Multivariate Fuzzy-Random and stochastic general sigmoid activation function induced Neural Network Approximations

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Abstract

In this article we research the degree of approximation of multivariate pointwise and uniform convergences in the q-mean to the Fuzzy-Random unit operator of multivariate Fuzzy-Random Quasi-Interpolation general sigmoid activation function based neural network operators. These multivariate Fuzzy-Random operators arise in a natural way among multivariate Fuzzy-Random neural networks. The rates are given through multivariate Probabilistic-Jackson type inequalities involving the multivariate Fuzzy-Random modulus of continuity of the engaged multivariate Fuzzy-Random function. The plain stochastic extreme analog of this theory is also met in detail for the stochastic analogs of the operators: the stochastic full quasi-interpolation operators, the stochastic Kantorovich type operators and the stochastic quadrature type operators.

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1 Fuzzy-Random Functions and Stochastic processes Background

See also [18], Ch. 22, pp. 497-501.

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We start with

Definition 1 *(see [28]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:*

(i) is normal, i.e., $\exists\, x_0 \in \mathbb{R} : \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\, x, y \in \mathbb{R}$, $\forall\, \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall\, x_0 \in \mathbb{R}$ and $\forall\, \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\, x \in V(x_0)$.

(iv) the set $\overline{supp(\mu)}$ is compact in $\mathbb{R}$ (where $supp(\mu) := \{x \in \mathbb{R}; \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ and $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall\, r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [28]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} < u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$$

by

$$D(u, v) := \sup_{r \in [0,1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\},$$

where $[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [28], with the properties

$$\begin{gathered} D(u \oplus w, v \oplus w) = D(u, v), \quad \forall\, u, v, w \in \mathbb{R}_{\mathcal{F}}, \\ D(k \odot u, k \odot v) = |k|\, D(u, v), \quad \forall\, u, v \in \mathbb{R}_{\mathcal{F}}, \forall\, k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e), \quad \forall\, u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{gathered} \tag{1}$$

Let (M, d) metric space and $f, g : M \to \mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between f, g is defined by

$$D^*(f, g) := \sup_{x \in M} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_{-}^{(r)} \leq v_{-}^{(r)}$ and $u_{+}^{(r)} \leq v_{+}^{(r)}$, $\forall\, r \in [0,1]$.

$\sum\limits^{*}$ denotes the fuzzy summation, $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ the neutral element with respect to $\oplus$. For more see also [29], [30].

We need

Definition 2 *(see also [24], Definition 13.16, p. 654) Let $(X, \mathcal{B}, P)$ be a probability space. A fuzzy-random variable is a $\mathcal{B}$-measurable mapping $g : X \to \mathbb{R}_{\mathcal{F}}$ (i.e., for any open set $U \subseteq \mathbb{R}_{\mathcal{F}}$, in the topology of $\mathbb{R}_{\mathcal{F}}$ generated by the metric D, we have*

$$g^{-1}(U) = \{s \in X; g(s) \in U\} \in \mathcal{B}). \tag{2}$$

The set of all fuzzy-random variables is denoted by $\mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Let $g_n, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $n \in \mathbb{N}$ and $0 < q < +\infty$. We say $g_n(s) \overset{\text{"q-mean"}}{\underset{n \to +\infty}{\to}} g(s)$ if

$$\lim_{n \to +\infty} \int_X D(g_n(s), g(s))^q\, P(ds) = 0. \tag{3}$$

Remark 3 *(see [24], p. 654) If $f, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, let us denote $F : X \to \mathbb{R}_+ \cup \{0\}$ by $F(s) = D(f(s), g(s))$, $s \in X$. Here, F is $\mathcal{B}$-measurable, because $F = G \circ H$, where $G(u, v) = D(u, v)$ is continuous on $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, and $H : X \to \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, $H(s) = (f(s), g(s))$, $s \in X$, is $\mathcal{B}$-measurable. This shows that the above convergence in q-mean makes sense.*

Definition 4 *(see [24], p. 654, Definition 13.17) Let $(T, \mathcal{T})$ be a topological space. A mapping $f : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$ will be called fuzzy-random function (or fuzzy-stochastic process) on T. We denote $f(t)(s) = f(t, s)$, $t \in T$, $s \in X$.*

Remark 5 *(see [24], p. 655) Any usual fuzzy real function $f : T \to \mathbb{R}_{\mathcal{F}}$ can be identified with the degenerate fuzzy-random function $f(t, s) = f(t)$, $\forall\, t \in T$, $s \in X$.*

Remark 6 *(see [24], p. 655) Fuzzy-random functions that coincide with probability one for each $t \in T$ will be consider equivalent.*

Remark 7 *(see [24], p. 655) Let $f, g : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Then $f \oplus g$ and $k \odot f$ are defined pointwise, i.e.,*

$$(f \oplus g)(t, s) = f(t, s) \oplus g(t, s),$$
$$(k \odot f)(t, s) = k \odot f(t, s), \quad t \in T, s \in X, \ k \in \mathbb{R}.$$

Definition 8 *(see also Definition 13.18, pp. 655-656, [24]) For a fuzzy-random function $f : W \subseteq \mathbb{R}^N \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, we define the (first) fuzzy-random modulus of continuity*

$$\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} =$$

$$\sup\left\{\left(\int_X D^q\left(f\left(x,s\right),f\left(y,s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}}:x,y\in W,\ \ \left\|x-y\right\|_\infty\leq\delta\right\},$$

$0<\delta$, $1\leq q<\infty$.

Definition 9 *([16]) Here $1\leq q<+\infty$. Let $f:W\subseteq\mathbb{R}^N\to\mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$, $N\in\mathbb{N}$, be a fuzzy random function. We call f a (q-mean) uniformly continuous fuzzy random function over W, iff $\forall\ \varepsilon>0\ \exists\ \delta>0$:whenever $\left\|x-y\right\|_\infty\leq\delta$, $x,y\in W$, implies that*

$$\int_X\left(D\left(f\left(x,s\right),f\left(y,s\right)\right)\right)^q P\left(ds\right)\leq\varepsilon.$$

We denote it as $f\in C_{FR}^{U_q}\left(W\right)$.

Proposition 10 *([16]) Let $f\in C_{FR}^{U_q}\left(W\right)$, where $W\subseteq\mathbb{R}^N$ is convex.*
Then $\Omega_1^{(\mathcal{F})}\left(f,\delta\right)_{L^q}<\infty$, any $\delta>0$.

