Multivariate Fuzzy Approximation by Neural Network Operators activated by a general sigmoid function

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Abstract

Here is studied in detail the multivariate fuzzy approximation to the multivariate unit by multivariate fuzzy neural network operators activated by a general sigmoid function. These operators are multivariate fuzzy analogs of earlier studied multivariate Banach space valued ones. The derived results generalize earlier Banach space valued ones into the fuzzy level. Here the high order multivariate fuzzy pointwise and uniform convergences with rates to the multivariate fuzzy unit operator are given through multivariate fuzzy Jackson type inequalities involving the multivariate fuzzy moduli of continuity of the mth order $(m \geq 0)$ H-fuzzy partial derivatives, of the involved multivariate fuzzy number valued function. The treated operators are of averaged, quasi-interpolation, Kantorovich and quadrature types at the multivariate fuzzy setting.

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1 Introduction

The author in [1], [2] and [3], see chapters 2-5, was the first to derive neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types

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[25], by employing the modulus of continuity of the engaged function or its high order derivative, and deriving very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" function are assumed to be of compact support.

The author motivated by [26], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [12] - [20], by treating both the univariate and multivariate cases.

Continuation of the author's works ([4] - [11], [22] and especially of [21], Ch. 21) is this article, where the multivariate fuzzy neural network approximation is based on a general sigmoid activation function, which among others, may result into higher rates of approximation. We involve the fuzzy partial derivatives of the multivariate fuzzy function under approximation or itself, and we establish tight multivariate fuzzy Jackson type inequalities. An extensive background is given on fuzzy multivariate analysis and real neural network approximation, all needed to present our results.

Our fuzzy multivariate feed-forward neural networks (FFNNs) are with one hidden layer. For neural networks in general you may read [29], [32], [33]. For the fractional aspect see [34].

2 Fuzzy Real Analysis background

See also [21], Ch. 21, pp. 466-473.

We need the following background

Definition 1 *(see [36]) Let* $\mu : \mathbb{R} \to [0,1]$ *with the following properties*

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R}$; $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ *(μ is called a convex fuzzy subset).*

(iii) μ *is upper semicontinuous on* $\mathbb{R}$, *i.e.* $\forall\ x_0 \in \mathbb{R}$ *and* $\forall\ \varepsilon > 0$, $\exists$ *neighborhood* $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(iv) The set $\overline{\sup p(\mu)}$ *is compact in* $\mathbb{R}$, *(where* $\sup p(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$*).*

We call μ *a fuzzy real number. Denote the set of all* μ *with* $\mathbb{R}_{\mathcal{F}}$.

E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, *for any* $x_0 \in \mathbb{R}$, *where* $\chi_{\{x_0\}}$ *is the characteristic function at* x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\} \tag{1}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval on $\mathbb{R}$ ([28]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \ \forall\, r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as substes of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [36]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \ \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_{\pm}^{(r)} = (\lambda \odot u)_{\pm}^{(r)}$, respectively.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by

$$D(u,v) := \sup_{r \in [0,1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\}, \tag{2}$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \ \ u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [36], [37].

Let $f, g : W \subseteq \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$. We define the distance

$$D^*(f,g) = \sup_{x \in W} D(f(x), g(x)).$$

Remark 2 *We determine and use*

$$D^*(f, \widetilde{o}) = \sup_{x \in W} D(f(x), \widetilde{o}) =$$

$$\sup_{x \in W} \sup_{r \in [0,1]} \max \left\{ \left| f_-^{(r)}(x) \right|, \left| f_+^{(r)}(x) \right| \right\}.$$

By the principle of iterated suprema we find that

$$D^*(f, \widetilde{o}) = \sup_{r \in [0,1]} \max \left\{ \left\| f_-^{(r)} \right\|_\infty, \left\| f_+^{(r)} \right\|_\infty \right\}, \tag{3}$$

under the assumption $D^*(f, \widetilde{o}) < \infty$, that is f is a fuzzy bounded function.

Above $\|\cdot\|_\infty$ is the supremum norm of the function over $W \subseteq \mathbb{R}^m$.

Here Σ^* stands for fuzzy summation and $\widetilde{0} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \ \forall \ u \in \mathbb{R}_{\mathcal{F}}.$$

We need

Remark 3 *([5]). Here* $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, ..., m \in \mathbb{N}$. *Suppose that*

$$\sup_{r\in[0,1]} \max \left(x_i^{(r)}, y_i^{(r)} \right) \in \mathbb{R}, \text{ for } i = 1, ..., m.$$

Then one sees easily that

$$\sup_{r\in[0,1]} \max \left(\sum_{i=1}^{m} x_i^{(r)}, \sum_{i=1}^{m} y_i^{(r)} \right) \leq \sum_{i=1}^{m} \sup_{r\in[0,1]} \max \left(x_i^{(r)}, y_i^{(r)} \right). \tag{4}$$

Definition 4 *Let* $f \in C(W)$, $W \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$, *which is bounded or uniformly continuous, we define* $(h > 0)$

$$\omega_1(f, h) := \sup_{x,y\in W, \ \|x-y\|_\infty \leq h} |f(x) - f(y)|, \tag{5}$$

where $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m)$.

Definition 5 *Let* $f : W \to \mathbb{R}_{\mathcal{F}}$, $W \subseteq \mathbb{R}^m$, *we define the fuzzy modulus of continuity of* f *by*

$$\omega_1^{(\mathcal{F})}(f, h) = \sup_{x,y\in W, \ \|x-y\|_\infty \leq h} D(f(x), f(y)), \ \ h > 0. \tag{6}$$

where $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m)$.

