$$

is the usual *Dougal transform.*

For $t \in (0,1)$

$$\Delta_{t,U}(T) = |T|^t U |T|^{1-t} =: \Delta_t(T)$$

is the *generalized Aluthge transform* introduced in by Cho and Tanahashi in [6].

Abu-Omar and Kittaneh [2] improved on inequality (1.10) using generalized Aluthge transform to prove that

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \min_{t \in [0,1]} \omega(\Delta_t(T))\right).$$

For $t = 1$ this also gives the following result for the *Dougal transform*

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \omega\left(\widehat{T}\right)\right). \tag{1.11}$$

In [3] Bunia et al. also proved that

$$\omega(T) \leq \min_{t \in [0,1]} \left\{ \frac{1}{2}\omega(\Delta_t(T)) + \frac{1}{4}\left(\|T\|^{2t} + \|T\|^{2(1-t)}\right) \right\},$$

which for $t = 1/2$ gives (1.10) as well.

If V is a contraction, then $\|V\| \leq 1$ and since $\|V^*\| = \|V\|$, hence V^* is also a contraction. Observe that

$$\Delta_{t,V}^*(T) := \left(|T|^t V |T|^{1-t}\right)^* = |T|^{1-t} V^* |T|^t = \Delta_{1-t,V^*}(T)$$

for all $t \in [0,1]$. Therefore

$$(T_V)^* = \widehat{T}_{V^*}, \quad \left(\widehat{T}_V\right)^* = T_{V^*}$$

and

$$\left(\widetilde{T}_V\right)^* = \widetilde{T}_{V^*}.$$

Since $\|V^*V\| = \|VV^*\| = \|V\|^2$ and V is a contraction, then

$$\left\| \frac{V^*V \pm VV^*}{2} \right\| \le \|V\|^2 \le 1$$

showing that

$$W := \frac{V^*V \pm VV^*}{2}$$

is a contraction and we can consider the transform

$$\Delta_{t, \frac{V^*V \pm VV^*}{2}} (T) := |T|^t \left(\frac{V^*V \pm VV^*}{2} \right) |T|^{1-t}$$

for $t \in [0, 1]$.

For a contraction V, we have

$$\operatorname{Im}(V) := \frac{V - V^*}{2i}, \ \operatorname{Re}(V) := \operatorname{Re}\left(\frac{V + V^*}{2} \right)$$

and since

$$\|\operatorname{Im}(V)\| = \left\| \frac{V - V^*}{2i} \right\| \le \|V\| \le 1 \text{ and } \|\operatorname{Re}(V)\| \le \|V\| \le 1,$$

hence $\operatorname{Im}(V)$ and $\operatorname{Re}(V)$ are contractions as well. We can then consider the transforms

$$\Delta_{t, \operatorname{Im}(V)} (T) := |T|^t \operatorname{Im}(V) |T|^{1-t} \text{ and } \Delta_{t, \operatorname{Re}(V)} (T) := |T|^t \operatorname{Re}(V) |T|^{1-t}$$

for $t \in [0, 1]$.

For $T \in \mathcal{B}(H)$ we define

$$T_+ := \frac{1}{2} (|T| + T) \text{ and } T_- := \frac{1}{2} (|T| - T).$$

If U is the partial isometry in the polar representation of T, then

$$V := \frac{I \pm U}{2}$$

is a contraction and

$$\Delta_{t, \frac{I \pm U}{2}} (T) := |T|^t \frac{I \pm U}{2} |T|^{1-t} = \frac{|T| \pm \Delta_t (T)}{2}.$$

In particular, we get

$$T_{\frac{I \pm U}{2}} = \frac{|T| \pm T}{2} = T_{\pm}, \widehat{T}_{\frac{I \pm U}{2}} = \frac{|T| \pm \widehat{T}}{2}$$

and

$$\widetilde{T}_{\frac{I \pm U}{2}} = \frac{|T| \pm \widetilde{T}}{2}$$

for any operator $T \in \mathcal{B}(H)$.

Motivated by the above results, in this paper we provide several upper bounds for the extended generalized Aluthge transform $\Delta_{t,V} (T)$. The cases of usual generalized Aluthge, Dougal and Aluthge transforms are also presented.

