SOME INEQUALITIES FOR THE EXTENDED GENERALIZED ALUTHGE TRANSFORM OF BOUNDED OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a *contraction* $V \in \mathcal{B}(H)$, i.e. $0 \leq V^*V \leq I$, an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$ we define the operator

$$\Delta_{t,V}(T) := |T|^t V |T|^{1-t}$$

that we *call the extended generalized Aluthge transform.* In this paper we provide several inequalities concerning the extended generalized Aluthge transform $\Delta_{t,V}(T)$. The cases of usual generalized Aluthge, Dougal and Aluthge transforms are also presented.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leq w(T)\|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda|\,\omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T + V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{1.3}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [9], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [10] improved the inequality (1.3) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{1.5}$$

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for any operator $T\in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [8]:

If for an operator $T\in B(H)$ we denote $|T|:=(T^{*}T)^{1/2}$, then

$$\omega^{r}(T)\leq\frac{1}{2}\left\Vert |T|^{2\alpha r}+|T^{*}|^{2(1-\alpha)r}\right\Vert \tag{1.6}$$

and

$$\omega^{2r}(T)\leq\left\Vert \alpha|T|^{2r}+(1-\alpha)|T^{*}|^{2r}\right\Vert , \tag{1.7}$$

where $\alpha\in(0,1)$ and $r\geq1$.

If we take $\alpha=\frac{1}{2}$ and $r=1$ we get from (1.6) that

$$\omega(T)\leq\frac{1}{2}\left\Vert |T|+|T^{*}|\right\Vert \tag{1.8}$$

and from (1.7) that

$$\omega^{2}(T)\leq\frac{1}{2}\left\Vert |T|^{2}+|T^{*}|^{2}\right\Vert . \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [7] and [4].

Let $T=U|T|$ be the *polar decomposition* of the bounded linear operator T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T}:=|T|^{1/2}U|T|^{1/2}$, see [1].

The following properties of $\widetilde{T}$ are as follows:

(i) $\left\Vert \widetilde{T}\right\Vert \leq\Vert T\Vert$,

(ii) $w\left(\widetilde{T}\right)\leq\omega(T)$,

(iii) $r\left(\widetilde{T}\right)=\omega(T)$,

(iv) $\omega\left(\widetilde{T}\right)\leq\left\Vert T^{2}\right\Vert^{1/2}(\leq\Vert T\Vert)$, [12].

Utilizing this transform T. Yamazaki, [12] obtained in 2007 the following refinement of Kittaneh's inequality (1.4):

$$\omega(T)\leq\frac{1}{2}\left(\Vert T\Vert+\omega\left(\widetilde{T}\right)\right)\leq\frac{1}{2}\left(\Vert T\Vert+\left\Vert T^{2}\right\Vert^{1/2}\right) \tag{1.10}$$

for any operator $T\in B(H)$.

We remark that if $\widetilde{T}=0$, then obviously $w(T)=\frac{1}{2}\Vert T\Vert$.

For a *contraction* $V\in\mathcal{B}(H)$, i.e. $0\leq V^{*}V\leq I$ and an operator $T\in\mathcal{B}(H)$ and $t\in[0,1]$ we define the operator

$$\Delta_{t,V}(T):=|T|^{t}V|T|^{1-t}$$

that we *call the extended generalized Aluthge transform.*

We assume in what follows that $|T|^{0}:=I$.

For $t=1$ we have

$$\widehat{T}_{V}:=\Delta_{1,V}(T)=|T|V,$$

that we call *the extended Dougal transform*, for $t=1/2$,

$$\widetilde{T}_{V}=\Delta_{1/2,V}(T):=|T|^{1/2}V|T|^{1/2},$$

that we *call the extended Aluthge transform* and for $t=0$,

$$T_{V}:=\Delta_{0,V}(T)=V|T|.$$

An operator $U \in \mathcal{B}(H)$ is called a *partial isometry* if $\|Ux\| = \|x\|$ for all $x \in \mathcal{N}^{\perp}(U)$.

Now, let $x \in H$, then there exists a unique $x_1 \in \mathcal{N}(U)$ and a unique $x_2 \in \mathcal{N}^{\perp}(U)$ such that $x = x_1 + x_2$. Then

$$0 \leq \langle U^*Ux, x\rangle = \|Ux\|^2 = \|Ux_1 + Ux_2\|^2 = \|Ux_2\|^2 = \|x_2\|^2 .$$

By the fact that $x_1 \perp x_2$,

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 .$$

Therefore

$$0 \leq \langle U^*Ux, x\rangle \leq \|x\|^2 ,$$

which shows that U is a contraction on H.