For $f : W \to \mathbb{R}_{\mathcal{F}}$, $W \subseteq \mathbb{R}^m$, we use

$$[f]^r = \left[f_-^{(r)}, f_+^{(r)} \right], \tag{7}$$

where $f_\pm^{(r)} : W \to \mathbb{R}$, $\forall \ r \in [0,1]$.
We need

Proposition 6 *([5]) Let* $f : W \to \mathbb{R}_{\mathcal{F}}$. *Assume that* $\omega_1^{\mathcal{F}}(f, \delta)$, $\omega_1\left(f_-^{(r)}, \delta\right)$, $\omega_1\left(f_+^{(r)}, \delta\right)$ *are finite for any* $\delta > 0$, $r \in [0,1]$.
Then

$$\omega_1^{(\mathcal{F})}(f, \delta) = \sup_{r\in[0,1]} \max \left\{ \omega_1\left(f_-^{(r)}, \delta\right), \omega_1\left(f_+^{(r)}, \delta\right) \right\}. \tag{8}$$

We denote by $C_{\mathcal{F}}^{U}(W)$ the space of fuzzy uniformly continuous functions from $W \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(W)$ is the space of fuzzy continuous functions on $W \subseteq \mathbb{R}^m$, and $C_B(W, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 7 *([7]) Let $f \in C_{\mathcal{F}}^{U}(W)$, where $W \subseteq \mathbb{R}^m$ is convex. Then $\omega_1^{(\mathcal{F})}(f, \delta) < \infty$, for any $\delta > 0$.*

Proposition 8 *([7]) It holds*

$$\lim_{\delta \to 0} \omega_1^{(\mathcal{F})}(f, \delta) = \omega_1^{(\mathcal{F})}(f, 0) = 0, \tag{9}$$

iff $f \in C_{\mathcal{F}}^{U}(W)$, $W \subseteq \mathbb{R}^m$, where W is convex and compact.

Proposition 9 *([7]) Let $f \in C_{\mathcal{F}}(W)$, $W \subseteq \mathbb{R}^m$ open or compact. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over W, respectively in $\pm$.*

Note 10 *It is clear by Propositions 6, 8, that if $f \in C_{\mathcal{F}}^{U}(W)$, then $f_{\pm}^{(r)} \in C_U(W)$ (uniformly continuous on W). Also if $f \in C_B(W, \mathbb{R}_{\mathcal{F}})$, it implies by (3) that $f_{\pm}^{(r)} \in C_B(W)$ (continuous and bounded functions on W).*

We need

Definition 11 *Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.*

Definition 12 *([36]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-difference at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)*

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h} \tag{10}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exists in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Definition 13 *We denote by $C_{\mathcal{F}}^{N^*}(W)$, $N^* \in \mathbb{N}$, the space of all N^*-times fuzzy continuously differentiable functions from W into $\mathbb{R}_{\mathcal{F}}$, $W \subseteq \mathbb{R}^m$ open or compact which is convex.*

Here fuzzy partial derivatives are defined via Definition 12 in the obvious way as in the ordinary real case.

We mention

Theorem 14 *([30]) Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in [a,b]$, $0 \leq r \leq 1$. Clearly*

$$[f(t)]^r = \left[f(t)_-^{(r)}, f(t)_+^{(r)} \right] \subseteq \mathbb{R}.$$

Then $(f(t))_\pm^{(r)}$ are differentiable and

$$[f'(t)]^r = \left[\left(f(t)_-^{(r)} \right)', \left(f(t)_+^{(r)} \right)' \right].$$

I.e.

$$(f')_\pm^{(r)} = \left(f_\pm^{(r)} \right)', \quad \forall\, r \in [0,1]. \tag{11}$$

Remark 15 *(see also [6]) Let $f \in C^{N^*}([a,b], \mathbb{R}_{\mathcal{F}})$, $N^* \geq 1$. Then by Theorem 14 we obtain $f_\pm^{(r)} \in C^{N^*}([a,b])$ and*

$$\left[f^{(i)}(t) \right]^r = \left[\left(f(t)_-^{(r)} \right)^{(i)}, \left(f(t)_+^{(r)} \right)^{(i)} \right],$$

for $i = 0,1,2,...,N^*$, and in particular we have

$$\left(f^{(i)} \right)_\pm^{(r)} = \left(f_\pm^{(r)} \right)^{(i)}, \tag{12}$$

for any $r \in [0,1]$.

Let $f \in C_{\mathcal{F}}^{N^*}(W)$, $W \subseteq \mathbb{R}^m$, open or compact, which is convex, denote $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha} := (\widetilde{\alpha_1}, ..., \widetilde{\alpha_m})$, $\widetilde{\alpha}_i \in \mathbb{Z}^+$, $i = 1,...,m$ and

$$0 < |\widetilde{\alpha}| := \sum_{i=1}^m \widetilde{\alpha}_i \leq N^*, \quad N^* > 1.$$

Then by Theorem 14 we get that

$$\left(f_\pm^{(r)} \right)_{\widetilde{\alpha}} = (f_{\widetilde{\alpha}})_\pm^{(r)}, \ \ \forall\, r \in [0,1], \tag{13}$$

and any $\widetilde{\alpha} : |\widetilde{\alpha}| \leq N^*$. Here $f_\pm^{(r)} \in C^{N^*}(W)$.

Notation 16 *We denote*

$$\left(\sum_{i=1}^2 D\left(\frac{\partial}{\partial x_i}, \widetilde{0} \right) \right)^2 f(\overrightarrow{x}) := \tag{14}$$

$$D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}, \widetilde{0} \right) + D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}, \widetilde{0} \right) + 2D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}, \widetilde{0} \right).$$

In general we denote ($j = 1, ..., N^*$)

$$\left(\sum_{i=1}^{m} D \left(\frac{\partial}{\partial x_i}, \widetilde{0} \right) \right)^j f(\overrightarrow{x}) := \tag{15}$$

$$\sum_{(j_1,...,j_m) \in \mathbb{Z}_+^m : \sum_{i=1}^m j_i = j} \frac{j!}{j_1! j_2! ... j_m!} D \left(\frac{\partial^j f(x_1, ..., x_m)}{\partial x_1^{j_1} \partial x_2^{j_2} ... \partial x_m^{j_m}}, \widetilde{0} \right).$$

We mention also a particular case of the Fuzzy Henstock integral ($\delta(x) = \frac{\delta}{2}$), see [36].