2. Main Results

We need the following fact, see [5]:

Lemma 1. *For any A, $B \in \mathcal{B}(H)$,*

$$\omega(AB) \le \frac{1}{2}\omega(BA) + \frac{1}{4}(\|AB\| + \|A\|\|B\|) \le \frac{1}{2}(\omega(BA) + \|A\|\|B\|). \tag{2.1}$$

We have the following first result:

Theorem 1. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$,*

$$\omega(\Delta_{t,V}(T)) \le \frac{1}{2}\omega(T_V) + \frac{1}{4}(\|\Delta_{t,V}(T)\| + \|V\|\|T\|) \le \frac{1}{2}[\omega(T_V) + \|V\|\|T\|]. \tag{2.2}$$

In particular,

$$\omega\left(\widehat{T}_V\right) \le \frac{1}{2}\omega(T_V) + \frac{1}{4}\left(\left\|\widehat{T}_V\right\| + \|V\|\|T\|\right) \le \frac{1}{2}[\omega(T_V) + \|V\|\|T\|]$$

and

$$\omega\left(\widetilde{T}_V\right) \le \frac{1}{2}\omega(T_V) + \frac{1}{4}\left(\left\|\widetilde{T}_V\right\| + \|V\|\|T\|\right) \le \frac{1}{2}[\omega(T_V) + \|V\|\|T\|].$$

Proof. If we take $A = |T|^{1-t}$ and $B = V|T|^t$ in (2.1), then we get

$$\begin{aligned}\omega(\Delta_{t,V}(T)) &= \omega\left(|T|^{1-t}V|T|^t\right)\\ &\le \frac{1}{2}\omega(T_V) + \frac{1}{4}\left(\|\Delta_{t,V}(T)\| + \left\||T|^{1-t}\right\|\left\|V|T|^t\right\|\right)\\ &\le \frac{1}{2}\omega(T_V) + \frac{1}{4}\left(\|\Delta_{t,V}(T)\| + \|T\|^{1-t}\|V\|\|T\|^t\right)\\ &= \frac{1}{2}\omega(T_V) + \frac{1}{4}(\|\Delta_{t,V}(T)\| + \|V\|\|T\|)\end{aligned}$$

for $t \in [0,1]$, which proves the first part of (2.2)

Since

$$\|\Delta_{t,V}(T)\| = \left\||T|^{1-t}V|T|^t\right\| \le \|T\|^{1-t}\|V\|\|T\|^t = \|T\|$$

hence the proof is concluded. □

Remark 1. *For an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$ we derive the following inequalities:*

$$\omega(\Delta_t(T)) \le \frac{1}{2}\omega(T) + \frac{1}{4}(\|\Delta_t(T)\| + \|T\|) \le \frac{1}{2}[\omega(T) + \|T\|].$$

In particular,

$$\omega\left(\widehat{T}\right) \le \frac{1}{2}\omega(T) + \frac{1}{4}\left(\left\|\widehat{T}\right\| + \|T\|\right) \le \frac{1}{2}[\omega(T) + \|T\|],$$

and

$$\omega\left(\widetilde{T}\right) \le \frac{1}{2}\omega(T) + \frac{1}{4}\left(\left\|\widetilde{T}\right\| + \|T\|\right) \le \frac{1}{2}\left[\omega(T) + \|T\|\right].$$

If $T = U|T|$ is the polar decomposition of T with U a partial isometry, then by putting $V = U$ in (2.2) we deduce the inequalities listed above in the Remark 1.

Corollary 1. *For a contraction $V \in \mathcal{B}(H)$ and an operator $T \in \mathcal{B}(H)$ we have*

$$\begin{aligned}
&\omega\left(|T|^{1-t}(V^*V \pm VV^*)|T|^t\right) \\
&\le \frac{1}{2}\omega\left((V^*V \pm VV^*)|T|\right) \\
&+ \frac{1}{4}\left(\left\||T|^{1-t}(V^*V \pm VV^*)|T|^t\right\| + 2\|V\|\|T\|\right) \\
&\le \frac{1}{2}\omega\left((V^*V \pm VV^*)|T|\right) + \|V\|\|T\|.
\end{aligned}$$

Proof. Follows by (2.2) for the contraction

$$W = \frac{V^*V \pm VV^*}{2}.$$

□

We observe that for $t = 1/2$,

$$|T|^{1/2}\left(\frac{V^*V \pm VV^*}{2}\right)|T|^{1/2}$$

is selfadjoint and then

$$\omega\left(|T|^{1/2}\left(\frac{V^*V \pm VV^*}{2}\right)|T|^{1/2}\right) = \left\||T|^{1/2}\left(\frac{V^*V \pm VV^*}{2}\right)|T|^{1/2}\right\|$$

and from the first inequality in (2.2) we derive

$$\left\||T|^{1/2}(V^*V \pm VV^*)|T|^{1/2}\right\| \le \frac{2}{3}\left[\omega\left((V^*V \pm VV^*)|T|\right) + \|V\|\|T\|\right]. \tag{2.3}$$