Let $T \in \mathcal{B}(H)$ and $T = U|T|$ the polar decomposition of T with U a partial isometry. Then

$$T_U = U|T| = T,$$

$$\widetilde{T}_U = |T|^{1/2} U |T|^{1/2} = \widetilde{T}$$

is the usual *Aluthge transform* and

$$\widehat{T}_U = |T| U = \widehat{T}$$

is the usual *Dougal transform.*

For $t \in (0,1)$

$$\Delta_{t,U}(T) = |T|^t U |T|^{1-t} =: \Delta_t(T)$$

is the *generalized Aluthge transform* introduced in by Cho and Tanahashi in [6].

Abu-Omar and Kittaneh [2] improved on inequality (1.10) using generalized Aluthge transform to prove that

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \min_{t\in[0,1]} \omega(\Delta_t(T))\right). \tag{1.11}$$

For $t = 1$ this also gives the following result for the *Dougal transform*

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \omega\left(\widehat{T}\right)\right). \tag{1.12}$$

In [3] Bunia et al. also proved that

$$\omega(T) \leq \min_{t\in[0,1]}\left\{\frac{1}{2}\omega(\Delta_t(T)) + \frac{1}{4}\left(\|T\|^{2t} + \|T\|^{2(1-t)}\right)\right\}, \tag{1.13}$$

which for $t = 1/2$ gives (1.10) as well.

If V is a contraction, then $\|V\| \leq 1$ and since $\|V^*\| = \|V\|$, hence V^* is also a contraction. Observe that

$$\Delta^*_{t,V}(T) := \left(|T|^t V |T|^{1-t}\right)^* = |T|^{1-t} V^* |T|^t = \Delta_{1-t,V^*}(T)$$

for all $t \in [0,1]$. Therefore

$$(T_V)^* = \widehat{T}_{V^*},\ \left(\widehat{T}_V\right)^* = T_{V^*}$$

and

$$\left(\widetilde{T}_V\right)^* = \widetilde{T}_{V^*}.$$

Since $\|V^*V\| = \|VV^*\| = \|V\|^2$ and V is a contraction, then

$$\left\| \frac{V^*V \pm VV^*}{2} \right\| \leq \|V\|^2 \leq 1$$

showing that

$$W := \frac{V^*V \pm VV^*}{2}$$

is a contraction and we can consider the transform

$$\Delta_{t, \frac{V^*V \pm VV^*}{2}} (T) := |T|^t \left(\frac{V^*V \pm VV^*}{2} \right) |T|^{1-t}$$

for $t \in [0, 1]$.

For a contraction V, we have

$$\operatorname{Im}(V) := \frac{V - V^*}{2i}, \quad \operatorname{Re}(V) := \operatorname{Re}\left(\frac{V + V^*}{2} \right)$$

and since

$$\|\operatorname{Im}(V)\| = \left\| \frac{V - V^*}{2i} \right\| \leq \|V\| \leq 1 \text{ and } \|\operatorname{Re}(V)\| \leq \|V\| \leq 1,$$

hence $\operatorname{Im}(V)$ and $\operatorname{Re}(V)$ are contractions as well. We can then consider the transforms

$$\Delta_{t,\operatorname{Im}(V)} (T) := |T|^t \operatorname{Im}(V) |T|^{1-t} \text{ and } \Delta_{t,\operatorname{Re}(V)} (T) := |T|^t \operatorname{Re}(V) |T|^{1-t}$$

for $t \in [0, 1]$.

For $T \in \mathcal{B}(H)$ we define

$$T_+ := \frac{1}{2} (|T| + T) \text{ and } T_- := \frac{1}{2} (|T| - T).$$

If U is the partial isometry in the polar representation of T, then

$$V := \frac{I \pm U}{2}$$

is a contraction and

$$\Delta_{t, \frac{I \pm U}{2}} (T) := |T|^t \frac{I \pm U}{2} |T|^{1-t} = \frac{|T| \pm \Delta_t(T)}{2}.$$

In particular, we get

$$T_{\frac{I \pm U}{2}} = \frac{|T| \pm T}{2} = T_{\pm}, \widehat{T}_{\frac{I \pm U}{2}} = \frac{|T| \pm \widehat{T}}{2}$$

and

$$\widetilde{T}_{\frac{I \pm U}{2}} = \frac{|T| \pm \widetilde{T}}{2}$$

for any operator $T \in \mathcal{B}(H)$.

Motivated by the above results, in this paper we provide several inequalities concerning the extended generalized Aluthge transform $\Delta_{t,V}(T)$. The cases of usual generalized Aluthge, Dougal and Aluthge transforms are also presented.