Definition 17 *([27], p. 644) Let* $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$. *We say that* f *is Fuzzy-Riemann integrable to* $I \in \mathbb{R}_{\mathcal{F}}$ *if for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any division* $P = \{[u, v]; \xi\}$ *of* $[a, b]$ *with the norms* $\Delta(P) < \delta$, *we have*

$$D \left(\sum_{P}^{*} (v - u) \odot f(\xi), I \right) < \varepsilon.$$

We write

$$I := (FR) \int_a^b f(x) \, dx. \tag{16}$$

We mention

Theorem 18 *([28]) Let* $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ *be fuzzy continuous. Then*

$$(FR) \int_a^b f(x) \, dx$$

exists and belongs to $\mathbb{R}_{\mathcal{F}}$, *furthermore it holds*

$$\left[(FR) \int_a^b f(x) \, dx \right]^r = \left[\int_a^b (f)_-^{(r)}(x) \, dx, \int_a^b (f)_+^{(r)}(x) \, dx \right],$$

$\forall \, r \in [0, 1]$.

For the definition of general fuzzy integral we follow [31] next.

Definition 19 *Let* (Ω, Σ, μ) *be a complete* σ*-finite measure space. We call* $F : \Omega \to R_{\mathcal{F}}$ *measurable iff* $\forall$ *closed* $B \subseteq \mathbb{R}$ *the function* $F^{-1}(B) : \Omega \to [0, 1]$ *defined by*

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \text{ all } w \in \Omega$$

is measurable, see [31].

Theorem 20 *([31]) For* $F : \Omega \to \mathbb{R}_{\mathcal{F}}$*,*

$$F(w) = \left\{ \left(F_{-}^{(r)}(w), F_{+}^{(r)}(w) \right) | 0 \leq r \leq 1 \right\},$$

the following are equivalent

(1) F *is measurable,*

(2) $\forall\ r \in [0, 1]$, $F_{-}^{(r)}$, $F_{+}^{(r)}$ *are measurable.*

Following [31], given that for each $r \in [0, 1]$, $F_{-}^{(r)}$, $F_{+}^{(r)}$ are integrable we have that the parametrized representation

$$\left\{ \left(\int_A F_{-}^{(r)} d\mu, \int_A F_{+}^{(r)} d\mu \right) | 0 \leq r \leq 1 \right\} \tag{17}$$

is a fuzzy real number for each $A \in \Sigma$.

The last fact leads to

Definition 21 *([31]) A measurable function* $F : \Omega \to \mathbb{R}_{\mathcal{F}}$*,*

$$F(w) = \left\{ \left(F_{-}^{(r)}(w), F_{+}^{(r)}(w) \right) | 0 \leq r \leq 1 \right\}$$

is integrable if for each $r \in [0, 1]$, $F_{\pm}^{(r)}$ *are integrable, or equivalently, if* $F_{\pm}^{(0)}$ *are integrable.*

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_{-}^{(r)} d\mu, \int_A F_{+}^{(r)} d\mu \right) | 0 \leq r \leq 1 \right\}. \tag{18}$$

By [31], F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here denote

$$\|u\|_{\mathcal{F}} := D\left(u, \widetilde{0}\right), \ \ \forall\ u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 22 *([31]) Let* $F, G : \Omega \to \mathbb{R}_{\mathcal{F}}$ *be integrable. Then*

(1) Let $a, b \in \mathbb{R}$*, then* $aF + bG$ *is integrable and for each* $A \in \Sigma$*,*

$$\int_A (aF + bG)\, d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F, G)$ *is a real- valued integrable function and for each* $A \in \Sigma$*,*

$$D\left(\int_A F d\mu, \int_A G d\mu \right) \leq \int_A D(F, G)\, d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}}\, d\mu.$$

Above μ could be the multivariate Lebesgue measure, which we use in this article, with all the basic properties valid here too. Notice by [31], Fubini's theorem is valid for fuzzy integral (18).

Basically here we have that

$$\left[\int_A F d\mu\right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right], \tag{19}$$

i.e.

$$\left(\int_A F d\mu\right)_\pm^{(r)} = \int_A F_\pm^{(r)} d\mu, \quad \forall\, r \in [0,1]. \tag{20}$$

3 About real neural networks background

Here we follow [24].

Let $h : \mathbb{R} \to [-1,1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0$, $h(-x) = -h(x)$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also h is strictly convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.

We consider the activation function

$$\psi(x) := \frac{1}{4}\left(h(x+1) - h(x-1)\right), \quad x \in \mathbb{R}, \tag{21}$$

As in [23], p. 88, we get that $\psi(-x) = \psi(x)$, thus ψ is an even function. Since $x+1 > x-1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi(0) = \frac{h(1)}{2}. \tag{22}$$

Let $x > 1$, we have that

$$\psi'(x) = \frac{1}{4}\left(h'(x+1) - h'(x-1)\right) < 0,$$

by h' being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1 - x > 0$ and $0 < 1 - x < 1 + x$. It holds $h'(x-1) = h'(1-x) > h'(x+1)$, so that again $\psi'(x) < 0$. Consequently ψ is stritly decreasing on $(0, +\infty)$.

Clearly, ψ is strictly increasing on $(-\infty, 0)$, and $\psi'(0) = 0$.