Corollary 2. *For a contraction $V \in \mathcal{B}(H)$ and an operator $T \in \mathcal{B}(H)$ we have*

$$\begin{aligned}
\omega\left(\Delta_{t,\mathrm{Im}(V)}(T)\right) &\le \frac{1}{2}\omega\left(T_{\mathrm{Im}(V)}\right) + \frac{1}{4}\left(\left\|\Delta_{t,\mathrm{Im}(V)}(T)\right\| + \|\mathrm{Im}(V)\|\|T\|\right) \\
&\le \frac{1}{2}\left[\omega\left(T_{\mathrm{Im}(V)}\right) + \|\mathrm{Im}(V)\|\|T\|\right]
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
\omega\left(\Delta_{t,\mathrm{Re}(V)}(T)\right) &\le \frac{1}{2}\omega\left(T_{\mathrm{Re}(V)}\right) + \frac{1}{4}\left(\left\|\Delta_{t,\mathrm{Re}(V)}(T)\right\| + \|\mathrm{Re}(V)\|\|T\|\right) \\
&\le \frac{1}{2}\left[\omega\left(T_{\mathrm{Re}(V)}\right) + \|\mathrm{Re}(V)\|\|T\|\right]
\end{aligned} \tag{2.5}$$

for all $t \in [0,1]$.

Proof. Follows by (2.2) for the contractions $\mathrm{Im}(V)$ and $\mathrm{Re}(V)$. □

Observe that

$$|T|^{1/2}(\operatorname{Im}(V))|T|^{1/2} = |T|^{1/2}\left(\frac{V-V^*}{2i}\right)|T|^{1/2} = \frac{\widetilde{T}_V - \left(\widetilde{T}_V\right)^*}{2i}$$
$$= \operatorname{Im}\left(\widetilde{T}_V\right).$$

Since $\operatorname{Im}\left(\widetilde{T}_V\right)$ is selfadjoint, then by (2.4) for $t = 1/2$ we get

$$\left\|\operatorname{Im}\left(\widetilde{T}_V\right)\right\| \leq \frac{2}{3}\omega\left(\operatorname{Im}(V)|T|\right) + \frac{1}{3}\|\operatorname{Im}(V)\|\|T\|. \tag{2.6}$$

Similarly, we can show that

$$\left\|\operatorname{Re}\left(\widetilde{T}_V\right)\right\| \leq \frac{2}{3}\omega\left(\operatorname{Re}(V)|T|\right) + \frac{1}{3}\|\operatorname{Re}(V)\|\|T\|. \tag{2.7}$$

For $T \in \mathcal{B}(H)$ we define

$$T_+ := \frac{1}{2}(|T| + T) \text{ and } T_- := \frac{1}{2}(|T| - T).$$

Corollary 3. *For an operator $T \in \mathcal{B}(H)$ we have for $t \in [0,1]$ that*

$$\begin{aligned}\omega(|T| - \Delta_t(T)) &\leq \omega(T_-) + \frac{1}{4}(\||T| - \Delta_t(T)\| + \|U - I\|\|T\|)\\ &\leq \omega(T_-) + \left\|\frac{U-I}{2}\right\|\|T\|\end{aligned}$$

and

$$\begin{aligned}\omega(|T| + \Delta_t(T)) &\leq \omega(T_+) + \frac{1}{4}(\||T| + \Delta_t(T)\| + \|U + I\|\|T\|)\\ &\leq \omega(T_+) + \left\|\frac{U+I}{2}\right\|\|T\|.\end{aligned}$$

Proof. From (2.2) we get for the contraction $V := \frac{I\pm U}{2}$ that

$$\begin{aligned}&\omega\left(\Delta_{t,\frac{I\pm U}{2}}(T)\right)\\ &\leq \frac{1}{2}\omega\left(T_{\frac{I\pm U}{2}}\right) + \frac{1}{4}\left(\left\|\Delta_{t,\frac{I\pm U}{2}}(T)\right\| + \left\|\frac{I\pm U}{2}\right\|\|T\|\right)\\ &\leq \frac{1}{2}\left[\omega\left(T_{\frac{I\pm U}{2}}\right) + \left\|\frac{I\pm U}{2}\right\|\|T\|\right],\end{aligned}$$

which proves the corollary. □

Theorem 2. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$ we have*

$$\begin{aligned}\omega(\Delta_{t,V}(T)) &\leq \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}(\|\Delta_{t,V}(T)\| + \|V\|\|T\|)\\ &\leq \frac{1}{2}\left[\omega\left(\widehat{T}_V\right) + \|V\|\|T\|\right].\end{aligned} \tag{2.8}$$

In particular,

$$\begin{aligned}\omega(T_V) &\leq \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}(\|T_V\| + \|V\|\|T\|)\\ &\leq \frac{1}{2}\left[\omega\left(\widehat{T}_V\right) + \|V\|\|T\|\right],\end{aligned}$$

and

$$\omega\left(\widetilde{T}_V\right) \le \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left(\left\|\widetilde{T}_V\right\| + \|V\|\,\|T\|\right) \le \frac{1}{2}\left[\omega\left(\widehat{T}_V\right) + \|V\|\,\|T\|\right].$$

Proof. If we take $A = |T|^{1-t} V$ and $B = |T|^t$ in (2.1), then we get

$$\omega\left(\Delta_{t,V}(T)\right) \le \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left(\|\Delta_{t,V}(T)\| + \left\||T|^{1-t} V\right\|\left\||T|^t\right\|\right) \le \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left(\|\Delta_{t,V}(T)\| + \|V\|\,\|T\|\right)$$

for $t \in [0,1]$.

