SOME INEQUALITIES FOR THE SPECTRAL RADIUS IN TERMS OF THE EXTENDED GENERALIZED ALUTHGE TRANSFORM OF BOUNDED OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a *contraction* $V \in \mathcal{B}(H)$, i.e. $0 \leq V^*V \leq I$, an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$ we define the operator

$$\Delta_{t,V}(T) := |T|^t V |T|^{1-t}$$

that we *call the extended generalized Aluthge transform.* In this paper we show among others that, for all $t \in [0,1]$

$$r(T_V) \leq \frac{1}{2}\max\left\{\omega(T_V), \omega\left(\Delta_{t,V}(T)\right)\right\} + \frac{1}{2}\|V\|^{1/2}\|T\|^{1/2}\min\left\{\|T_V\|^{1/2}, \left\|\Delta_{t,V}(T)\right\|^{1/2}\right\},$$

$$r(T_V) \leq \frac{1}{2}\max\left\{\omega\left(\widehat{T}_V\right), \omega\left(\Delta_{t,V}(T)\right)\right\} + \frac{1}{2}\|V\|^{1/2}\|T\|^{1/2}\min\left\{\left\|\widehat{T}_V\right\|^{1/2}, \left\|\Delta_{t,V}(T)\right\|^{1/2}\right\}$$

and

$$r(T_V) \leq \frac{1}{2}\max\left\{\omega\left(\widehat{T}_V\right), \omega\left(\widetilde{T}_V\right)\right\} + \frac{1}{2}\|V\|^{1/2}\|T\|^{1/2}\min\left\{\left\|\widehat{T}_V\right\|^{1/2}, \left\|\widetilde{T}_V\right\|^{1/2}\right\},$$

where $r(\cdot)$ is the spectral radius, $T_V := V|T|$, $\widetilde{T}_V := |T|^{1/2} V |T|^{1/2}$ and $\widehat{T}_V := |T| V$. The cases of usual generalized Aluthge, Dougal and Aluthge transforms are also presented.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda|\,\omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T + V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

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This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \le \|T\| \le 2\omega(T) \tag{1.2}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [10], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.2):

$$\omega(T) \le \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.3}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [11] improved the inequality (1.2) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \le \omega^2(T) \le \frac{1}{2}\|T^*T + TT^*\| \tag{1.4}$$

for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [9]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \le \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{1.5}$$

and

$$\omega^{2r}(T) \le \left\|\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|, \tag{1.6}$$

where $\alpha \in (0,1)$ and $r \ge 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (1.5) that

$$\omega(T) \le \frac{1}{2}\||T| + |T^*|\| \tag{1.7}$$

and from (1.6) that

$$\omega^2(T) \le \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{1.8}$$

For more related results, see the recent books on inequalities for numerical radii [7] and [4].

We denote by $r(T)$ the spectral radius of the operator T. It is well known that $r(T) \le \omega(T)$ for any T and

$$r(AB) = r(BA) \text{ for every } A,\ B \in B(H). \tag{1.9}$$

It is well known that if $AB = BA$, then also

$$r(A+B) \le r(A) + r(B) \text{ and } r(AB) \le r(A)\,r(B).$$

Let $T = U|T|$ be the *polar decomposition* of the bounded linear operator T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T} := |T|^{1/2} U |T|^{1/2}$, see [1].

The following properties of $\widetilde{T}$ are as follows:

(i) $\left\|\widetilde{T}\right\| \le \|T\|$,

(ii) $\omega\left(\widetilde{T}\right) \le \omega(T)$,

(iii) $r\left(\widetilde{T}\right) = r(T)$,

(iv) $\omega\left(\widetilde{T}\right) \le \left\|T^2\right\|^{1/2} (\le \|T\|)$, [13].

Utilizing this transform T. Yamazaki, [13] obtained in 2007 the following refinement of Kittaneh's inequality (1.3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \omega\left(\widetilde{T}\right)\right) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right) \tag{1.10}$$

for any operator $T \in B(H)$.

We remark that if $\widetilde{T} = 0$, then obviously $\omega(T) = \frac{1}{2}\|T\|$.

For a *contraction* $V \in \mathcal{B}(H)$, i.e. $0 \leq V^*V \leq I$ and an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$, in [8] we introduced the two variables transform,

$$\Delta_{t,V}(T) := |T|^t V |T|^{1-t}$$

that we call *the extended generalized Aluthge transform.*

We assume in what follows that $|T|^0 := I$.

For $t = 1$ we have

$$\widehat{T}_V := \Delta_{1,V}(T) = |T| V,$$

that we call *the extended Dougal transform*, for $t = 1/2$,

$$\widetilde{T}_V = \Delta_{1/2,V}(T) := |T|^{1/2} V |T|^{1/2},$$

that we call *the extended Aluthge transform* and for $t = 0$,

$$T_V := \Delta_{0,V}(T) = V|T|.$$

An operator $U \in \mathcal{B}(H)$ is called a *partial isometry* if $\|Ux\| = \|x\|$ for all $x \in \mathcal{N}^{\perp}(U)$.