See that

$$\lim_{x\to+\infty} \psi(x) = \frac{1}{4}\left(h(+\infty) - h(+\infty)\right) = 0, \tag{23}$$

and

$$\lim_{x\to-\infty} \psi(x) = \frac{1}{4}\left(h(-\infty) - h(-\infty)\right) = 0. \tag{24}$$

That is the x-axis is the horizontal asymptote on ψ.

Conclusion, ψ is a bell symmetric function with maximum

$$\psi(0) = \frac{h(1)}{2}.$$

We need

Theorem 23 *([24]) We have that*

$$\sum_{i=-\infty}^{\infty} \psi(x-i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{25}$$

Theorem 24 *([24]) It holds*

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 1. \tag{26}$$

Thus $\psi(x)$ is a density function on $\mathbb{R}$.
We give

Theorem 25 *([24]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi(nx-k) < \frac{\left(1 - h\left(n^{1-\alpha}-2\right)\right)}{2}. \tag{27}$$

Notice that

$$\lim_{n \to +\infty} \frac{\left(1 - h\left(n^{1-\alpha}-2\right)\right)}{2} = 0.$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.
We further give

Theorem 26 *([24]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} < \frac{1}{\psi(1)}, \quad \forall\, x \in [a,b]. \tag{28}$$

Remark 27 *([24]) i) We have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \neq 1, \tag{29}$$

for at least some $x \in [a,b]$.

ii) For large enough $n \in \mathbb{N}$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general, by Theorem 23, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx-k\right) \leq 1. \tag{30}$$

We introduce

$$Z\left(x_1,...,x_N\right) := Z\left(x\right) := \prod_{i=1}^{N} \psi\left(x_i\right), \quad x = \left(x_1,...,x_N\right) \in \mathbb{R}^N,\ N \in \mathbb{N}. \tag{31}$$

It has the properties:

(i) $Z\left(x\right) > 0,\ \ \forall\ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z\left(x-k\right) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z\left(x_1-k_1,...,x_N-k_N\right) = 1, \tag{32}$$

where $k := \left(k_1,...,k_n\right) \in \mathbb{Z}^N$, $\forall\ x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z\left(nx-k\right) = 1, \tag{33}$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z\left(x\right) dx = 1, \tag{34}$$

that is Z is a multivariate density function.

Here denote $\|x\|_\infty := \max\left\{|x_1|,...,|x_N|\right\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty,...,\infty)$, $-\infty := (-\infty,...,-\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= \left(\lceil na_1 \rceil,...,\lceil na_N \rceil\right), \\ \lfloor nb \rfloor &:= \left(\lfloor nb_1 \rfloor,...,\lfloor nb_N \rfloor\right), \end{aligned} \tag{35}$$

where $a := \left(a_1,...,a_N\right)$, $b := \left(b_1,...,b_N\right)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \psi\left(nx_i-k_i\right)\right) =$$

$$\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \dots \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} \left(\prod_{i=1}^{N} \psi(nx_i - k_i) \right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} \psi(nx_i - k_i) \right). \quad (36)$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) =$$

$$\sum_{\begin{cases} k = \lceil na\rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\lfloor nb\rfloor} Z(nx-k) + \sum_{\begin{cases} k = \lceil na\rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb\rfloor} Z(nx-k). \quad (37)$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, where $r \in \{1, ..., N\}$.

(v) As in [21], pp. 379-380, we derive that

$$\sum_{\begin{cases} k = \lceil na\rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb\rfloor} Z(nx-k) \overset{(7)}{<} \frac{1 - h\left(n^{1-\beta} - 2\right)}{2}, \quad 0 < \beta < 1, \quad (38)$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N} [a_i, b_i]$.

(vi) By Theorem 26 we get that

$$0 < \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)} < \frac{1}{(\psi(1))^N} =: \gamma(N), \quad (39)$$

$\forall\, x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $n \in \mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z(nx-k) < \frac{1 - h\left(n^{1-\beta} - 2\right)}{2} =: c(\beta, n), \quad (40)$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) \neq 1, \quad (41)$$

for at least some $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Let $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We define the multivariate averaged positive linear neural network operators ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$):

$$A_n(f, x_1, ..., x_N) := A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} = \tag{42}$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right)\left(\prod_{i=1}^{N} \psi(nx_i - k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i)\right)}.$$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

When $f \in C_B\left(\mathbb{R}^N\right)$ we define

$$B_n(f, x) := B_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) := \tag{43}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right)\left(\prod_{i=1}^{N} \psi(nx_i - k_i)\right),$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operators.

Also for $f \in C_B\left(\mathbb{R}^N\right)$ we define the multivariate Kantorovich type neural network operators

$$C_n(f, x) := C_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty}\left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt\right) Z(nx-k) := \tag{44}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty}\left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \cdots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, ..., t_N)\, dt_1 ... dt_N\right)$$

$$\cdot\left(\prod_{i=1}^{N} \psi(nx_i - k_i)\right),$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operators of quadrature type $D_n(f, x)$, $n \in \mathbb{N}$, as follows. Let $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $\overline{r} = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_{\overline{r}} = w_{r_1, r_2, ... r_N} \geq 0$, such that

$\sum\limits_{\overline{r}=0}^{\theta} w_{\overline{r}} = \sum\limits_{r_1=0}^{\theta_1} \sum\limits_{r_2=0}^{\theta_2} ... \sum\limits_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n,k_1,k_2,...,k_N}(f) := \sum_{\overline{r}=0}^{\theta} w_{\overline{r}} f\left(\frac{k}{n} + \frac{\overline{r}}{n\theta}\right) :=$$

$$\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, ..., \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \tag{45}$$

where $\frac{\overline{r}}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$.