Also

$$\|\Delta_{t,V}(T)\| = \left\||T|^t V |T|^{1-t}\right\| \le \|V\|\,\|T\|$$

and the inequality (2.8) is proved. □

Remark 2. *For an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$ we have*

$$\omega\left(\Delta_t(T)\right) \le \frac{1}{2}\omega\left(\widehat{T}\right) + \frac{1}{4}\left(\|\Delta_t(T)\| + \|T\|\right) \le \frac{1}{2}\left[\omega\left(\widehat{T}\right) + \|T\|\right]. \tag{2.9}$$

In particular,

$$\omega(T) \le \frac{1}{2}\left[\omega\left(\widehat{T}\right) + \|T\|\right], \text{ see also (1.11)}$$

and

$$\omega\left(\widetilde{T}\right) \le \frac{1}{2}\omega\left(\widehat{T}\right) + \frac{1}{4}\left(\left\|\widetilde{T}\right\| + \|T\|\right) \le \frac{1}{2}\left[\omega\left(\widehat{T}\right) + \|T\|\right].$$

If $T = U|T|$ is the polar decomposition of T with U a partial isometry, then by putting $V = U$ in (2.8) we deduce the inequalities listed above in the Remark 2.

One can obtain other results for the difference $V^*V - VV^*$. For instance, by using the inequality from Theorem 2

$$\omega\left(\Delta_{t,V}(T)\right) \le \frac{1}{2}\left[\omega\left(\widehat{T}_V\right) + \|V\|\,\|T\|\right] \tag{2.10}$$

for $\frac{1}{2}(V^*V - VV^*)$, we obtain

$$\omega\left(|T|^t (V^*V - VV^*) |T|^{1-t}\right) \le \frac{1}{2}\left[\omega\left(|T|(V^*V - VV^*)\right) + \|V^*V - VV^*\|\,\|T\|\right] \tag{2.11}$$

for $t \in [0,1]$.

Also, if we take $\frac{1}{2}(V - V^*)$ in (2.10) we get

$$\omega\left(|T|^t \operatorname{Im}(V) |T|^{1-t}\right) \le \frac{1}{2}\left[\omega\left(|T| \operatorname{Im}(V)\right) + \|\operatorname{Im}(V)\|\,\|T\|\right] \tag{2.12}$$

while for $\frac{1}{2}(V + V^*)$ in (2.10) we obtain

$$\omega\left(|T|^t \operatorname{Re}(V) |T|^{1-t}\right) \le \frac{1}{2}\left[\omega\left(|T| \operatorname{Re}(V)\right) + \|\operatorname{Re}(V)\|\,\|T\|\right] \tag{2.13}$$

for $t \in [0,1]$.

We also have:

Theorem 3. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$,*

$$\omega\left(\Delta_{t,V}(T)\right) \le \frac{1}{2}\omega\left(T_V\right) + \frac{1}{4}\left\| \|T\|^{2(1-t)} |V^*|^2 + |T|^{2t} \right\| \le \frac{1}{2}\omega\left(T_V\right) + \frac{1}{4}\left(\|T\|^{2(1-t)} \|V\|^2 + \|T\|^{2t} \right). \tag{2.14}$$

In particular,

$$\omega\left(\widetilde{T}_V\right) \le \frac{1}{2}\omega\left(T_V\right) + \frac{1}{4}\left\| \|T\| |V^*|^2 + |T| \right\| \le \frac{1}{2}\omega\left(T_V\right) + \frac{1}{4}\left(\|V\|^2 + 1\right)\|T\|$$

and

$$\omega\left(\widehat{T}_V\right) \le \frac{1}{2}\omega\left(T_V\right) + \frac{1}{4}\left\| |V^*|^2 + |T|^2 \right\| \le \frac{1}{2}\omega\left(T_V\right) + \frac{1}{4}\left(\|V\|^2 + \|T\|^2\right).$$

Proof. We use the following inequality obtained in [11]

$$\omega(AB) \le \frac{1}{2}\omega(BA) + \frac{1}{4}\|BB^* + A^*A\| \tag{2.15}$$

for all $A,\ B \in \mathbb{B}(\mathcal{H})$.