Proposition 11 *([16]) Let $f,g:W\subseteq\mathbb{R}^N\to\mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$, $N\in\mathbb{N}$, be fuzzy random functions. It holds*
(i) $\Omega_1^{(\mathcal{F})}\left(f,\delta\right)_{L^q}$ is nonnegative and nondecreasing in $\delta>0$.
(ii) $\lim_{\delta\downarrow 0}\Omega_1^{(\mathcal{F})}\left(f,\delta\right)_{L^q}=\Omega_1^{(\mathcal{F})}\left(f,0\right)_{L^q}=0$, iff $f\in C_{FR}^{U_q}\left(W\right)$.

We mention

Definition 12 *(see also [6]) Let $f\left(t,s\right)$ be a random function (stochastic process) from $W\times\left(X,\mathcal{B},P\right)$, $W\subseteq\mathbb{R}^N$, into $\mathbb{R}$, where $\left(X,\mathcal{B},P\right)$ is a probability space. We define the q-mean multivariate first modulus of continuity of f by*

$$\Omega_1\left(f,\delta\right)_{L^q}:=$$

$$\sup\left\{\left(\int_X\left|f\left(x,s\right)-f\left(y,s\right)\right|^q P\left(ds\right)\right)^{\frac{1}{q}}:x,y\in W,\ \ \left\|x-y\right\|_\infty\leq\delta\right\},\qquad(4)$$

$\delta>0$, $1\leq q<\infty$.

The concept of f being (q-mean) uniformly continuous random function is defined the same way as in Definition 9, just replace D by $\left|\cdot\right|$, etc. We denote it as $f\in C_{\mathbb{R}}^{U_q}\left(W\right)$.

Similar properties as in Propositions 10, 11 are valid for $\Omega_1\left(f,\delta\right)_{L^q}$.

Also we have

Proposition 13 *([3]) Let $Y\left(t,\omega\right)$ be a real valued stochastic process such that Y is continuous in $t\in[a,b]$. Then Y is jointly measurable in $\left(t,\omega\right)$.*

According to [23], p. 94 we have the following

Definition 14 *Let* $(Y, \mathcal{T})$ *be a topological space, with its* σ*-algebra of Borel sets* $\mathcal{B} := \mathcal{B}(Y, \mathcal{T})$ *generated by* $\mathcal{T}$*. If* $(X, \mathcal{S})$ *is a measurable space, a function* $f : X \to Y$ *is called measurable iff* $f^{-1}(B) \in \mathcal{S}$ *for all* $B \in \mathcal{B}$*.*

By Theorem 4.1.6 of [23], p. 89 f as above is measurable iff

$$f^{-1}(C) \in \mathcal{S} \text{ for all } C \in \mathcal{T}.$$

We mention

Theorem 15 *(see [23], p. 95) Let* $(X, \mathcal{S})$ *be a measurable space and* (Y, d) *be a metric space. Let* f_n *be measurable functions from* X *into* Y *such that for all* $x \in X$*,* $f_n(x) \to f(x)$ *in* Y*. Then* f *is measurable. I.e.,* $\lim_{n \to \infty} f_n = f$ *is measurable.*

We need also

Proposition 16 *([16]) Let* f, g *be fuzzy random variables from* $\mathcal{S}$ *into* $\mathbb{R}_{\mathcal{F}}$*. Then*

(i) Let $c \in \mathbb{R}$*, then* $c \odot f$ *is a fuzzy random variable.*

(ii) $f \oplus g$ *is a fuzzy random variable.*

Proposition 17 *Let* $Y\left(\overrightarrow{t}, \omega\right)$ *be a real valued multivariate random function (stochastic process) such that* Y *is continuous in* $\overrightarrow{t} \in \prod_{i=1}^{N}[a_i, b_i]$*. Then* Y *is jointly measurable in* $\left(\overrightarrow{t}, \omega\right)$ *and* $\int_{\prod_{i=1}^{N}[a_i,b_i]} Y\left(\overrightarrow{t}, \omega\right) d\overrightarrow{t}$ *is a real valued random variable.*

Proof. Similar to Proposition 18.14, p. 353 of [7]. ∎

2 About real neural networks background

Here we follow [21].

Let $h : \mathbb{R} \to [-1, 1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0$, $h(-x) = -h(x)$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also h is strictly convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.

We consider the activation function

$$\psi(x) := \frac{1}{4}\left(h(x+1) - h(x-1)\right), \quad x \in \mathbb{R}, \tag{5}$$

As in [20], p. 88, we get that $\psi(-x) = \psi(x)$, thus ψ is an even function. Since $x + 1 > x - 1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi(0) = \frac{h(1)}{2}. \tag{6}$$

Let $x > 1$, we have that

$$\psi'(x) = \frac{1}{4}\left(h'(x+1) - h'(x-1)\right) < 0,$$

by h' being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1 - x > 0$ and $0 < 1 - x < 1 + x$. It holds $h'(x-1) = h'(1-x) > h'(x+1)$, so that again $\psi'(x) < 0$. Consequently ψ is stritly decreasing on $(0, +\infty)$.

Clearly, ψ is strictly increasing on $(-\infty, 0)$, and $\psi'(0) = 0$.

See that

$$\lim_{x \to +\infty} \psi(x) = \frac{1}{4}\left(h(+\infty) - h(+\infty)\right) = 0, \tag{7}$$

and

$$\lim_{x \to -\infty} \psi(x) = \frac{1}{4}\left(h(-\infty) - h(-\infty)\right) = 0. \tag{8}$$

That is the x-axis is the horizontal asymptote on ψ.

Conclusion, ψ is a bell symmetric function with maximum

$$\psi(0) = \frac{h(1)}{2}.$$

We need

Theorem 18 *([21]) We have that*

$$\sum_{i=-\infty}^{\infty} \psi(x - i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{9}$$

Theorem 19 *([21]) It holds*

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 1. \tag{10}$$

Thus $\psi(x)$ is a density function on $\mathbb{R}$.