We put

$$D_n(f,x) := D_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx-k) := \tag{46}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \delta_{n,k_1,k_2,...,k_N}(f) \left(\prod_{i=1}^{N} \psi(nx_i - k_i)\right),$$

$\forall\, x \in \mathbb{R}^N$.

Let $f \in C^m\left(\prod_{i=1}^N [a_i, b_i]\right)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, ..., N$, and $|\alpha| := \sum_{i=1}^N \alpha_i = l$, where $l = 0, 1, ..., m$. We write also $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is of order l.

We denote

$$\omega_{1,m}^{\max}(f_\alpha, h) := \max_{\alpha:|\alpha|=m} \omega_1(f_\alpha, h).$$

Call also

$$\|f_\alpha\|_{\infty,m}^{\max} := \max_{\alpha:|\alpha|=m} \{\|f_\alpha\|_\infty\},$$

where $\|\cdot\|_\infty$ is the supremum norm.

In [21], [23], we studied the basic approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates for Banach space valued functions. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I.

We need

Theorem 28 *Let $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*

1)

$$|A_n(f,x) - f(x)| \leq \gamma(N)\left[\omega_1\left(f, \frac{1}{n^\beta}\right) + 2c(\beta, n)\|f\|_\infty\right] =: \lambda_1, \tag{47}$$

and

2)
$$\left\|A_n(f)-f\right\|_\infty \leq \lambda_1. \tag{48}$$
We notice that $\lim_{n\to\infty} A_n(f)=f$*, pointwise and uniformly.*

Proof. Similar to [23], p. 118. ■

We need

Theorem 29 *Let* $f\in C_B\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x\in\mathbb{R}^N$, $N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$. *Then*

1)
$$\left|B_n(f,x)-f(x)\right| \leq \omega_1\left(f,\frac{1}{n^\beta}\right)+2c(\beta,n)\left\|f\right\|_\infty =: \lambda_2, \tag{49}$$

2)
$$\left\|B_n(f)-f\right\|_\infty \leq \lambda_2. \tag{50}$$
Given that $f\in\left(C_U\left(\mathbb{R}^N\right)\cap C_B\left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n\to\infty} B_n(f)=f$*, uniformly.*

Proof. Similar to [23], p. 128. ■

We also need

Theorem 30 *Let* $f\in C_B\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x\in\mathbb{R}^N$, $N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$. *Then*

1)
$$\left|C_n(f,x)-f(x)\right| \leq \omega_1\left(f,\frac{1}{n}+\frac{1}{n^\beta}\right)+2c(\beta,n)\left\|f\right\|_\infty =: \lambda_3, \tag{51}$$

2)
$$\left\|C_n(f)-f\right\|_\infty \leq \lambda_3. \tag{52}$$
Given that $f\in\left(C_U\left(\mathbb{R}^N\right)\cap C_B\left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n\to\infty} C_n(f)=f$*, uniformly.*

Proof. Similar to [23], p. 129. ■

We also need

Theorem 31 *Let* $f\in C_B\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x\in\mathbb{R}^N$, $N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$. *Then*

1)
$$\left|D_n(f,x)-f(x)\right| \leq \omega_1\left(f,\frac{1}{n}+\frac{1}{n^\beta}\right)+2c(\beta,n)\left\|f\right\|_\infty = \lambda_3, \tag{53}$$

2)
$$\left\|D_n(f)-f\right\|_\infty \leq \lambda_3. \tag{54}$$

Given that $f\in\left(C_U\left(\mathbb{R}^N\right)\cap C_B\left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n\to\infty} D_n(f)=f$*, uniformly.*

Proof. Similar to [23], p. 131. ■

We finally mention (similar to [21], p. 481)

Theorem 32 *Let* $f \in C^m\left(\prod_{i=1}^N [a_i, b_i]\right)$, $0 < \beta < 1$, $n, m, N \in \mathbb{N}$, $n^{1-\beta} > 2$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$. *Then*

i)

$$\left| A_n(f, x) - f(x) - \sum_{j_*=1}^{m} \left(\sum_{|\alpha|=j_*} \left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) A_n\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) \right| \leq \tag{55}$$

$$\gamma(N) \cdot \left\{ \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left(\frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2c(\beta, n) \right\},$$

ii)

$$|A_n(f, x) - f(x)| \leq \gamma(N) \cdot \tag{56}$$

$$\left\{ \sum_{j_*=1}^{m} \left(\sum_{|\alpha|=j_*} \left(\frac{|f_\alpha(x)|}{\prod_{i=1}^N \alpha_i!} \right) \left[\frac{1}{n^{\beta j_*}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) c(\beta, n) \right] \right) \right.$$

$$\left. + \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left(\frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2c(\beta, n) \right\},$$

iii)

$$\|A_n(f) - f\|_\infty \leq \gamma(N) \cdot \tag{57}$$

$$\left\{ \sum_{j_*=1}^{m} \left(\sum_{|\alpha|=j_*} \left(\frac{\|f_\alpha\|_\infty}{\prod_{i=1}^N \alpha_i!} \right) \left[\frac{1}{n^{\beta j_*}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) c(\beta, n) \right] \right) \right.$$

$$\left. + \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left(\frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2c(\beta, n) \right\},$$

iv) additionally assume $f_\alpha(x_0) = 0$, *for all* $\alpha : |\alpha| = 1, ..., m$; $x_0 \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, *then*

$$|A_n(f, x_0) - f(x_0)| \leq \gamma(N) \left\{ \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \right. \tag{58}$$

$$\left. \left(\frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2c(\beta, n) \right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

4 Main Results: Fuzzy multivariate Neural Network Approximation based on a general sigmoid function

We define the following General Fuzzy multivariate Neural Network Operators $A_n^{\mathcal{F}}$, $B_n^{\mathcal{F}}$, $C_n^{\mathcal{F}}$, $D_n^{\mathcal{F}}$, based on a general sigmoid activation function. These are analogs of the real A_n, B_n, C_n, D_n, see (42), (43), (44) and (46), respectively.