2. Main Results

We need the following fact, see [5]:

Lemma 1. *For any* $A,\, B\in\mathcal{B}(H)$,

$$\omega(AB)\le\frac{1}{2}\omega(BA)+\frac{1}{4}(\|AB\|+\|A\|\,\|B\|)\le\frac{1}{2}(\omega(BA)+\|A\|\,\|B\|).\tag{2.1}$$

We have the following first result:

Theorem 1. *For a contraction* $V\in\mathcal{B}(H)$, *an operator* $T\in\mathcal{B}(H)$ *we have that*

$$\omega(T_V)\le\frac{1}{2}\min_{t\in[0,1]}\omega(\Delta_{t,V}(T))+\frac{1}{4}(\|T_V\|+\|V\|\,\|T\|)\le\frac{1}{2}\left(\min_{t\in[0,1]}\omega(\Delta_{t,V}(T))+\|V\|\,\|T\|\right).\tag{2.2}$$

In particular,

$$\omega(T_V)\le\frac{1}{2}\omega\left(\widetilde{T}_V\right)+\frac{1}{4}(\|T_V\|+\|V\|\,\|T\|)\le\frac{1}{2}\left(\omega\left(\widetilde{T}_V\right)+\|V\|\,\|T\|\right)\tag{2.3}$$

and

$$\omega(T_V)\le\frac{1}{2}\omega\left(\widehat{T}_V\right)+\frac{1}{4}(\|T_V\|+\|V\|\,\|T\|)\le\frac{1}{2}\left(\omega\left(\widehat{T}_V\right)+\|V\|\,\|T\|\right).\tag{2.4}$$

Also,

$$\omega\left(\widehat{T}_V\right)\le\frac{1}{2}\min_{t\in[0,1]}\omega(\Delta_{t,V}(T))+\frac{1}{4}\left(\left\|\widehat{T}_V\right\|+\|V\|\,\|T\|\right)\le\frac{1}{2}\left(\min_{t\in[0,1]}\omega(\Delta_{t,V}(T))+\|V\|\,\|T\|\right).\tag{2.5}$$

In particular,

$$\omega\left(\widehat{T}_V\right)\le\frac{1}{2}\omega(T_V)+\frac{1}{4}\left(\left\|\widehat{T}_V\right\|+\|V\|\,\|T\|\right)\le\frac{1}{2}(\omega(T_V)+\|V\|\,\|T\|)\tag{2.6}$$

and

$$\omega\left(\widehat{T}_V\right)\le\frac{1}{2}\omega\left(\widetilde{T}_V\right)+\frac{1}{4}\left(\left\|\widehat{T}_V\right\|+\|V\|\,\|T\|\right)\le\frac{1}{2}\left(\omega\left(\widetilde{T}_V\right)+\|V\|\,\|T\|\right).\tag{2.7}$$

Proof. If we take $A = V|T|^{1-t}$ and $B = |T|^t$ in (2.1) then we get

$$\begin{aligned}
\omega(T_V) &= \omega(V|T|) = \omega\left(V|T|^{1-t}|T|^t\right) \\
&\leq \frac{1}{2}\omega\left(|T|^t V |T|^{1-t}\right) + \frac{1}{4}\left(\|V|T|\| + \left\|V|T|^{1-t}\right\|\left\||T|^t\right\|\right) \\
&\leq \frac{1}{2}\omega\left(|T|^t V |T|^{1-t}\right) + \frac{1}{4}\left(\|V|T|\| + \|V\|\left\||T|^{1-t}\right\|\left\||T|^t\right\|\right) \\
&= \frac{1}{2}\omega\left(\Delta_{t,V}(T)\right) + \frac{1}{4}\left(\|T_V\| + \|V\|\|T\|^{1-t}\|T\|^t\right) \\
&= \frac{1}{2}\omega\left(\Delta_{t,V}(T)\right) + \frac{1}{4}\left(\|T_V\| + \|V\|\|T\|\right)
\end{aligned}$$

for all $t \in [0,1]$, which proves (2.2).

If we take in Lemma 1 $A = |T|^{1-t}$, $B = |T|^t V$, then we derive

$$\begin{aligned}
\omega\left(\widehat{T}_V\right) &= \omega(|P|V) = \omega\left(|T|^{1-t}|T|^t V\right) \\
&\leq \frac{1}{2}\omega\left(|T|^t V |T|^{1-t}\right) + \frac{1}{4}\left(\||P|V\| + \left\||T|^{1-t}\right\|\left\||T|^t V\right\|\right) \\
&\leq \frac{1}{2}\omega\left(\Delta_{t,V}(T)\right) + \frac{1}{4}\left(\left\|\widehat{T}_V\right\| + \|T\|^{1-t}\|V\|\|T\|^t\right) \\
&= \frac{1}{2}\omega\left(\Delta_{t,V}(T)\right) + \frac{1}{4}\left(\left\|\widehat{T}_V\right\| + \|V\|\|T\|\right)
\end{aligned}$$

for all $t \in [0,1]$, which proves (2.2). □

Remark 1. *For an operator $T \in \mathcal{B}(H)$ we recapture some known results*

$$\omega(T) \leq \frac{1}{2}\left(\min_{t\in[0,1]} \omega\left(\Delta_t(T)\right) + \|T\|\right), \text{ see also (1.11).} \tag{2.8}$$