If we take $A = |T|^t$ and $B = V|T|^{1-t}$, then we get

$$\begin{aligned} \omega\left(\Delta_{t,V}(T)\right) &= \omega\left(|T|^{1-t} V |T|^t\right) \\ &\le \frac{1}{2}\omega\left(V|T|\right) + \frac{1}{4}\left\| V|T|^{2(1-t)}V^* + |T|^{2t} \right\| \\ &= \frac{1}{2}\omega\left(T_V\right) + \frac{1}{4}\left\| V|T|^{2(1-t)}V^* + |T|^{2t} \right\|. \end{aligned} \tag{2.16}$$

Observe that

$$0 \le |T|^{2(1-t)} \le \left\| |T|^{2(1-t)} \right\| I = \|T\|^{2(1-t)} I.$$

Then by multiplying to the left with V and to the right with V^* we get

$$0 \le V|T|^{2(1-t)}V^* \le \|T\|^{2(1-t)} VV^* = \|T\|^{2(1-t)} |V^*|^2.$$

Therefore

$$0 \le V|T|^{2(1-t)}V^* + |T|^{2t} \le \|T\|^{2(1-t)}|V^*|^2 + |T|^{2t},$$

which implies that

$$\left\| V|T|^{2(1-t)}V^* + |T|^{2t} \right\| \le \left\| \|T\|^{2(1-t)}|V^*|^2 + |T|^{2t} \right\|$$

and by (2.16) we derive the first part of (2.14).

Finally, since

$$\begin{aligned} \left\| \|T\|^{2(1-t)}|V^*|^2 + |T|^{2t} \right\| &\le \|T\|^{2(1-t)} \left\| |V^*|^2 \right\| + \left\| |T|^{2t} \right\| \\ &= \|T\|^{2(1-t)}\|V\|^2 + \|T\|^{2t}, \end{aligned}$$

hence the proof is completed. □

Remark 3. *For an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$ we have*

$$\begin{aligned} \omega(\Delta_t(T)) &\le \frac{1}{2}\omega(T) + \frac{1}{4}\left\| \|T\|^{2(1-t)} + |T|^{2t} \right\| \\ &\le \frac{1}{2}\omega(T) + \frac{1}{4}\left(\|T\|^{2(1-t)} + \|T\|^{2t} \right). \end{aligned} \tag{2.17}$$

In particular,

$$\omega\left(\widetilde{T}\right) \le \frac{1}{2}\omega(T) + \frac{1}{4}\left\| \|T\| I + |T| \right\| \le \frac{1}{2}\left(\omega(T) + \|T\|\right)$$

and

$$\omega\left(\widehat{T}\right) \le \frac{1}{2}\omega(T) + \frac{1}{4}\left\| I + |T|^2 \right\| \le \frac{1}{2}\omega(T) + \frac{1}{4}\left(1 + \|T\|^2\right).$$

If $T = U|T|$ is the polar decomposition of T with U a partial isometry, then by putting $V = U$ in (2.14) and observing that $|U^*|^2 \le I$ we obtain the desired results.

Moreover, we can also state:

Theorem 4. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$ we have*

$$\begin{aligned} \omega(\Delta_{t,V}(T)) &\le \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left\| |T|^{2(1-t)} + \|T\|^{2t} |V|^2 \right\| \\ &\le \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left(\|T\|^{2(1-t)} + \|T\|^{2t} \|V\|^2 \right). \end{aligned} \tag{2.18}$$

In particular,

$$\begin{aligned} \omega(T_V) &\le \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left\| |T|^2 + |V|^2 \right\| \\ &\le \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left(\|T\|^2 + \|V\|^2 \right), \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \omega\left(\widetilde{T}_V\right) &\le \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left\| |T| + \|T\| |V|^2 \right\| \\ &\le \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left(1 + \|V\|^2\right)\|T\|. \end{aligned} \tag{2.20}$$

Proof. If we take $A = |T|^t V$ and $B = |T|^{1-t}$ in (2.15), then we get

$$\begin{aligned} \omega(\Delta_{t,V}(T)) &= \omega\left(|T|^t V |T|^{1-t}\right) \\ &\le \frac{1}{2}\omega(|T| V) + \frac{1}{4}\left\| |T|^{2(1-t)} + V^* |T|^{2t} V \right\|. \end{aligned} \tag{2.21}$$

Now, observe that, since

$$0 \le |T|^{2t} \le \left\| |T|^{2t} \right\| I = \|T\|^{2t} I,$$

hence

$$0 \le V^* |T|^{2t} V \le \|T\|^{2t} V^* V = \|T\|^{2t} |V|^2 \le \|T\|^{2t} \left\| |V|^2 \right\| I = \|T\|^{2t} \|V\|^2 I.$$