Let $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $N \in \mathbb{N}$, we set

$$A_n^{\mathcal{F}}(f, x_1, ..., x_N) := A_n^{\mathcal{F}}(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{k}{n}\right) \odot Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}$$

$$= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor *} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor *} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \odot \left(\prod_{i=1}^{N} \psi(nx_i - k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i)\right)}, \tag{59}$$

$x \in \prod_{i=1}^{N}[a_i, b_i]$, $n \in \mathbb{N}$.

Let $f \in C_B\left(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}}\right)$, we put

$$B_n^{\mathcal{F}}(f, x) := B_n^{\mathcal{F}}(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty *} f\left(\frac{k}{n}\right) \odot Z(nx - k)$$

$$:= \sum_{k_1=-\infty}^{\infty *} ... \sum_{k_N=-\infty}^{\infty *} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \odot \left(\prod_{i=1}^{N} \psi(nx_i - k_i)\right), \tag{60}$$

$x \in \mathbb{R}^N$, $n \in \mathbb{N}$.

Let $f \in C_B\left(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}}\right)$, we define the multivariate fuzzy Kantorovich type neural network operator,

$$C_n^{\mathcal{F}}(f, x) := C_n^{\mathcal{F}}(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty *} \left(n^N \odot \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt\right) \odot Z(nx - k) :=$$

$$\sum_{k_1=-\infty}^{\infty *} ... \sum_{k_N=-\infty}^{\infty *} \left(n^N \odot \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, ..., t_N)\, dt_1 ... dt_N\right)$$

$$\odot \left(\prod_{i=1}^{N} \psi(nx_i - k_i)\right), \tag{61}$$

$x \in \mathbb{R}^N$, $n \in \mathbb{N}$.

Let $f \in C_B\left(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}}\right)$, we define the multivariate fuzzy quadrature type neural network operator. Let here

$$\delta_{nk}^{\mathcal{F}}(f) := \delta_{n,k_1,\ldots,k_N}^{\mathcal{F}}(f) := \sum_{\overline{r}=0}^{\theta *} w_{\overline{r}} \odot f\left(\frac{k}{n} + \frac{\overline{r}}{n\theta}\right) := \tag{62}$$

$$\sum_{r_1=0}^{\theta_1 *} \ldots \sum_{r_N=0}^{\theta_N *} w_{r_1,\ldots r_N} \odot f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \ldots, \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right).$$

We put

$$D_n^{\mathcal{F}}(f, x) := D_n^{\mathcal{F}}(f, x_1, \ldots, x_N) := \sum_{k=-\infty}^{\infty *} \delta_{nk}^{\mathcal{F}}(f) \odot Z(nx - k) :=$$

$$\sum_{k_1=-\infty}^{\infty *} \ldots \sum_{k_N=-\infty}^{\infty *} \delta_{n,k_1,\ldots,k_N}^{\mathcal{F}}(f) \odot \left(\prod_{i=1}^{N} \psi(nx_i - k_i)\right), \tag{63}$$

$x \in \mathbb{R}^N$, $n \in \mathbb{N}$.

We can put together all $B_n^{\mathcal{F}}$, $C_n^{\mathcal{F}}$, $D_n^{\mathcal{F}}$ fuzzy operators as follows:

$$L_n^{\mathcal{F}}(f, x) := \sum_{k=-\infty}^{\infty *} l_{nk}^{\mathcal{F}}(f) \odot Z(nx - k), \tag{64}$$

where

$$l_{nk}^{\mathcal{F}}(f) := \begin{cases} f\left(\frac{k}{n}\right), \text{ if } L_n^{\mathcal{F}} = B_n^{\mathcal{F}}, \\ n^N \odot \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt, \text{ if } L_n^{\mathcal{F}} = C_n^{\mathcal{F}}, \\ \delta_{nk}^{\mathcal{F}}(f), \text{ if } L_n^{\mathcal{F}} = D_n^{\mathcal{F}}, \end{cases} \tag{65}$$

$x \in \mathbb{R}^N$, $n \in \mathbb{N}$.

Similarly, we can put together all B_n, C_n, D_n real operators as

$$L_n(f, x) := \sum_{k=-\infty}^{\infty} l_{nk}(f)\, Z(nx - k), \tag{66}$$

where

$$l_{nk}(f) := \begin{cases} f\left(\frac{k}{n}\right), \text{ if } L_n = B_n, \\ n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt, \text{ if } L_n = C_n, \\ \delta_{nk}(f), \text{ if } L_n = D_n, \end{cases} \tag{67}$$

$x \in \mathbb{R}^N$, $n \in \mathbb{N}$.

Let $r \in [0, 1]$, we observe that

$$\left[A_n^{\mathcal{F}}(f, x)\right]^r = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left[f\left(\frac{k}{n}\right)\right]^r \left(\frac{Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}\right) =$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left[f_{-}^{(r)}\left(\frac{k}{n}\right),f_{+}^{(r)}\left(\frac{k}{n}\right)\right]\left(\frac{Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)}\right)= \tag{68}$$

$$\left[\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f_{-}^{(r)}\left(\frac{k}{n}\right)\left(\frac{Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)}\right),\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f_{+}^{(r)}\left(\frac{k}{n}\right)\left(\frac{Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)}\right)\right]$$

$$=\left[A_n\left(f_{-}^{(r)},x\right),A_n\left(f_{+}^{(r)},x\right)\right]. \tag{69}$$

We have proved that

$$\left(A_n^{\mathcal{F}}(f,x)\right)_{\pm}^{(r)}=A_n\left(f_{\pm}^{(r)},x\right), \tag{70}$$

$\forall\ r\in[0,1]$, $\forall\ x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Similarly, as in [21], pp. 485-489, a lengthy proof (see Remark 21.31 and proof of (21.76) there) it holds that

$$\left(L_n^{\mathcal{F}}(f,x)\right)_{\pm}^{(r)}=L_n\left(f_{\pm}^{(r)},x\right), \tag{71}$$

$\forall\ r\in[0,1]$, $\forall\ x\in\mathbb{R}^N$.