We give

Theorem 20 *([21]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi(nx - k) < \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2}. \tag{11}$$

Notice that

$$\lim_{n \to +\infty} \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} = 0.$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We further give

Theorem 21 *([21]) Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx - k\right)} < \frac{1}{\psi\left(1\right)}, \quad \forall\, x \in [a, b]\,. \tag{12}$$

Remark 22 *([21]) i) We have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx - k\right) \neq 1, \tag{13}$$

for at least some $x \in [a, b]$.

ii) For large enough $n \in \mathbb{N}$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general, by Theorem 18, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi\left(nx - k\right) \leq 1. \tag{14}$$

We introduce

$$Z\left(x_1, ..., x_N\right) := Z\left(x\right) := \prod_{i=1}^{N} \psi\left(x_i\right), \quad x = \left(x_1, ..., x_N\right) \in \mathbb{R}^N,\ N \in \mathbb{N}. \tag{15}$$

It has the properties:

(i) $Z\left(x\right) > 0,\ \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z\left(x - k\right) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z\left(x_1 - k_1, ..., x_N - k_N\right) = 1, \tag{16}$$

where $k := \left(k_1, ..., k_n\right) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z\left(nx - k\right) = 1, \tag{17}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z\left(x\right) dx = 1, \tag{18}$$

that is Z is a multivariate density function.

Here denote $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= (\lceil na_1 \rceil, ..., \lceil na_N \rceil), \\ \lfloor nb \rfloor &:= (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor), \end{aligned} \tag{19}$$

where $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \psi(nx_i - k_i) \right) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^{N} \psi(nx_i - k_i) \right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i) \right). \tag{20}$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) =$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \le \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) + \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k). \tag{21}$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, where $r \in \{1, ..., N\}$.

(v) As in [18], pp. 379-380, we derive that

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) \overset{(11)}{<} \frac{1 - h\left(n^{1-\beta} - 2\right)}{2}, \quad 0 < \beta < 1, \tag{22}$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N} [a_i, b_i]$.

(vi) By Theorem 21 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} < \frac{1}{(\psi(1))^N} =: \gamma(N), \tag{23}$$

$\forall\, x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $n \in \mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ \left\| \frac{k}{n}-x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{array}\right.}^{\infty} Z(nx-k) < \frac{1-h\left(n^{1-\beta}-2\right)}{2} =: c(\beta,n), \tag{24}$$

$0<\beta<1$, $n\in\mathbb{N}: n^{1-\beta}>2$, $x\in\mathbb{R}^N$.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) \neq 1, \tag{25}$$

for at least some $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Let $f\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, and $n\in\mathbb{N}$ such that $\lceil na_i\rceil \leq \lfloor nb_i\rfloor$, $i=1,...,N$.

We define the multivariate averaged positive linear neural network operators ($x:=(x_1,...,x_N)\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$):

$$A_n(f,x_1,...,x_N) := A_n(f,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)} = \tag{26}$$

$$\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor} ... \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\psi(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\psi(nx_i-k_i)\right)}.$$

For large enough $n\in\mathbb{N}$ we always obtain $\lceil na_i\rceil \leq \lfloor nb_i\rfloor$, $i=1,...,N$. Also $a_i\leq\frac{k_i}{n}\leq b_i$, iff $\lceil na_i\rceil\leq k_i\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

When $f\in C_B\left(\mathbb{R}^N\right)$ we define

$$B_n(f,x) := B_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) := \tag{27}$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\psi(nx_i-k_i)\right),$$

$n\in\mathbb{N}$, $\forall\ x\in\mathbb{R}^N$, $N\in\mathbb{N}$, the multivariate quasi-interpolation neural network operators.

Also for $f\in C_B\left(\mathbb{R}^N\right)$ we define the multivariate Kantorovich type neural network operators

$$C_n(f,x) := C_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty}\left(n^N\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt\right) Z(nx-k) := \tag{28}$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\left(n^N\int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}}...\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}}f\left(t_1,...,t_N\right)dt_1...dt_N\right)$$

$$\cdot\left(\prod_{i=1}^{N}\psi\left(nx_i-k_i\right)\right),$$

$n\in\mathbb{N}$, $\forall\ x\in\mathbb{R}^N$.

Again for $f\in C_B\left(\mathbb{R}^N\right)$, $N\in\mathbb{N}$, we define the multivariate neural network operators of quadrature type $D_n\left(f,x\right)$, $n\in\mathbb{N}$, as follows. Let $\theta=\left(\theta_1,...,\theta_N\right)\in\mathbb{N}^N$, $\overline{r}=\left(r_1,...,r_N\right)\in\mathbb{Z}_+^N$, $w_{\overline{r}}=w_{r_1,r_2,...r_N}\geq 0$, such that $\sum_{\overline{r}=0}^{\theta}w_{\overline{r}}=\sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}...\sum_{r_N=0}^{\theta_N}w_{r_1,r_2,...r_N}=1$; $k\in\mathbb{Z}^N$ and

$$\delta_{nk}\left(f\right):=\delta_{n,k_1,k_2,...,k_N}\left(f\right):=\sum_{\overline{r}=0}^{\theta}w_{\overline{r}}f\left(\frac{k}{n}+\frac{\overline{r}}{n\theta}\right):=$$

$$\sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}...\sum_{r_N=0}^{\theta_N}w_{r_1,r_2,...r_N}f\left(\frac{k_1}{n}+\frac{r_1}{n\theta_1},\frac{k_2}{n}+\frac{r_2}{n\theta_2},...,\frac{k_N}{n}+\frac{r_N}{n\theta_N}\right),\qquad(29)$$

where $\frac{\overline{r}}{\theta}:=\left(\frac{r_1}{\theta_1},\frac{r_2}{\theta_2},...,\frac{r_N}{\theta_N}\right)$.

We put

$$D_n\left(f,x\right):=D_n\left(f,x_1,...,x_N\right):=\sum_{k=-\infty}^{\infty}\delta_{nk}\left(f\right)Z\left(nx-k\right):=\qquad(30)$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\delta_{n,k_1,k_2,...,k_N}\left(f\right)\left(\prod_{i=1}^{N}\psi\left(nx_i-k_i\right)\right),$$

$\forall\ x\in\mathbb{R}^N$.

For the next we need, for $f\in C\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$ the first multivariate modulus of continuity

$$\omega_1\left(f,h\right):=\sup_{\substack{x,y\in\prod_{i=1}^{N}\left[a_i,b_i\right]\\ \|x-y\|_\infty\leq h}}\left|f\left(x\right)-f\left(y\right)\right|,\ \ h>0.$$

A totally similar definition applies to $f\in C_B\left(\mathbb{R}^N\right)$.

Above $\|\cdot\|_\infty$ is the supremum norm.

In [20] we studied the basic approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates for Banach space valued functions. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I.