In particular,

$$\omega(T) \leq \frac{1}{2}\left(\omega\left(\widetilde{T}\right) + \|T\|\right), \text{ see also (1.10)}$$

and

$$\omega(T) \leq \frac{1}{2}\left(\omega\left(\widehat{T}\right) + \|T\|\right) \text{ see also (1.12).}$$

Also, we obtain the following bounds for the Dougal transform

$$\begin{aligned}
\omega\left(\widehat{T}\right) &\leq \frac{1}{2}\min_{t\in[0,1]} \omega\left(\Delta_t(T)\right) + \frac{1}{4}\left(\left\|\widehat{T}\right\| + \|T\|\right) \\
&\leq \frac{1}{2}\left(\min_{t\in[0,1]} \omega\left(\Delta_t(T)\right) + \|T\|\right).
\end{aligned} \tag{2.9}$$

In particular,

$$\omega\left(\widehat{T}\right) \leq \frac{1}{2}\omega(T) + \frac{1}{4}\left(\left\|\widehat{T}\right\| + \|T\|\right) \leq \frac{1}{2}\left(\omega(T) + \|T\|\right)$$

and

$$\omega\left(\widehat{T}\right) \leq \frac{1}{2}\omega\left(\widetilde{T}\right) + \frac{1}{4}\left(\left\|\widehat{T}\right\| + \|T\|\right) \leq \frac{1}{2}\left(\omega\left(\widetilde{T}\right) + \|T\|\right).$$

The proof follows from Theorem 1 applied for the polar decomposition $T = U|T|$ where U is a partial isometry and observing that for $V = U$, $\|V\| \leq 1$.

Theorem 2. *With the assumptions of Theorem 1, we have*

$$\omega(T_V) \leq \frac{1}{2}\omega(\Delta_{t,V}(T)) + \frac{1}{4}\left\| |T|^{2t} + |T|^{1-t}|V|^2|T|^{1-t} \right\| \tag{2.10}$$
$$\leq \frac{1}{2}\omega(\Delta_{t,V}(T)) + \frac{1}{4}\left\| |T|^{2t} + |T|^{2(1-t)} \right\|$$

for all $t \in [0,1]$.

In particular

$$\omega(T_V) \leq \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left\| |T|^2 + |V|^2 \right\| \tag{2.11}$$
$$\leq \frac{1}{2}\left(\omega\left(\widehat{T}_V\right) + \frac{1}{2}\left\| |T|^{2t} + I \right\|\right)$$

and

$$\omega(T_V) \leq \frac{1}{2}\omega\left(\widetilde{T}_V\right) + \frac{1}{4}\left\| |T| + |T|^{1/2}|V|^2|T|^{1/2} \right\| \tag{2.12}$$
$$\leq \frac{1}{2}\left(\omega\left(\widetilde{T}_V\right) + \|T\|\right).$$

Also

$$\omega\left(\widehat{T}_V\right) \leq \frac{1}{2}\omega(\Delta_{t,V}(T)) + \frac{1}{4}\left\| |T|^t|V^*|^2|T|^t + |T|^{2(1-t)} \right\| \tag{2.13}$$
$$\leq \frac{1}{2}\omega(\Delta_{t,V}(T)) + \frac{1}{4}\left\| |T|^{2t} + |T|^{2(1-t)} \right\|$$

for all $t \in [0,1]$.

In particular,

$$\omega\left(\widehat{T}_V\right) \leq \frac{1}{2}\omega(T_V) + \frac{1}{4}\left\| |V^*|^2 + |T|^2 \right\| \tag{2.14}$$
$$\leq \frac{1}{2}\left(\omega(T_V) + \frac{1}{2}\left\| I + |T|^2 \right\|\right)$$

and

$$\omega\left(\widehat{T}_V\right) \leq \frac{1}{2}\omega\left(\widetilde{T}_V\right) + \frac{1}{4}\left\| |T|^{1/2}|V^*|^2|T|^{1/2} + |T| \right\| \tag{2.15}$$
$$\leq \frac{1}{2}\left(\omega\left(\widetilde{T}_V\right) + \|T\|\right).$$

Proof. We use the following inequality obtained in [11]

$$\omega(AB) \leq \frac{1}{4}\|BB^* + A^*A\| + \frac{1}{2}\omega(BA) \tag{2.16}$$

for all $A,\ B \in \mathbb{B}(\mathcal{H})$.