Based on (70) and (71) now one may write

$$D\left(A_n^{\mathcal{F}}(f,x),f(x)\right)= \tag{72}$$

$$\sup_{r\in[0,1]}\max\left\{\left|\left(A_n\left(f_{-}^{(r)},x\right)\right)-f_{-}^{(r)}(x)\right|,\left|A_n\left(f_{+}^{(r)},x\right)-f_{+}^{(r)}(x)\right|\right\},$$

and

$$D\left(L_n^{\mathcal{F}}(f,x),f(x)\right)=$$

$$\sup_{r\in[0,1]}\max\left\{\left|\left(L_n\left(f_{-}^{(r)},x\right)\right)-f_{-}^{(r)}(x)\right|,\left|L_n\left(f_{+}^{(r)},x\right)-f_{+}^{(r)}(x)\right|\right\}. \tag{73}$$

We present

Theorem 33 *Let* $f\in C_{\mathcal{F}}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta<1$, $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$. *Then*

1)

$$D\left(A_n^{\mathcal{F}}(f,x),f(x)\right)\leq$$

$$\gamma(N)\left[\omega_1^{(\mathcal{F})}\left(f,\frac{1}{n^{\beta}}\right)+2c(\beta,n)D^{*}(f,\widetilde{o})\right]=:\rho_1, \tag{74}$$

and

2)

$$D^*\left(A_n^{\mathcal{F}}(f), f\right) \leq \rho_1. \tag{75}$$

We notice that $A_n^{\mathcal{F}}(f,x) \overset{D}{\to} f(x)$, *and* $A_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, *as* $n \to \infty$,*quantitatively with rates.*

Proof. Since $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{N}[a_i, b_i]\right)$ we have that $f_{\pm}^{(r)} \in C\left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $\forall$ $r \in [0,1]$. Hence by (47) we obtain

$$\left|A_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq \gamma(N)\left[\omega_1\left(f_{\pm}^{(r)}, \frac{1}{n^{\beta}}\right) + 2c(\beta,n)\left\|f_{\pm}^{(r)}\right\|_{\infty}\right]$$

$$\overset{\text{(by (8), (3))}}{\leq} \gamma(N)\left[\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^{\beta}}\right) + 2c(\beta,n)\, D^*(f, \widetilde{o})\right]. \tag{76}$$

By (72) now we are proving the claim. ■

We give

Theorem 34 *Let* $f \in C_B\left(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}}\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$, *with* $n^{1-\beta} > 2$. *Then*

1)

$$D\left(B_n^{\mathcal{F}}(f,x), f(x)\right) \leq \tag{77}$$

$$\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^{\beta}}\right) + 2c(\beta,n)\, D^*(f, \widetilde{o}) =: \rho_2,$$

and

2)

$$D^*\left(B_n^{\mathcal{F}}(f), f\right) \leq \rho_2. \tag{78}$$

Proof. Similar to Theorem 33. We use (49) and (73), along with (3) and (8). ■

We also present

Theorem 35 *All as in Theorem 34. Then*

1)

$$D\left(C_n^{\mathcal{F}}(f,x), f(x)\right) \leq$$

$$\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + 2c(\beta,n)\, D^*(f, \widetilde{o}) =: \rho_3, \tag{79}$$

and

2)

$$D^*\left(C_n^{\mathcal{F}}(f), f\right) \leq \rho_3. \tag{80}$$

Proof. Similar to Theorem 33. We use (51) and (73), along with (3) and (8). ■

We also give

Theorem 36 *All as in Theorem 34. Then*

1)

$$D\left(D_n^{\mathcal{F}}(f,x), f(x)\right) \leq$$
$$\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n}+\frac{1}{n^{\beta}}\right) + 2c(\beta,n)\, D^*\left(f,\widetilde{o}\right) = \rho_3, \tag{81}$$

and

2)

$$D^*\left(D_n^{\mathcal{F}}(f), f\right) \leq \rho_3. \tag{82}$$

Proof. Similar to Theorem 33. We use (53) and (73), along with (3) and (8). ■

Note 37 *By Theorems 34, 35, 36 for* $f \in \left(C_B\left(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}}\right) \cap C_{\mathcal{F}}^U\left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n\to\infty} D\left(L_n^{\mathcal{F}}(f,x), f(x)\right) = 0$, *and* $\lim_{n\to\infty} D^*\left(L_n^{\mathcal{F}}(f), f\right) = 0$, *quantitatively with rates, where* $L_n^{\mathcal{F}}$ *is as in (64) and (65).*

Notation 38 *Let* $f \in C_{\mathcal{F}}^m\left(\prod_{i=1}^N [a_i,b_i]\right)$, $m, N \in \mathbb{N}$. *Here* f_α *denotes a fuzzy partial derivative with all related notation similar to the real case, see also Remark 15 and Notation 16. We denote*

$$\omega_{1,m}^{(\mathcal{F})\max}(f_\alpha, h) := \max_{\alpha:|\alpha|=m} \omega_1^{(\mathcal{F})}(f_\alpha, h), \quad h > 0. \tag{83}$$

Call also

$$D_m^{*\max}(f_\alpha, \widetilde{o}) := \max_{\alpha:|\alpha|=m} \left\{D^*\left(f_\alpha, \widetilde{o}\right)\right\}. \tag{84}$$