We need

Theorem 23 *Let* $f \in C\left(\prod_{i=1}^{N}\left[a_i, b_i\right]\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^{N}\left[a_i, b_i\right]\right)$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$. *Then*

1)

$$\left|A_n\left(f, x\right) - f\left(x\right)\right| \leq \gamma\left(N\right)\left[\omega_1\left(f, \frac{1}{n^{\beta}}\right) + 2c\left(\beta, n\right)\left\|f\right\|_{\infty}\right] =: \lambda_1, \tag{31}$$

and

2)

$$\left\|A_n\left(f\right) - f\right\|_{\infty} \leq \lambda_1. \tag{32}$$

We notice that $\lim_{n \to \infty} A_n\left(f\right) = f$, *pointwise and uniformly.*

Proof. Similar to [20], p. 118. ■

We need

Theorem 24 *Let* $f \in C_B\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$. *Then*

1)

$$\left|B_n\left(f, x\right) - f\left(x\right)\right| \leq \omega_1\left(f, \frac{1}{n^{\beta}}\right) + 2c\left(\beta, n\right)\left\|f\right\|_{\infty} =: \lambda_2, \tag{33}$$

2)

$$\left\|B_n\left(f\right) - f\right\|_{\infty} \leq \lambda_2. \tag{34}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N\right) \cap C_B\left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n \to \infty} B_n\left(f\right) = f$, *uniformly.*

Proof. Similar to [20], p. 128. ■

We also need

Theorem 25 *Let* $f \in C_B\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$. *Then*

1)

$$\left|C_n\left(f, x\right) - f\left(x\right)\right| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + 2c\left(\beta, n\right)\left\|f\right\|_{\infty} =: \lambda_3, \tag{35}$$

2)

$$\left\|C_n\left(f\right) - f\right\|_{\infty} \leq \lambda_3. \tag{36}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N\right) \cap C_B\left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n \to \infty} C_n\left(f\right) = f$, *uniformly.*

Proof. Similar to [20], p. 129. ■

We also need

Theorem 26 *Let* $f \in C_B\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta}>2$. *Then*

1)

$$\left|D_n(f, x)-f(x)\right| \leq \omega_1\left(f, \frac{1}{n}+\frac{1}{n^\beta}\right)+2 c(\beta, n)\|f\|_{\infty}=\lambda_3, \tag{37}$$

2)

$$\left\|D_n(f)-f\right\|_{\infty} \leq \lambda_3. \tag{38}$$

Given that $f \in\left(C_U\left(\mathbb{R}^N\right) \cap C_B\left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n \rightarrow \infty} D_n(f)=f$, *uniformly.*

Proof. Similar to [20], p. 131. ■

In this article we extend Theorems 23, 24, 25, 26 to the random level.

We are also motivated by [1] - [16] and continuing [17]. For general knowledge on neural networks we recommend [25], [26], [27].

3 Main Results

I) q-mean Approximation by Fuzzy-Random general sigmoid activation function based Quasi-Interpolation Neural Network Operators

All terms and assumptions here as in Sections 1, 2.

Let $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^N\left[a_i, b_i\right]\right)$, $1 \leq q<+\infty$, $n, N \in \mathbb{N}$, $0<\beta<1$, $\overrightarrow{x} \in \left(\prod_{i=1}^N\left[a_i, b_i\right]\right)$, $(X, \mathcal{B}, P)$ probability space, $s \in X$.

We define the following multivariate fuzzy random general sigmoid activation function based quasi-interpolation linear neural network operators

$$\left(A_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x}, s):=\sum_{\overrightarrow{k}=\lceil n a\rceil}^{\lfloor n b\rfloor *} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot \frac{Z\left(n \overrightarrow{x}-\overrightarrow{k}\right)}{\sum_{\overrightarrow{k}=\lceil n a\rceil}^{\lfloor n b\rfloor} Z\left(n \overrightarrow{x}-\overrightarrow{k}\right)}, \tag{39}$$

(see also (26).

We present

Theorem 27 *Let* $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^N\left[a_i, b_i\right]\right)$, $0<\beta<1$, $\overrightarrow{x} \in\left(\prod_{i=1}^N\left[a_i, b_i\right]\right)$, $n, N \in \mathbb{N}$, *with* $n^{1-\beta}>2$, $1 \leq q<+\infty$. *Assume that* $\int_X\left(D^*\left(f(\cdot, s), \widetilde{o}\right)\right)^q P(d s)<\infty$. *Then*

1)

$$\left(\int_X D^q\left(\left(A_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x}, s), f(\overrightarrow{x}, s)\right) P(d s)\right)^{\frac{1}{q}} \leq \tag{40}$$

$$\gamma(N)\left\{\Omega_1\left(f,\frac{1}{n^{\beta}}\right)_{L^q}+2c(\beta,n)\left(\int_X\left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^q P(ds)\right)^{\frac{1}{q}}\right\}=:\lambda_1^{(\mathcal{FR})},$$

2)

$$\left\|\left(\int_X D^q\left(\left(A_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s),f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}}\right\|_{\infty,\left(\prod_{i=1}^{N}[a_i,b_i]\right)}\leq\lambda_1^{(\mathcal{FR})},\quad (41)$$

where $\gamma(N)$ as in (23) and $c(\beta,n)$ as in (24).

Proof. We notice that

$$D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f(\overrightarrow{x},s)\right)\leq D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),\widetilde{o}\right)+D\left(f(\overrightarrow{x},s),\widetilde{o}\right)\quad (42)$$

$$\leq 2D^*\left(f(\cdot,s),\widetilde{o}\right).$$

Hence

$$D^q\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f(\overrightarrow{x},s)\right)\leq 2^q D^{*q}\left(f(\cdot,s),\widetilde{o}\right),\quad (43)$$

and

$$\left(\int_X D^q\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}}\leq 2\left(\int_X\left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^q P(ds)\right)^{\frac{1}{q}}.\quad (44)$$

We observe that

$$D\left(\left(A_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s),f(\overrightarrow{x},s)\right)=\quad (45)$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{Z(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)},f(\overrightarrow{x},s)\odot 1\right)=$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{Z(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)},f(\overrightarrow{x},s)\odot\frac{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)}\right)=\quad (46)$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{Z(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)},\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f(\overrightarrow{x},s)\odot\frac{Z(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)}\right)$$

$$\leq \sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) D\left(f\left(\frac{\overrightarrow{k}}{n}, s \right), f(\overrightarrow{x}, s) \right). \tag{47}$$