If we take $A = V|T|^{1-t}$ and $B = |T|^t$ in (2.16), then we get

$$\omega(V|T|) \leq \frac{1}{4}\left\| |T|^{2t} + |T|^{1-t}V^*V|T|^{1-t} \right\| + \frac{1}{2}\omega\left(|T|^tV|T|^{1-t}\right). \tag{2.17}$$

Observe that

$$0 \leq V^*V = |V|^2 \leq \left\| |V|^2 \right\| I = \|V\|^2 I \leq I \tag{2.18}$$

since $\|V\| \leq 1$.

By multiplying both sides of (2.18) with $|T|^{1-t} \geq 0$ we get

$$0 \leq |T|^{1-t}V^*V|T|^{1-t} \leq |T|^{2(1-t)},$$

which gives that

$$0 \leq |T|^{2t} + |T|^{1-t} V^* V |T|^{1-t} \leq |T|^{2t} + |T|^{2(1-t)}.$$

By taking the norm we obtain

$$\left\| |T|^{2t} + |T|^{1-t} V^* V |T|^{1-t} \right\| \leq \left\| |T|^{2t} + |T|^{2(1-t)} \right\|$$

and by (2.17) we obtain (2.10).

Further, if we take in (2.16) $A = |T|^{1-t}$, $B = |T|^t V$, then we have

$$\omega (|T| V) \leq \frac{1}{4} \left\| |T|^t V V^* |T|^t + |T|^{2(1-t)} \right\| + \frac{1}{2} \omega \left(|T|^t V |T|^{1-t} \right). \tag{2.19}$$

Since, as above,

$$0 \leq |T|^t V V^* |T|^t \leq |T|^{2t},$$

hence

$$\left\| |T|^t V V^* |T|^t + |T|^{2(1-t)} \right\| \leq \left\| |T|^{2t} + |T|^{2(1-t)} \right\|$$

and by (2.19) we get (2.13). □

Remark 2. *For an operator $T \in \mathcal{B}(H)$ we have*

$$\omega (T) \leq \frac{1}{2} \omega (\Delta_t (T)) + \frac{1}{4} \left\| |T|^{2t} + |T|^{2(1-t)} \right\| \tag{2.20}$$

for all $t \in [0, 1]$. The inequality (2.20) is better than (1.13) since

$$\left\| |T|^{2t} + |T|^{2(1-t)} \right\| \leq \|T\|^{2t} + \|T\|^{2(1-t)}.$$

In particular we get

$$\omega (T) \leq \frac{1}{2} \left(\omega \left(\widehat{T}_V \right) + \frac{1}{2} \left\| |T|^{2t} + I \right\| \right) \tag{2.21}$$

and the known inequality

$$\omega (T) \leq \frac{1}{2} \left(\omega \left(\widetilde{T} \right) + \|T\| \right), \text{ see (1.12).}$$

Also

$$\omega \left(\widehat{T} \right) \leq \frac{1}{2} \omega (\Delta_t (T)) + \frac{1}{4} \left\| |T|^{2t} + |T|^{2(1-t)} \right\| \tag{2.22}$$

for all $t \in [0, 1]$.

In particular,

$$\omega \left(\widehat{T} \right) \leq \frac{1}{2} \left(\omega (T) + \frac{1}{2} \left\| I + |T|^2 \right\| \right) \tag{2.23}$$

and

$$\omega \left(\widehat{T} \right) \leq \frac{1}{2} \left(\omega \left(\widetilde{T} \right) + \|T\| \right). \tag{2.24}$$

The proof follows from Theorem 2 applied for the polar decomposition $T = U |T|$ where U is a partial isometry and observing that for $V = U$, $\|V\| \leq 1$.

3. Some Related Results

We start to the following simple result:

Proposition 1. *For a contraction $V \in \mathcal{B}(H)$ and an operator $T \in \mathcal{B}(H)$, we have*

$$\omega(T_V),\ \omega\left(\widehat{T}_V\right) \leq \frac{\sqrt{2}}{2}\left\||T|^2 + \|T\|^2 I\right\|^{1/2} \leq \|T\|. \tag{3.1}$$

Proof. We use the following inequality obtained by Kittaneh in [10]

$$\omega^2(A) \leq \frac{1}{2}\|A^*A + AA^*\| \tag{3.2}$$

for all $A \in \mathbb{B}(\mathcal{H})$.