Therefore

$$\begin{aligned}\left\| |T|^{2(1-t)} + V^* |T|^{2t} V \right\| &\le \left\| |T|^{2(1-t)} + \|T\|^{2t} |V|^2 \right\| \\ &\le \left\| |T|^{2(1-t)} + \|T\|^{2t} \|V\|^2 I \right\| \\ &\le \|T\|^{2(1-t)} + \|T\|^{2t} \|V\|^2\end{aligned}$$

and by (2.21) we get (2.18). □

Remark 4. *For an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$ we have*

$$\begin{aligned}\omega(\Delta_t(T)) &\le \frac{1}{2}\omega\left(\widehat{T}\right) + \frac{1}{4}\left\| |T|^{2(1-t)} + \|T\|^{2t} \right\| \\ &\le \frac{1}{2}\omega\left(\widehat{T}\right) + \frac{1}{4}\left(\|T\|^{2(1-t)} + \|T\|^{2t} \right).\end{aligned} \tag{2.22}$$

In particular,

$$\omega(T) \le \frac{1}{2}\omega\left(\widehat{T}\right) + \frac{1}{4}\left\| |T|^2 + I \right\| \le \frac{1}{2}\omega\left(\widehat{T}\right) + \frac{1}{4}\left(\|T\|^2 + 1 \right)$$

and

$$\omega\left(\widetilde{T}\right) \le \frac{1}{2}\omega\left(\widehat{T}\right) + \frac{1}{4}\left\| |T| + \|T\| I \right\| \le \frac{1}{2}\left(\omega\left(\widehat{T}\right) + \|T\| \right).$$

If $T = U|T|$ is the polar decomposition of T with U a partial isometry, then by putting $V = U$ in (2.18) and observing that $|U|^2 \le I$ we obtain the desired inequalities.

3. Some Related Results

We also have:

Theorem 5. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$ we have*

$$\begin{aligned}\omega^2(\Delta_{t,V}(T)) &\le \frac{1}{2}\left\| |T|^{1-t} V^* |T|^{2t} V |T|^{1-t} + |T|^t V |T|^{2(1-t)} V^* |T|^t \right\| \\ &\le \frac{1}{2}\left\| \|T\|^{2t} |T|^{2(1-t)} + \|T\|^{2(1-t)} |T|^{2t} \right\| \le \|T\|^2.\end{aligned} \tag{3.1}$$

In particular,

$$\omega^2\left(\widehat{T}_V\right) \le \frac{1}{2}\left\| \|T\|^2 |V|^2 + |T| |V^*|^2 |T| \right\| \le \frac{1}{2}\left\| \|T\|^2 + |T|^2 \right\| \le \|T\|^2,$$

$$\omega^2\left(\widetilde{T}_V\right) \le \frac{1}{2}\left\| |T|^{1/2}\left(|V|^2 + |V^*|^2 \right) |T|^{1/2} \right\| \|T\| \le \|T\|^2$$

and

$$\omega^2(T_V) \le \frac{1}{2}\left\| |T| |V|^2 |T| + \|T\|^2 |V^*|^2 \right\| \le \frac{1}{2}\left\| |T|^2 + \|T\|^2 I \right\| \le \|T\|^2.$$

Proof. We use the following inequality obtained by Kittaneh in [10]

$$\omega^2(A) \le \frac{1}{2}\|A^*A + AA^*\| \tag{3.2}$$

for all $A \in \mathbb{B}(\mathcal{H})$.

If we put $A = |T|^t V |T|^{1-t}$ in (2.15), then we get

$$\omega^2\left(|T|^t V |T|^{1-t} \right) \le \frac{1}{2}\left\| |T|^{1-t} V^* |T|^{2t} V |T|^{1-t} + |T|^t V |T|^{2(1-t)} V^* |T|^t \right\| \tag{3.3}$$

for $t\in[0,1]$.

Since $|T|^{2t}\le\|T\|^{2t}$ then $V^{*}|T|^{2t}V\le\|T\|^{2t}|V|^{2}$, which implies that

$$|T|^{1-t}V^{*}|T|^{2t}V|T|^{1-t}\le\|T\|^{2t}|T|^{1-t}|V|^{2}|T|^{1-t}=\|T\|^{2t}|T|^{2(1-t)},$$

since $|V|^{2}\le I$.

Also $|T|^{2(1-t)}\le\|T\|^{2(1-t)}$ implies that $V|T|^{2(1-t)}V^{*}\le\|T\|^{2(1-t)}|V^{*}|^{2}$ and

$$|T|^{t}V|T|^{2(1-t)}V^{*}|T|^{t}\le\|T\|^{2(1-t)}|T|^{t}|V^{*}|^{2}|T|^{t}\le\|T\|^{2(1-t)}|T|^{2t}$$

since $|V^{*}|^{2}\le I$.