So that

$$D\left(\left({}_{j}A_{n}^{\mathcal{FR}}(f) \right)(\overrightarrow{x}, s), f(\overrightarrow{x}, s) \right) \leq$$

$$\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) D\left(f\left(\frac{\overrightarrow{k}}{n}, s \right), f(\overrightarrow{x}, s) \right) = \tag{48}$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na \rceil \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) D\left(f\left(\frac{\overrightarrow{k}}{n}, s \right), f(\overrightarrow{x}, s) \right) +$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na \rceil \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) D\left(f\left(\frac{\overrightarrow{k}}{n}, s \right), f(\overrightarrow{x}, s) \right).$$

Hence it holds

$$\left(\int_{X} D^{q}\left(\left(A_{n}^{\mathcal{FR}}(f) \right)(\overrightarrow{x}, s), f(\overrightarrow{x}, s) \right) P(ds) \right)^{\frac{1}{q}} \leq \tag{49}$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na \rceil \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) \left(\int_{X} D^{q}\left(f\left(\frac{\overrightarrow{k}}{n}, s \right), f(\overrightarrow{x}, s) \right) P(ds) \right)^{\frac{1}{q}} +$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na \rceil \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \left(\frac{Z(nx-k)}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) \left(\int_{X} D^{q}\left(f\left(\frac{\overrightarrow{k}}{n}, s \right), f(\overrightarrow{x}, s) \right) P(ds) \right)^{\frac{1}{q}} \leq$$

$$\left(\frac{1}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \right) \cdot \left\{ \Omega_{1}^{(\mathcal{F})}\left(f, \frac{1}{n^{\beta}} \right)_{L^{q}} + \right. \tag{50}$$

$$2\left(\int_X \left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^q P(ds)\right)^{\frac{1}{q}}\left(\sum_{\substack{\overrightarrow{k}=\lceil na\rceil \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor} Z(nx-k)\right)\Bigg)\Bigg\}$$

(by (23), (24))

$$\leq \gamma(N)\left\{\Omega_1^{(\mathcal{F})}\left(f,\frac{1}{n^{\beta}}\right)_{L^q}+2c(\beta,n)\left(\int_X \left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^q P(ds)\right)^{\frac{1}{q}}\right\}. \quad (51)$$

We have proved claim. ■

Conclusion 28 *By Theorem 27 we obtain the pointwise and uniform convergences with rates in the q-mean and D-metric of the operator* $A_n^{\mathcal{FR}}$ *to the unit operator for* $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

II) 1-mean Approximation by Stochastic general sigmoid activation function based full Quasi-Interpolation Neural Network Operators

Let $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0<\beta<1$, $\overrightarrow{x}\in\mathbb{R}^N$, $n,N\in\mathbb{N}$, with $\|g\|_{\infty,\mathbb{R}^N,X}<\infty$, $(X,\mathcal{B},P)$ probability space, $s\in X$.

We define

$$B_n^{(\mathcal{R})}(g)\left(\overrightarrow{x},s\right):=\sum_{\overrightarrow{k}=-\infty}^{\infty} g\left(\frac{\overrightarrow{k}}{n},s\right)Z\left(n\overrightarrow{x}-\overrightarrow{k}\right), \quad (52)$$

(see also (27)).

We give

Theorem 29 *Let* $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0<\beta<1$, $\overrightarrow{x}\in\mathbb{R}^N$, $n,N\in\mathbb{N}$, *with* $n^{1-\beta}>2$, $\|g\|_{\infty,\mathbb{R}^N,X}<\infty$. *Then*

1)

$$\int_X \left|\left(B_n^{(\mathcal{R})}(g)\right)\left(\overrightarrow{x},s\right)-g\left(\overrightarrow{x},s\right)\right| P(ds)\leq \quad (53)$$

$$\left\{\Omega_1\left(g,\frac{1}{n^{\beta}}\right)_{L^1}+2c(\beta,n)\|g\|_{\infty,\mathbb{R}^N,X}\right\}=:\mu_1^{(\mathcal{R})},$$

2)

$$\left\|\int_X \left|\left(B_n^{(\mathcal{R})}(g)\right)\left(\overrightarrow{x},s\right)-g\left(\overrightarrow{x},s\right)\right| P(ds)\right\|_{\infty,\mathbb{R}^N}\leq \mu_1^{(\mathcal{R})}. \quad (54)$$

Proof. Since $\|g\|_{\infty,\mathbb{R}^N,X}<\infty$, then

$$\left|g\left(\frac{\overrightarrow{k}}{n},s\right)-g\left(\overrightarrow{x},s\right)\right|\leq 2\|g\|_{\infty,\mathbb{R}^N,X}<\infty. \quad (55)$$

Hence

$$\int_X \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| P(ds) \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{56}$$

We observe that

$$\left(B_n^{(\mathcal{R})}(g)\right)\left(\overrightarrow{x}, s\right) - g\left(\overrightarrow{x}, s\right) =$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} g\left(\frac{\overrightarrow{k}}{n}, s\right) Z(nx-k) - g\left(\overrightarrow{x}, s\right) \sum_{\overrightarrow{k}=-\infty}^{\infty} Z(nx-k) = \tag{57}$$

$$\left(\sum_{\overrightarrow{k}=-\infty}^{\infty} g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right)\right) Z(nx-k).$$

However it holds

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| Z(nx-k) \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{58}$$

Hence

$$\left| \left(B_n^{(\mathcal{R})}(g)\right)\left(\overrightarrow{x}, s\right) - g\left(\overrightarrow{x}, s\right) \right| \leq$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| Z(nx-k) = \tag{59}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| Z(nx-k) +$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| Z(nx-k).$$

Furthermore it holds

$$\left(\int_X \left| \left(B_n^{(\mathcal{R})}(g)\right)\left(\overrightarrow{x}, s\right) - g\left(\overrightarrow{x}, s\right) \right| P(ds) \right) \leq$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left(\int_X \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| P(ds) \right) Z(nx-k) + \tag{60}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left(\int_X \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| P(ds) \right) Z(nx-k) \leq$$

$$\Omega_1\left(g,\frac{1}{n^\beta}\right)_{L^1}+2\left\|g\right\|_{\infty,\mathbb{R}^N,X}\sum_{\substack{\overrightarrow{k}=-\infty\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty>\frac{1}{n^\beta}}}^{\infty}Z\left(nx-k\right)\leq$$

$$\Omega_1\left(g,\frac{1}{n^\beta}\right)_{L^1}+2c\left(\beta,n\right)\left\|g\right\|_{\infty,\mathbb{R}^N,X},$$

proving the claim. ■

Conclusion 30 *By Theorem 29 we obtain pointwise and uniform convergences with rates in the* 1*-mean of random operators* $B_n^{(\mathcal{R})}$ *to the unit operator for* $g\in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$.