Now, let $x \in H$, then there exists a unique $x_1 \in \mathcal{N}(U)$ and a unique $x_2 \in \mathcal{N}^{\perp}(U)$ such that $x = x_1 + x_2$. Then

$$0 \leq \langle U^*Ux, x\rangle = \|Ux\|^2 = \|Ux_1 + Ux_2\|^2 = \|Ux_2\|^2 = \|x_2\|^2.$$

By the fact that $x_1 \perp x_2$, $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$. Therefore $0 \leq \langle U^*Ux, x\rangle \leq \|x\|^2$, which shows that U is a contraction on H.

Let $T \in \mathcal{B}(H)$ and $T = U|T|$ the polar decomposition of T with U a partial isometry. Then $T_U = U|T| = T$,

$$\widetilde{T}_U = |T|^{1/2} U |T|^{1/2} = \widetilde{T}$$

is the usual *Aluthge transform* and

$$\widehat{T}_U = |T| U = \widehat{T}$$

is the usual *Dougal transform.*

For $t \in (0,1)$

$$\Delta_{t,U}(T) = |T|^t U |T|^{1-t} =: \Delta_t(T)$$

is the *generalized Aluthge transform* introduced in by Cho and Tanahashi in [6].

Abu-Omar and Kittaneh [2] improved on inequality (1.10) using generalized Aluthge transform to prove that

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \min_{t\in[0,1]} \omega(\Delta_t(T))\right). \tag{1.11}$$

For $t = 1$ this also gives the following result for the *Dougal transform*

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \omega\left(\widehat{T}\right)\right). \tag{1.12}$$

In [3] Bunia et al. also proved that

$$\omega(T) \leq \min_{t\in[0,1]} \left\{ \frac{1}{2}\omega(\Delta_t(T)) + \frac{1}{4}\left(\|T\|^{2t} + \|T\|^{2(1-t)} \right) \right\}, \tag{1.13}$$

which for $t = 1/2$ gives (1.10) as well.

If V is a contraction, then $\|V\| \leq 1$ and since $\|V^*\| = \|V\|$, hence V^* is also a contraction. Observe that

$$\Delta_{t,V}^*(T) := \left(|T|^t V |T|^{1-t} \right)^* = |T|^{1-t} V^* |T|^t = \Delta_{1-t,V^*}(T)$$

for all $t \in [0,1]$. Therefore

$$(T_V)^* = \widehat{T}_{V^*}, \quad \left(\widehat{T}_V\right)^* = T_{V^*} \text{ and } \left(\widetilde{T}_V\right)^* = \widetilde{T}_{V^*}.$$

Since $\|V^*V\| = \|VV^*\| = \|V\|^2$ and V is a contraction, then $\left\| \frac{V^*V \pm VV^*}{2} \right\| \leq \|V\|^2 \leq 1$ showing that $W := \frac{V^*V \pm VV^*}{2}$ is a contraction and we can consider the transform

$$\Delta_{t,\frac{V^*V\pm VV^*}{2}}(T) := |T|^t \left(\frac{V^*V \pm VV^*}{2} \right) |T|^{1-t}$$

for $t \in [0,1]$.

For a contraction V, we have

$$\operatorname{Im}(V) := \frac{V - V^*}{2i}, \quad \operatorname{Re}(V) := \operatorname{Re}\left(\frac{V + V^*}{2} \right)$$

and since

$$\|\operatorname{Im}(V)\| = \left\| \frac{V - V^*}{2i} \right\| \leq \|V\| \leq 1 \text{ and } \|\operatorname{Re}(V)\| \leq \|V\| \leq 1,$$

hence $\operatorname{Im}(V)$ and $\operatorname{Re}(V)$ are contractions as well. We can then consider the transforms

$$\Delta_{t,\operatorname{Im}(V)}(T) := |T|^t \operatorname{Im}(V) |T|^{1-t} \text{ and } \Delta_{t,\operatorname{Re}(V)}(T) := |T|^t \operatorname{Re}(V) |T|^{1-t}$$

for $t \in [0,1]$.

For $T \in \mathcal{B}(H)$ we define $T_+ := \frac{1}{2}(|T| + T)$ and $T_- := \frac{1}{2}(|T| - T)$. If U is the partial isometry in the polar representation of T, then $V := \frac{I\pm U}{2}$ is a contraction and

$$\Delta_{t,\frac{I\pm U}{2}}(T) := |T|^t \frac{I \pm U}{2} |T|^{1-t} = \frac{|T| \pm \Delta_t(T)}{2}.$$

In particular, we get

$$T_{\frac{I\pm U}{2}} = \frac{|T| \pm T}{2} = T_\pm, \widehat{T}_{\frac{I\pm U}{2}} = \frac{|T| \pm \widehat{T}}{2} \text{ and } \widetilde{T}_{\frac{I\pm U}{2}} = \frac{|T| \pm \widetilde{T}}{2}$$

for any operator $T \in \mathcal{B}(H)$.