If we take $A = V|T|$ in (3.2) then we get

$$\omega^2(V|T|) \leq \frac{1}{2}\left\||T| V^*V|T| + V|T|^2 V^*\right\|.$$

Since $0 \leq V^*V \leq I$, then by multiplying both sides with $|T| \geq 0$ we get $|T| V^*V |T| \leq |T|^2$. Also since $|T|^2 \leq \|T\|^2 I$ hence by multiplying at the left with$V$ and at the right with V^*, we get $V|T|^2 V^* \leq \|T\|^2 VV^* \leq \|T\|^2 I$. Therefore

$$0 \leq |T| V^*V|T| + V|T|^2 V^* \leq |T|^2 + \|T\|^2 I,$$

which implies that

$$\frac{1}{2}\left\||T| V^*V|T| + V|T|^2 V^*\right\| \leq \frac{1}{2}\left\||T|^2 + \|T\|^2 I\right\|.$$

This proves the first part of (3.1).

If we take $A = |T| V$ in (3.2) then we get

$$\omega^2(|T| V) \leq \frac{1}{2}\left\|V^*|T|^2 V + |T| VV^*|T|\right\| \leq \frac{1}{2}\left\|\|T\|^2 I + |T|^2\right\|,$$

which proves the second part of (3.1). □

We also have:

Theorem 3. *For a contraction $V \in \mathcal{B}(H)$ and an operator $T \in \mathcal{B}(H)$, we have for $r \geq 2$ and $\alpha \in [0,1]$ that*

$$\begin{aligned} \omega(T_V),\ \omega\left(\widehat{T}_V\right) &\leq \|V\|\left\|\alpha I + (1-\alpha)|T|^{\frac{r}{1-\alpha}}\right\|^{1/r} \\ &\leq \left\|\alpha I + (1-\alpha)|T|^{\frac{r}{1-\alpha}}\right\|^{1/r}. \end{aligned} \tag{3.3}$$

Moreover, if T is invertible, then

$$\omega(T_V) \leq \|\Delta_{t,V}(T)\|\left\|\alpha|T|^{-\frac{rt}{\alpha}} + (1-\alpha)|T|^{\frac{rt}{1-\alpha}}\right\|^{1/r} \tag{3.4}$$

for all $t \in [0,1]$.

In particular,

$$\omega(T_V) \leq \left\|\widehat{T}_V\right\|\left\|\alpha|T|^{-\frac{r}{\alpha}} + (1-\alpha)|T|^{\frac{r}{1-\alpha}}\right\|^{1/r}$$

and

$$\omega(T_V) \leq \left\|\widetilde{T}_V\right\|\left\|\alpha|T|^{-\frac{rt}{2\alpha}} + (1-\alpha)|T|^{\frac{r}{2(1-\alpha)}}\right\|^{1/r}.$$

Also,

$$\omega\left(\widehat{T}_V\right) \leq \|\Delta_{t,V}(T)\| \left\|\alpha |T|^{\frac{(1-t)r}{\alpha}} + (1-\alpha)|T|^{-\frac{(1-t)r}{1-\alpha}}\right\|^{1/r} \tag{3.5}$$

for all $t \in [0,1]$.

In particular

$$\omega\left(\widehat{T}_V\right) \leq \|T_V\| \left\|\alpha |T|^{\frac{r}{\alpha}} + (1-\alpha)|T|^{-\frac{r}{1-\alpha}}\right\|^{1/r}$$

and

$$\omega\left(\widehat{T}_V\right) \leq \left\|\widetilde{T}_V\right\| \left\|\alpha |T|^{\frac{r}{2\alpha}} + (1-\alpha)|T|^{-\frac{r}{2(1-\alpha)}}\right\|^{1/r}.$$

Proof. We use the following inequality obtained in [11]

$$\omega^r\left(A^\alpha X B^{1-\alpha}\right) \leq \|X\|^r \|\alpha A^r + (1-\alpha)B^r\|, \tag{3.6}$$

where $A, B \geq 0$, $X \in \mathbb{B}(\mathcal{H})$, $r \geq 2$ and $\alpha \in [0,1]$.

Then

$$\begin{aligned}\omega^r(T_V) &= \omega^r(V|T|) = \omega^r\left(I^\alpha V\left(|T|^{\frac{1}{1-\alpha}}\right)^{1-\alpha}\right)\\ &\leq \|V\|^r \left\|\alpha I + (1-\alpha)\left(|T|^{\frac{1}{1-\alpha}}\right)^r\right\| \leq \left\|\alpha I + (1-\alpha)|T|^{\frac{r}{1-\alpha}}\right\|,\end{aligned}$$

which proves the first part of (3.3).