III) 1-mean Approximation by Stochastic general sigmoid activation function based multivariate Kantorovich type neural network operator

Let $g\in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0<\beta<1$, $\overrightarrow{x}\in\mathbb{R}^N$, $n,N\in\mathbb{N}$, with $\left\|g\right\|_{\infty,\mathbb{R}^N,X}<\infty$, $(X,\mathcal{B},P)$ probability space, $s\in X$.

We define

$$C_n^{(\mathcal{R})}\left(g\right)\left(\overrightarrow{x},s\right):=\sum_{\overrightarrow{k}=-\infty}^{\infty}\left(n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}g\left(\overrightarrow{t},s\right)d\overrightarrow{t}\right)Z\left(n\overrightarrow{x}-\overrightarrow{k}\right),\qquad(61)$$

(see also (28).

We present

Theorem 31 *Let* $g\in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0<\beta<1$, $\overrightarrow{x}\in\mathbb{R}^N$, $n,N\in\mathbb{N}$, *with* $n^{1-\beta}>2$, $\left\|g\right\|_{\infty,\mathbb{R}^N,X}<\infty$. *Then*

1)

$$\int_X\left|\left(C_n^{(\mathcal{R})}\left(g\right)\right)\left(\overrightarrow{x},s\right)-g\left(\overrightarrow{x},s\right)\right|P\left(ds\right)\leq$$

$$\left[\Omega_1\left(g,\frac{1}{n}+\frac{1}{n^\beta}\right)_{L^1}+2c\left(\beta,n\right)\left\|g\right\|_{\infty,\mathbb{R}^N,X}\right]=:\gamma_1^{(\mathcal{R})},\qquad(62)$$

2)

$$\left\|\int_X\left|\left(C_n^{(\mathcal{R})}\left(g\right)\right)\left(\overrightarrow{x},s\right)-g\left(\overrightarrow{x},s\right)\right|P\left(ds\right)\right\|_{\infty,\mathbb{R}^N}\leq\gamma_1^{(\mathcal{R})}.\qquad(63)$$

Proof. Since $\left\|g\right\|_{\infty,\mathbb{R}^N,X}<\infty$, then

$$\left|n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}g\left(\overrightarrow{t},s\right)d\overrightarrow{t}-g\left(\overrightarrow{x},s\right)\right|=\left|n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}\left(g\left(\overrightarrow{t},s\right)-g\left(\overrightarrow{x},s\right)\right)d\overrightarrow{t}\right|\leq$$

$$n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left| g\left(\vec{t}, s\right) - g\left(\vec{x}, s\right) \right| d\vec{t} \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{64}$$

Hence

$$\int_X \left| n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} g\left(\vec{t}, s\right) d\vec{t} - g\left(\vec{x}, s\right) \right| P(ds) \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{65}$$

We observe that

$$\left(C_n^{(\mathcal{R})}(g)\right)\left(\vec{x}, s\right) - g\left(\vec{x}, s\right) =$$

$$\sum_{\vec{k}=-\infty}^{\infty} \left(n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} g\left(\vec{t}, s\right) d\vec{t} \right) Z\left(n\vec{x} - \vec{k}\right) - g\left(\vec{x}, s\right) =$$

$$\sum_{\vec{k}=-\infty}^{\infty} \left(n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} g\left(\vec{t}, s\right) d\vec{t} \right) Z\left(n\vec{x} - \vec{k}\right) - g\left(\vec{x}, s\right) \sum_{\vec{k}=-\infty}^{\infty} Z\left(n\vec{x} - \vec{k}\right) = \tag{66}$$

$$\sum_{\vec{k}=-\infty}^{\infty} \left[\left(n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} g\left(\vec{t}, s\right) d\vec{t} \right) - g\left(\vec{x}, s\right) \right] Z\left(n\vec{x} - \vec{k}\right) =$$

$$\sum_{\vec{k}=-\infty}^{\infty} \left[n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left(g\left(\vec{t}, s\right) - g\left(\vec{x}, s\right) \right) d\vec{t} \right] Z\left(n\vec{x} - \vec{k}\right).$$

However it holds

$$\sum_{\vec{k}=-\infty}^{\infty} \left[n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left| g\left(\vec{t}, s\right) - g\left(\vec{x}, s\right) \right| d\vec{t} \right] Z\left(n\vec{x} - \vec{k}\right) \leq 2 \|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{67}$$

Hence

$$\left| \left(C_n^{(\mathcal{R})}(g)\right)\left(\vec{x}, s\right) - g\left(\vec{x}, s\right) \right| \leq$$

$$\sum_{\vec{k}=-\infty}^{\infty} \left[n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left| g\left(\vec{t}, s\right) - g\left(\vec{x}, s\right) \right| d\vec{t} \right] Z\left(n\vec{x} - \vec{k}\right) = \tag{68}$$

$$\sum_{\substack{\vec{k}=-\infty \\ \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left[n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left| g\left(\vec{t}, s\right) - g\left(\vec{x}, s\right) \right| d\vec{t} \right] Z\left(n\vec{x} - \vec{k}\right) + \tag{69}$$

$$\sum_{\substack{\vec{k}=-\infty \\ \left\| \frac{\vec{k}}{n} - \vec{x} \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left[n^N \int_{\frac{\vec{k}}{n}}^{\frac{\vec{k}+1}{n}} \left| g\left(\vec{t}, s\right) - g\left(\vec{x}, s\right) \right| d\vec{t} \right] Z\left(n\vec{x} - \vec{k}\right) =$$

$$\sum_{\substack{\vec{k}=-\infty \\ \left\|\frac{\vec{k}}{n}-\vec{x}\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left| g\left(\vec{t}+\frac{\vec{k}}{n},s\right) - g\left(\vec{x},s\right)\right| d\vec{t}\right] Z\left(n\vec{x}-\vec{k}\right) + \tag{70}$$

$$\sum_{\substack{\vec{k}=-\infty \\ \left\|\frac{\vec{k}}{n}-\vec{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left| g\left(\vec{t}+\frac{\vec{k}}{n},s\right) - g\left(\vec{x},s\right)\right| d\vec{t}\right] Z\left(n\vec{x}-\vec{k}\right).$$