By using (3.3) we derive (3.1). □

Remark 5. *For an operator $T\in\mathcal{B}(H)$ and $t\in[0,1]$ we have*

$$\omega(\Delta_{t}(T))\le\frac{\sqrt{2}}{2}\left\|\|T\|^{2t}|T|^{2(1-t)}+\|T\|^{2(1-t)}|T|^{2t}\right\|^{1/2}\le\|T\|.$$

In particular,

$$\omega\left(\widehat{T}\right)\le\frac{\sqrt{2}}{2}\left\|\|T\|^{2}+|T|^{2}\right\|^{1/2}\le\|T\|$$

and

$$\omega(T)\le\frac{\sqrt{2}}{2}\left\||T|^{2}+\|T\|^{2}I\right\|^{1/2}\le\|T\|.$$

Finally, we can also state:

Theorem 6. *For a contraction $V\in\mathcal{B}(H)$, an operator $T\in\mathcal{B}(H)$ and $t\in[0,1]$ we have for $r\ge2$ and $\alpha\in[0,1]$ that*

$$\begin{aligned}\omega^{r}(\Delta_{t,V}(T))&\le\|V\|^{r}\left\|\alpha|T|^{\frac{tr}{\alpha}}+(1-\alpha)|T|^{\frac{(1-t)r}{1-\alpha}}\right\|\\&\le\left\|\alpha|T|^{\frac{tr}{\alpha}}+(1-\alpha)|T|^{\frac{(1-t)r}{1-\alpha}}\right\|.\end{aligned}\tag{3.4}$$

In particular

$$\omega^{r}\left(\widehat{T}_{V}\right)\le\|V\|^{r}\left\|\alpha|T|^{\frac{r}{\alpha}}+(1-\alpha)I\right\|\le\left\|\alpha|T|^{\frac{r}{\alpha}}+(1-\alpha)I\right\|,$$

$$\begin{aligned}\omega^{r}\left(\widetilde{T}_{V}\right)&\le\|V\|^{r}\left\|\alpha|T|^{\frac{r}{2\alpha}}+(1-\alpha)|T|^{\frac{r}{2(1-\alpha)}}\right\|\\&\le\left\|\alpha|T|^{\frac{r}{2\alpha}}+(1-\alpha)|T|^{\frac{r}{2(1-\alpha)}}\right\|\end{aligned}$$

and

$$\omega^{r}(T_{V})\le\|V\|^{r}\left\|\alpha I+(1-\alpha)|T|^{\frac{r}{1-\alpha}}\right\|\le\left\|\alpha I+(1-\alpha)|T|^{\frac{r}{1-\alpha}}\right\|.$$

If $r\ge0$, $\alpha\in(0,1]$ and $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\,qr\ge2$, we also have that

$$\begin{aligned}\omega^{r}(\Delta_{t,V}(T))&\le\|V\|^{r}\left\|\frac{1}{p}|T|^{\frac{prt}{\alpha}}+\frac{1}{q}|T|^{\frac{qr(1-t)}{\alpha}}\right\|^{\alpha}\\&\le\left\|\frac{1}{p}|T|^{\frac{prt}{\alpha}}+\frac{1}{q}|T|^{\frac{qr(1-t)}{\alpha}}\right\|^{\alpha}.\end{aligned}\tag{3.5}$$

In particular,

$$\omega^{r}\left(\widehat{T}_{V}\right)\le\|V\|^{r}\left\|\frac{1}{p}|T|^{\frac{pr}{\alpha}}+\frac{1}{q}I\right\|^{\alpha}\le\left\|\frac{1}{p}|T|^{\frac{pr}{\alpha}}+\frac{1}{q}I\right\|^{\alpha},$$

$$\omega^r\left(\widetilde{T}_V\right)\leq\|V\|^r\left\|\frac{1}{p}|T|^{\frac{pr}{2\alpha}}+\frac{1}{q}|T|^{\frac{qr}{2\alpha}}\right\|^{\alpha}\leq\left\|\frac{1}{p}|T|^{\frac{pr}{2\alpha}}+\frac{1}{q}|T|^{\frac{qr}{2\alpha}}\right\|^{\alpha}$$

and

$$\omega^r\left(T_V\right)\leq\|V\|^r\left\|\frac{1}{p}I+\frac{1}{q}|T|^{\frac{qr}{\alpha}}\right\|^{\alpha}\leq\left\|\frac{1}{p}I+\frac{1}{q}|T|^{\frac{qr}{\alpha}}\right\|^{\alpha}.$$

Proof. We use the following inequality obtained in [11]

$$\omega^r\left(A^{\alpha}XB^{1-\alpha}\right)\leq\|X\|^r\left\|\alpha A^r+(1-\alpha)B^r\right\|, \tag{3.6}$$

where $A,\ B\geq 0,\ X\in\mathbb{B}(\mathcal{H}),\ r\geq 2$ and $\alpha\in[0,1]$.