Also,

$$\begin{aligned}\omega^r\left(\widehat{T}_V\right) &= \omega(|T|V) = \omega\left(\left(|T|^{\frac{1}{\alpha}}\right)^\alpha V I^{1-\alpha}\right)\\ &\leq \|V\|^r \left\|\alpha\left(|T|^{\frac{1}{\alpha}}\right)^r + (1-\alpha)I\right\| \leq \left\|\alpha\left(|T|^{\frac{1}{\alpha}}\right)^r + (1-\alpha)I\right\|,\end{aligned}$$

which, by changing α with $1-\alpha$, proves the second part of (3.3).

Now, if T is invertible, then $|T|$ is invertible and we have for $t \in [0,1]$ that

$$\begin{aligned}\omega^r(T_V) &= \omega^r(V|T|) = \omega^r\left(|T|^{-t}|T|^t V|T|^{1-t}|T|^t\right)\\ &= \omega^r\left(\left(|T|^{-\frac{t}{\alpha}}\right)^\alpha |T|^t V |T|^{1-t}\left(|T|^{\frac{t}{1-\alpha}}\right)^{1-\alpha}\right).\end{aligned}$$

If we take $X = |T|^t V|T|^{1-t}$, $A = |T|^{-\frac{t}{\alpha}}$ and $B = |T|^{\frac{t}{1-\alpha}}$ in (3.6), then we get

$$\begin{aligned}\omega^r(T_V) &\leq \left\||T|^t V|T|^{1-t}\right\|^r \left\|\alpha\left(|T|^{-\frac{t}{\alpha}}\right)^r + (1-\alpha)\left(|T|^{\frac{t}{1-\alpha}}\right)^r\right\|\\ &= \|\Delta_{t,V}(T)\|^r \left\|\alpha|T|^{-\frac{rt}{\alpha}} + (1-\alpha)|T|^{\frac{rt}{1-\alpha}}\right\|,\end{aligned}$$

which proves (3.4).

Also,

$$\begin{aligned}\omega^r\left(\widehat{T}_V\right) &= \omega(|T|V) = \omega\left(|T|^{1-t}|T|^t V|T|^{1-t}|T|^{-(1-t)}\right)\\ &= \omega\left(\left(|T|^{\frac{1-t}{\alpha}}\right)^\alpha |T|^t V|T|^{1-t}\left(|T|^{-\frac{1-t}{1-\alpha}}\right)^{1-\alpha}\right)\\ &\leq \left\||T|^t V|T|^{1-t}\right\|^r \left\|\alpha\left(|T|^{\frac{1-t}{\alpha}}\right)^r + (1-\alpha)\left(|T|^{-\frac{1-t}{1-\alpha}}\right)^r\right\|\\ &= \|\Delta_{t,V}(T)\|^r \left\|\alpha|T|^{\frac{(1-t)r}{\alpha}} + (1-\alpha)|T|^{-\frac{(1-t)r}{1-\alpha}}\right\|,\end{aligned}$$

which proves (3.5). □

Finally, we have

Theorem 4. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$, $0 \leq \alpha \leq 1$, $r \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$ we have that*

$$\omega(T_V), \ \omega\left(\widehat{T}_V\right) \leq \|V\| \left\| \frac{1}{p} I + \frac{1}{q} |T|^{\frac{qr}{\alpha}} \right\|^{\frac{\alpha}{r}}. \tag{3.7}$$

Moreover, if T is invertible, then

$$\omega(T_V) \leq \|\Delta_{t,V}(T)\| \left\| \frac{1}{p} |T|^{-\frac{tpr}{\alpha}} + \frac{1}{q} |T|^{\frac{tqr}{\alpha}} \right\|^{\alpha/r} \tag{3.8}$$

for all $t \in [0,1]$.

In particular,

$$\omega(T_V) \leq \left\|\widehat{T}_V\right\| \left\| \frac{1}{p} |T|^{-\frac{pr}{\alpha}} + \frac{1}{q} |T|^{\frac{qr}{\alpha}} \right\|^{\alpha/r}$$

and

$$\omega(T_V) \leq \left\|\widetilde{T}_V\right\| \left\| \frac{1}{p} |T|^{-\frac{pr}{2\alpha}} + \frac{1}{q} |T|^{\frac{qr}{2\alpha}} \right\|^{\alpha/r}.$$

Also

$$\omega\left(\widehat{T}_V\right) \leq \|\Delta_{t,V}(T)\| \left\| \frac{1}{p} |T|^{\frac{(1-t)pr}{\alpha}} + \frac{1}{q} |T|^{-\frac{(1-t)qr}{\alpha}} \right\|^{\alpha/r} \tag{3.9}$$

for all $t \in [0,1]$.