We finally present

Theorem 39 *Let* $f \in C_{\mathcal{F}}^m\left(\prod_{i=1}^N [a_i,b_i]\right)$, $0 < \beta < 1$, $n, m, N \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, *and* $x \in \left(\prod_{i=1}^N [a_i,b_i]\right)$. *Then*

1)

$$D\left(A_n^{\mathcal{F}}(f,x), f(x)\right) \leq \gamma(N)\cdot$$
$$\left\{\sum_{j_*=1}^{m}\left(\sum_{|\alpha|=j_*} \frac{D\left(f_\alpha(x), \widetilde{o}\right)}{\prod_{i=1}^N \alpha_i!}\left[\frac{1}{n^{\beta j_*}} + \left(\prod_{i=1}^N (b_i-a_i)^{\alpha_i}\right) c(\beta,n)\right]\right)\right.$$

$$+\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{\|b-a\|_\infty^m D_m^{*\max}\left(f_\alpha,\widetilde{o}\right)N^m}{m!}\right)2c\left(\beta,n\right)\Bigg\},\tag{85}$$

2)

$$D^*\left(A_n^{\mathcal{F}}\left(f\right),f\right)\leq\gamma\left(N\right)\cdot$$

$$\left\{\sum_{j_*=1}^m\left(\sum_{|\alpha|=j_*}\frac{D^*\left(f_\alpha,\widetilde{o}\right)}{\prod_{i=1}^N\alpha_i!}\left[\frac{1}{n^{\beta j_*}}+\left(\prod_{i=1}^N\left(b_i-a_i\right)^{\alpha_i}\right)c\left(\beta,n\right)\right]\right)\right.$$

$$\left.+\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{\|b-a\|_\infty^m D_m^{*\max}\left(f_\alpha,\widetilde{o}\right)N^m}{m!}\right)2c\left(\beta,n\right)\right\},\tag{86}$$

3) additionally assume that $f_\alpha\left(x_0\right)=\widetilde{o}$*, for all* $\alpha:|\alpha|=1,...,m;$ $x_0\in\left(\prod_{i=1}^N\left[a_i,b_i\right]\right)$*, then*

$$D\left(A_n^{\mathcal{F}}\left(f,x_0\right),f\left(x_0\right)\right)\leq\gamma\left(N\right)\left\{\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\right.$$

$$\left.\left(\frac{\|b-a\|_\infty^m D_m^{*\max}\left(f_\alpha,\widetilde{o}\right)N^m}{m!}\right)2c\left(\beta,n\right)\right\},\tag{87}$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

Above we derive quantitatively with rates the high speed approximation of $D\left(A_n^{\mathcal{F}}\left(f,x\right),f\left(x\right)\right)\to0$*, as* $n\to\infty$.

Also we establish with rates that $D^*\left(A_n^{\mathcal{F}}\left(f\right),f\right)\to0$*, as* $n\to\infty$*, involving the fuzzy smoothness of* f.

Proof. Here $f_\pm^{(r)}\in C^m\left(\prod_{i=1}^N\left[a_i,b_i\right]\right)$. We observe that

$$\left|A_n\left(f_\pm^{(r)},x\right)-f_\pm^{(r)}\left(x\right)\right|\overset{(56)}{\leq}\gamma\left(N\right)\cdot$$

$$\left\{\sum_{j_*=1}^m\left(\sum_{|\alpha|=j_*}\left(\frac{\left|\left(f_\pm^{(r)}\right)_\alpha\left(x\right)\right|}{\prod_{i=1}^N\alpha_i!}\right)\left[\frac{1}{n^{\beta j_*}}+\left(\prod_{i=1}^N\left(b_i-a_i\right)^{\alpha_i}\right)c\left(\beta,n\right)\right]\right)\right.$$

$$+\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(\left(f_\pm^{(r)}\right)_\alpha,\frac{1}{n^\beta}\right)+$$

$$\left.\left(\frac{\|b-a\|_\infty^m\left\|\left(f_\pm^{(r)}\right)_\alpha\right\|_{\infty,m}^{\max}N^m}{m!}\right)2c\left(\beta,n\right)\right\}\overset{(13)}{=}\tag{88}$$

$$\gamma(N)\left\{\sum_{j_*=1}^{m}\left(\sum_{|\alpha|=j_*}\left(\frac{\left|(f_\alpha)_\pm^{(r)}(x)\right|}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta j_*}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)c(\beta,n)\right]\right)\right.$$

$$+\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left((f_\alpha)_\pm^{(r)},\frac{1}{n^\beta}\right)+$$

$$\left.\left(\frac{\|b-a\|_\infty^m\left\|(f_\alpha)_\pm^{(r)}\right\|_{\infty,m}^{\max}N^m}{m!}\right)2c(\beta,n)\right\}\overset{\text{(by (3), (8))}}{\leq} \tag{89}$$

$$\gamma(N)\left\{\sum_{j_*=1}^{m}\left(\sum_{|\alpha|=j_*}\left(\frac{D(f_\alpha(x),\widetilde{o})}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta j_*}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)c(\beta,n)\right]\right)\right.$$

$$+\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+$$

$$\left.\left(\frac{\|b-a\|_\infty^m D_m^{*\max}(f_\alpha,\widetilde{o})N^m}{m!}\right)2c(\beta,n)\right\}=:T, \tag{90}$$

respectively in $\pm$.

We have proved that

$$\left|A_n\left(f_\pm^{(r)},x\right)-f_\pm^{(r)}(x)\right|\leq T, \tag{91}$$

$\forall\ r\in[0,1]$, respectively in $\pm$.

Using (72) we obtain

$$D\left(A_n^{\mathcal{F}}(f,x),f(x)\right)\leq T, \tag{92}$$

proving the theorem. ■

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