By utilizing (3.6) we get

$$\begin{aligned}\omega^r\left(|T|^t V|T|^{1-t}\right)&=\omega^r\left(\left(|T|^{\frac{t}{\alpha}}\right)^{\alpha}V\left(|T|^{\frac{1-t}{1-\alpha}}\right)^{1-\alpha}\right)\\&\leq\|V\|^r\left\|\alpha\left(|T|^{\frac{t}{\alpha}}\right)^r+(1-\alpha)\left(|T|^{\frac{1-t}{1-\alpha}}\right)^r\right\|\\&\leq\|V\|^r\left\|\alpha|T|^{\frac{tr}{\alpha}}+(1-\alpha)|T|^{\frac{(1-t)r}{1-\alpha}}\right\|\\&\leq\left\|\alpha|T|^{\frac{tr}{\alpha}}+(1-\alpha)|T|^{\frac{(1-t)r}{1-\alpha}}\right\|,\end{aligned}$$

which proves (3.4).

We use the following inequality obtained in [11]

$$\omega^r\left(A^{\alpha}XB^{\alpha}\right)\leq\|X\|^r\left\|\frac{1}{p}A^{pr}+\frac{1}{q}B^{qr}\right\|^{\alpha} \tag{3.7}$$

for $A,\ B\geq 0,\ X\in\mathbb{B}(\mathcal{H}),\ 0\leq\alpha\leq 1,\ r\geq 0$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\geq 2$.

We have by (3.7) that

$$\begin{aligned}\omega^r\left(|T|^t V|T|^{1-t}\right)&=\omega^r\left(\left(|T|^{\frac{t}{\alpha}}\right)^{\alpha}V\left(|T|^{\frac{1-t}{\alpha}}\right)^{\alpha}\right)\\&\leq\|V\|^r\left\|\frac{1}{p}\left(|T|^{\frac{t}{\alpha}}\right)^{pr}+\frac{1}{q}\left(|T|^{\frac{1-t}{\alpha}}\right)^{qr}\right\|^{\alpha}\\&=\|V\|^r\left\|\frac{1}{p}|T|^{\frac{prt}{\alpha}}+\frac{1}{q}|T|^{\frac{qr(1-t)}{\alpha}}\right\|^{\alpha}\\&\leq\left\|\frac{1}{p}|T|^{\frac{prt}{\alpha}}+\frac{1}{q}|T|^{\frac{qr(1-t)}{\alpha}}\right\|^{\alpha},\end{aligned}$$

which gives (3.5). □

By taking $V=U$ from the polar decomposition of T we can obtain the corresponding inequalities concerning the usual transforms, namely

$$\omega\left(\Delta_t(T)\right)\leq\left\|\alpha|T|^{\frac{tr}{\alpha}}+(1-\alpha)|T|^{\frac{(1-t)r}{1-\alpha}}\right\|^{1/r} \tag{3.8}$$

for all $r\geq 2$, $\alpha\in[0,1]$ and $t\in[0,1]$.

In particular

$$\omega\left(\widehat{T}\right)\leq\left\|\alpha|T|^{\frac{r}{\alpha}}+(1-\alpha)I\right\|^{1/r},\ \omega\left(\widetilde{T}\right)\leq\left\|\alpha|T|^{\frac{r}{2\alpha}}+(1-\alpha)|T|^{\frac{r}{2(1-\alpha)}}\right\|^{1/r}$$

and

$$\omega(T)\leq\left\|\alpha I+(1-\alpha)|T|^{\frac{r}{1-\alpha}}\right\|^{1/r}.$$

Also for $r>0$, $\alpha\in(0,1]$ and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\geq 2$, we also have that

$$\omega\left(\Delta_t\left(T\right)\right)\leq\left\Vert\frac{1}{p}\left\vert T\right\vert^{\frac{prt}{\alpha}}+\frac{1}{q}\left\vert T\right\vert^{\frac{qr(1-t)}{\alpha}}\right\Vert^{\alpha/r} \tag{3.9}$$

for all $t\in[0,1]$.

In particular,

$$\omega\left(\widehat{T}\right)\leq\left\Vert\frac{1}{p}\left\vert T\right\vert^{\frac{pr}{\alpha}}+\frac{1}{q}I\right\Vert^{\alpha/r},\ \omega\left(\widetilde{T}\right)\leq\left\Vert\frac{1}{p}\left\vert T\right\vert^{\frac{pr}{2\alpha}}+\frac{1}{q}\left\vert T\right\vert^{\frac{qr}{2\alpha}}\right\Vert^{\alpha/r}$$

and

$$\omega\left(T\right)\leq\left\Vert\frac{1}{p}I+\frac{1}{q}\left\vert T\right\vert^{\frac{qr}{\alpha}}\right\Vert^{\alpha/r}.$$

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