Furthermore it holds

$$\left(\int_X \left|\left(C_n^{(\mathcal{R})}(g)\right)(\vec{x},s) - g(\vec{x},s)\right| P(ds)\right) \underset{\text{(by Fubini's theorem)}}{\leq}$$

$$\sum_{\substack{\vec{k}=-\infty \\ \left\|\frac{\vec{k}}{n}-\vec{x}\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left(\int_X \left| g\left(\vec{t}+\frac{\vec{k}}{n},s\right) - g\left(\vec{x},s\right)\right| P(ds)\right) d\vec{t}\right] Z\left(n\vec{x}-\vec{k}\right) +$$

$$\sum_{\substack{\vec{k}=-\infty \\ \left\|\frac{\vec{k}}{n}-\vec{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left(\int_X \left| g\left(\vec{t}+\frac{\vec{k}}{n},s\right) - g\left(\vec{x},s\right)\right| P(ds)\right) d\vec{t}\right] Z\left(n\vec{x}-\vec{k}\right) \leq \tag{71}$$

$$\Omega_1\left(g,\frac{1}{n}+\frac{1}{n^{\beta}}\right)_{L^1} + 2\|g\|_{\infty,\mathbb{R}^N,X} \sum_{\substack{\vec{k}=-\infty \\ \left\|\frac{\vec{k}}{n}-\vec{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty} Z\left(n\vec{x}-\vec{k}\right) \leq$$

$$\Omega_1\left(g,\frac{1}{n}+\frac{1}{n^{\beta}}\right)_{L^1} + 2c(\beta,n)\|g\|_{\infty,\mathbb{R}^N,X}, \tag{72}$$

proving the claim. ■

Conclusion 32 *By Theorem 31 we obtain pointwise and uniform convergences with rates in the* 1*-mean of random operators* $C_n^{(\mathcal{R})}$ *to the unit operator for* $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$.

IV) 1-mean Approximation by Stochastic general sigmoid activation function based multivariate quadrature type neural network operator

Let $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0<\beta<1$, $\vec{x}\in\mathbb{R}^N$, $n, N \in \mathbb{N}$, with $\|g\|_{\infty,\mathbb{R}^N,X}<\infty$, $(X,\mathcal{B},P)$ probability space, $s \in X$.

We define

$$D_n^{(\mathcal{R})}(g)(\vec{x},s) := \sum_{\vec{k}=-\infty}^{\infty} \left(\delta_{n\vec{k}}(g)\right)(s)\, Z\left(n\vec{x}-\vec{k}\right), \tag{73}$$

where

$$\left(\delta_{n\overrightarrow{k}}(g)\right)(s) := \sum_{\overrightarrow{r}=0}^{\overrightarrow{\theta}} w_{\overrightarrow{r}}\, g\left(\frac{\overrightarrow{k}}{n} + \frac{\overrightarrow{r}}{n\overrightarrow{\theta}}, s\right), \tag{74}$$

(see also (29), (30)).

We finally give

Theorem 33 *Let $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $n^{1-\beta} > 2$, $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$. Then*

1)

$$\int_X \left|\left(D_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s)\right| P(ds) \leq$$

$$\left\{\Omega_1\left(g, \frac{1}{n} + \frac{1}{n^\beta}\right)_{L^1} + 2c(\beta, n)\|g\|_{\infty,\mathbb{R}^N,X}\right\} =: \gamma_1^{(\mathcal{R})}, \tag{75}$$

2)

$$\left\|\int_X \left|\left(D_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s)\right| P(ds)\right\|_{\infty,\mathbb{R}^N} \leq \gamma_1^{(\mathcal{R})}. \tag{76}$$

Proof. Notice that

$$\left|\left(\delta_{n\overrightarrow{k}}(g)\right)(s) - g(\overrightarrow{x}, s)\right| =$$

$$\left|\sum_{\overrightarrow{r}=0}^{\overrightarrow{\theta}} w_{\overrightarrow{r}}\left(g\left(\frac{\overrightarrow{k}}{n} + \frac{\overrightarrow{r}}{n\overrightarrow{\theta}}, s\right) - g(\overrightarrow{x}, s)\right)\right| \leq$$

$$\sum_{\overrightarrow{r}=0}^{\overrightarrow{\theta}} w_{\overrightarrow{r}}\left|g\left(\frac{\overrightarrow{k}}{n} + \frac{\overrightarrow{r}}{n\overrightarrow{\theta}}, s\right) - g(\overrightarrow{x}, s)\right| \leq 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{77}$$

Hence

$$\int_X \left|\left(\delta_{n\overrightarrow{k}}(g)\right)(s) - g(\overrightarrow{x}, s)\right| P(ds) \leq 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{78}$$

We observe that

$$\left(D_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) =$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left(\delta_{n\overrightarrow{k}}(g)\right)(s)\, Z\left(n\overrightarrow{x} - \overrightarrow{k}\right) - g(\overrightarrow{x}, s) =$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left(\left(\delta_{n\overrightarrow{k}}(g)\right)(s) - g(\overrightarrow{x}, s)\right) Z\left(n\overrightarrow{x} - \overrightarrow{k}\right). \tag{79}$$

Thus

$$\left|D_n^{(\mathcal{R})}(g)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s)\right| \leq$$

$$\sum_{\vec{k}=-\infty}^{\infty} \left|\left(\delta_{n\vec{k}}(g)\right)(s) - g(\vec{x},s)\right| Z\left(n\vec{x}-\vec{k}\right) \leq 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{80}$$

Hence it holds

$$\left|\left(D_n^{(\mathcal{R})}(g)\right)(\vec{x},s) - g(\vec{x},s)\right| \leq$$

$$\sum_{\vec{k}=-\infty}^{\infty} \left|\left(\delta_{n\vec{k}}(g)\right)(s) - g(\vec{x},s)\right| Z\left(n\vec{x}-\vec{k}\right) =$$

$$\sum_{\substack{\vec{k}=-\infty \\ \left\|\frac{\vec{k}}{n}-\vec{x}\right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty} \left|\left(\delta_{n\vec{k}}(g)\right)(s) - g(\vec{x},s)\right| Z\left(n\vec{x}-\vec{k}\right) +$$

$$\sum_{\substack{\vec{k}=-\infty \\ \left\|\frac{\vec{k}}{n}-\vec{x}\right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} \left|\left(\delta_{n\vec{k}}(g)\right)(s) - g(\vec{x},s)\right| Z\left(n\vec{x}-\vec{k}\right). \tag{81}$$