In particular,

$$\omega\left(\widehat{T}_V\right) \leq \|T_V\| \left\| \frac{1}{p} |T|^{\frac{pr}{\alpha}} + \frac{1}{q} |T|^{-\frac{qr}{\alpha}} \right\|^{\alpha/r}$$

and

$$\omega\left(\widehat{T}_V\right) \leq \left\|\widetilde{T}_V\right\| \left\| \frac{1}{p} |T|^{\frac{pr}{2\alpha}} + \frac{1}{q} |T|^{-\frac{qr}{2\alpha}} \right\|^{\alpha/r}.$$

Proof. We use the following inequality obtained in [11]

$$\omega^r(A^\alpha X B^\alpha) \leq \|X\|^r \left\| \frac{1}{p} A^{pr} + \frac{1}{q} B^{qr} \right\|^{\alpha} \tag{3.10}$$

for A, $B \geq 0$, $X \in \mathbb{B}(\mathcal{H})$, $0 \leq \alpha \leq 1$, $r \geq 0$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$.

Then

$$\begin{aligned}\omega^r(T_V) &= \omega^r(V|T|) = \omega^r\left(I^\alpha V \left(|T|^{\frac{1}{\alpha}}\right)^\alpha\right) \\ &\leq \|V\|^r \left\| \frac{1}{p} I + \frac{1}{q} |T|^{\frac{qr}{\alpha}} \right\|^{\alpha} \leq \left\| \frac{1}{p} I + \frac{1}{q} |T|^{\frac{qr}{\alpha}} \right\|^{\alpha},\end{aligned}$$

which proves the first part of (3.7).

Also,

$$\begin{aligned}\omega^r\left(\widehat{T}_V\right) &= \omega(|T| V) = \omega\left(\left(|T|^{\frac{1}{\alpha}}\right)^\alpha V I^\alpha\right) \\ &\leq \|V\|^r \left\| \frac{1}{p} \left(|T|^{\frac{1}{\alpha}}\right)^{pr} + \frac{1}{q} I \right\|^{\alpha} \leq \left\| \frac{1}{p} \left(|T|^{\frac{1}{\alpha}}\right)^{pr} + \frac{1}{q} I \right\|^{\alpha},\end{aligned}$$

which, by replacing p with q, gives the second part of (3.7).

Now, if T is invertible, then $|T|$ is invertible and we have for $t\in[0,1]$ that

$$\begin{aligned}\omega^r(T_V)&=\omega^r(V|T|)=\omega^r\left(|T|^{-t}|T|^tV|T|^{1-t}|T|^t\right)\\&=\omega^r\left(\left(|T|^{-\frac{t}{\alpha}}\right)^\alpha|T|^tV|T|^{1-t}\left(|T|^{\frac{t}{\alpha}}\right)^\alpha\right)\\&\le\left\||T|^tV|T|^{1-t}\right\|^r\left\|\frac{1}{p}\left(|T|^{-\frac{t}{\alpha}}\right)^{pr}+\frac{1}{q}\left(|T|^{\frac{t}{\alpha}}\right)^{qr}\right\|^\alpha\\&=\|\Delta_{t,V}(T)\|^r\left\|\frac{1}{p}|T|^{-\frac{tpr}{\alpha}}+\frac{1}{q}|T|^{\frac{tqr}{\alpha}}\right\|^\alpha,\end{aligned}$$

which gives (3.8).

Also,

$$\begin{aligned}\omega^r\left(\widehat{T}_V\right)&=\omega(|T|V)=\omega\left(|T|^{1-t}|T|^tV|T|^{1-t}|T|^{-(1-t)}\right)\\&=\omega\left(\left(|T|^{\frac{1-t}{\alpha}}\right)^\alpha|T|^tV|T|^{1-t}\left(|T|^{-\frac{1-t}{\alpha}}\right)^\alpha\right)\\&\le\left\||T|^tV|T|^{1-t}\right\|^r\left\|\frac{1}{p}\left(|T|^{\frac{1-t}{\alpha}}\right)^{pr}+\frac{1}{q}\left(|T|^{-\frac{1-t}{\alpha}}\right)^{qr}\right\|^\alpha\\&=\|\Delta_{t,V}(T)\|^r\left\|\frac{1}{p}|T|^{\frac{(1-t)pr}{\alpha}}+\frac{1}{q}|T|^{-\frac{(1-t)qr}{\alpha}}\right\|^\alpha,\end{aligned}$$

which proves (3.9). □

Remark 3. *One can state several similar inequalities for the usual generalized Aluthge transform* $\Delta_t(T)$ *or for the transforms*

$$\Delta_{t,\frac{V^*V\pm VV^*}{2}}(T),\ \Delta_{t,\mathrm{Im}(V)}(T)\ \text{ and } \Delta_{t,\mathrm{Re}(V)}(T)$$

defined in the introduction for any contraction V, *any operator* $T\in\mathcal{B}(H)$ *and a parameter* $t\in[0,1]$.

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