Furthermore we derive

$$\left(\int_X \left|\left(D_n^{(\mathcal{R})}(g)\right)(\vec{x},s) - g(\vec{x},s)\right| P(ds)\right) \leq$$

$$\sum_{\substack{\vec{k}=-\infty \\ \left\|\frac{\vec{k}}{n}-\vec{x}\right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty} \sum_{\vec{r}=0}^{\vec{\theta}} w_{\vec{r}} \left(\int_X \left|g\left(\frac{\vec{k}}{n}+\frac{\vec{r}}{n\vec{\theta}},s\right) - g(\vec{x},s)\right| P(ds)\right) Z\left(n\vec{x}-\vec{k}\right) \tag{82}$$

$$+\left(\sum_{\substack{\vec{k}=-\infty \\ \left\|\frac{\vec{k}}{n}-\vec{x}\right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} Z\left(n\vec{x}-\vec{k}\right)\right) 2\|g\|_{\infty,\mathbb{R}^N,X} \leq$$

$$\Omega_1\left(g, \frac{1}{n}+\frac{1}{n^{\beta}}\right)_{L^1} + 2c(\beta,n)\|g\|_{\infty,\mathbb{R}^N,X}, \tag{83}$$

proving the claim. ■

Conclusion 34 *From Theorem 33 we obtain pointwise and uniform convergences with rates in the* 1*-mean of random operators* $D_n^{(\mathcal{R})}$ *to the unit operator for* $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$.

References

[1] G.A. Anastassiou, *Rate of convergence of Fuzzy neural network operators, univariate case*, The Journal of Fuzzy Mathematics, 10, No. 3 (2002), 755-780.

[2] G.A. Anastassiou, *Higher order Fuzzy Approximation by Fuzzy Wavelet type and Neural Network Operators*, Computers and Mathematics, 48(2004), 1387-1401.

[3] G.A. Anastassiou, *Univariate fuzzy-random neural network approximation operators*, Computers and Mathematics with Applications, Special issue/ Proceedings edited by G. Anastassiou of special session "Computational Methods in Analysis", AMS meeting in Orlando. Florida, November 2002, Vol. 48 (2004), 1263-1283.

[4] G.A. Anastassiou, *Higher order Fuzzy Korovkin Theory via inequalities*, Communications in Applied Analysis, 10(2006), No. 2, 359-392.

[5] G.A. Anastassiou, *Fuzzy Korovkin Theorems and Inequalities*, Journal of Fuzzy Mathematics, 15(2007), No. 1, 169-205.

[6] G.A. Anastassiou, *Multivariate Stochastic Korovkin Theory given quantitatively*, Mathematics and Computer Modeling, 48 (2008), 558-580.

[7] G.A. Anastassiou, *Fuzzy Mathematics: Approximation Theory*, Springer, Heidelberg, New York, 2010.

[8] G.A. Anastassiou, *Inteligent Systems: Approximation by Artificial Neural Networks*, Springer, Heidelberg, 2011.

[9] G.A. Anastassiou, *Univariate hyperbolic tangent neural network approximation*, Mathematics and Computer Modelling, 53(2011), 1111-1132.

[10] G.A. Anastassiou, *Multivariate hyperbolic tangent neural network approximation*, Computers and Mathematics 61(2011), 809-821.

[11] G.A. Anastassiou, *Multivariate sigmoidal neural network approximation*, Neural Networks 24(2011), 378-386.

[12] G.A. Anastassiou, *Higher order multivariate fuzzy approximation by multivariate fuzzy wavelet type and neural network operators*, J. of Fuzzy Mathematics, 19(2011), no. 3, 601-618.

[13] G.A. Anastassiou, *Univariate sigmoidal neural network approximation*, J. of Computational Analysis and Applications, Vol. 14(4), 2012, 659-690.

[14] G.A. Anastassiou, *Rate of convergence of some multivariate neural network operators to the unit,revisited*, J. of Computational Analysis and Application, Vol. 15, No. 7, 2013, 1300-1309.

[15] G.A. Anastassiou, *Higher order multivariate fuzzy approximation by basic neural network operators*, CUBO, 16(3) 2014, 21-35.

[16] G.A. Anastassiou, *Multivariate Fuzzy-Random Quasi-interpolation neural network approximation operators*, J. Fuzzy Mathematics, Vol. 22, No. 1, 2014, 167-184.

[17] G. Anastassiou, *Multivariate Fuzzy-Random error function based Neural Network Approximation*, J. Fuzzy Mathematics, 23(4) (2015), 917-935.

[18] G.A. Anastassiou, *Intelligent Systems II: Complete Approximation by Neural Network Operators*, Springer, Heidelberg, New York, 2016.

[19] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.

[20] G.A. Anastassiou, *Banach Space Valued Neural Network*, Springer, Heidelberg, New York, 2023.

[21] G.A. Anastassiou, *General sigmoid based Banach space valued neural network approximation*, J. Computational Analysis and Applications, 31 (4) (2023), 520-534.

[22] Z. Chen and F. Cao, *The approximation operators with sigmoidal functions*, Computers and Mathematics with Applications, 58 (2009), 758-765.

[23] R.M. Dudley, *Real Analysis and Probability*, Wadsworth & Brooks / Cole Mathematics Series, Pacific Grove, California, 1989.

[24] S. Gal, *Approximation Theory in Fuzzy Setting*, Chapter 13 in Handbook of Analytic-Computational Methods in Applied Mathematics, pp. 617-666, edited by G. Anastassiou, Chapman & Hall/CRC, 2000, Boca Raton, New York.

[25] S. Haykin, *Neural Networks: A Comprehensive Foundation* (2 ed.), Prentice Hall, New York, 1998.

[26] T.M. Mitchell, *Machine Learning*, WCB-McGraw-Hill, New York, 1997.

[27] W. McCulloch and W. Pitts, *A logical calculus of the ideas immanent in nervous activity*, Bulletin of Mathematical Biophysics, 7 (1943), 115-133.

[28] Wu Congxin, Gong Zengtai, *On Henstock integral of interval-valued functions and fuzzy valued functions*, Fuzzy Sets and Systems, Vol. 115, No. 3, 2000, 377-391.

[29] C. Wu, Z. Gong, *On Henstock integral of fuzzy-number-valued functions (I)*, Fuzzy Sets and Systems, 120, No. 3, (2001), 523-532.

[30] C. Wu, M. Ma, *On embedding problem of fuzzy number space: Part 1*, Fuzzy Sets and Systems, 44 (1991), 33-38.