2. Main Results

We use the following result:

Lemma 1. *Let* A, $B \in \mathcal{B}(H)$, *then*

$$r(AB) \leq \frac{1}{4}(\omega(AB)+\omega(BA)) + \frac{1}{4}\left[(\omega(AB)-\omega(BA))^2 + 4\min\{\|A\|\|BAB\|, \|B\|\|ABA\|\}\right]^{1/2} \leq \frac{1}{2}\max\{\omega(AB), \omega(BA)\} + \frac{1}{2}\min\left\{\|A\|^{1/2}\|BAB\|^{1/2}, \|B\|^{1/2}\|ABA\|^{1/2}\right\}. \tag{2.1}$$

Proof. The first part was obtained by Abu-Omar and Kittaneh in [2].

Using the elementary inequality $(c+d)^{1/2} \leq c^{1/2} + d^{1/2}$ for $c, d \geq 0$, we have

$$\left[(\omega(AB)-\omega(BA))^2 + 4\min\{\|A\|\|BAB\|, \|B\|\|ABA\|\}\right]^{1/2} \leq |\omega(AB)-\omega(BA)| + 2\min\left\{\|A\|^{1/2}\|BAB\|^{1/2}, \|B\|^{1/2}\|ABA\|^{1/2}\right\}.$$

Since

$$\max\{a, b\} = \frac{1}{2}(a+b+|a-b|),$$

hence

$$\frac{1}{4}(\omega(AB)+\omega(BA)) + \frac{1}{4}|\omega(AB)-\omega(BA)| + \frac{1}{2}\min\left\{\|A\|^{1/2}\|BAB\|^{1/2}, \|B\|^{1/2}\|ABA\|^{1/2}\right\} = \frac{1}{2}\max\{\omega(AB), \omega(BA)\} + \frac{1}{2}\min\left\{\|A\|^{1/2}\|BAB\|^{1/2}, \|B\|^{1/2}\|ABA\|^{1/2}\right\}$$

and the last part is also proved. □

Now, we observe that by the property (1.9) we have the chain of equalities

$$r(T_V) = r\left(\widehat{T}_V\right) = r\left(\widetilde{T}_V\right) = r(\Delta_{t,V}(T))$$

for all $t \in [0,1]$.

We can state the following result:

Theorem 1. *For a contraction* $V \in \mathcal{B}(H)$, *an operator* $T \in \mathcal{B}(H)$ *we have that*

$$r(T_V) \leq \frac{1}{4}(\omega(T_V)+\omega(\Delta_{t,V}(T))) + \frac{1}{4}\left[(\omega(T_V)-\omega(\Delta_{t,V}(T)))^2 + 4\|V\|\|T\|\min\{\|T_V\|, \|\Delta_{t,V}(T)\|\}\right]^{1/2} \leq \frac{1}{2}\max\{\omega(T_V), \omega(\Delta_{t,V}(T))\} + \frac{1}{2}\|V\|^{1/2}\|T\|^{1/2}\min\left\{\|T_V\|^{1/2}, \|\Delta_{t,V}(T)\|^{1/2}\right\} \tag{2.2}$$

for all $t \in [0,1]$.

Also,

$$r(T_V) \le \frac{1}{4}\left(\omega\left(\widehat{T}_V\right) + \omega\left(\Delta_{t,V}(T)\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widehat{T}_V\right) - \omega\left(\Delta_{t,V}(T)\right)\right)^2 + 4\|V\|\|T\|\min\left\{\left\|\widehat{T}_V\right\|, \|\Delta_{t,V}(T)\|\right\}\right]^{1/2} \le \frac{1}{2}\max\left\{\omega\left(\widehat{T}_V\right), \omega\left(\Delta_{t,V}(T)\right)\right\} + \frac{1}{2}\|V\|^{1/2}\|T\|^{1/2}\min\left\{\left\|\widehat{T}_V\right\|^{1/2}, \|\Delta_{t,V}(T)\|^{1/2}\right\} \tag{2.3}$$

for all $t \in [0,1]$.

Proof. Now, since

$$\min\{\|A\|\|BAB\|, \|B\|\|ABA\|\} \le \min\{\|A\|\|B\|\|AB\|, \|B\|\|A\|\|BA\|\} = \|A\|\|B\|\min\{\|AB\|, \|BA\|\},$$

then from (2.1) we have

$$r(AB) \le \frac{1}{4}\left(\omega(AB) + \omega(BA)\right) + \frac{1}{4}\left[\left(\omega(AB) - \omega(BA)\right)^2 + 4\|A\|\|B\|\min\{\|AB\|, \|BA\|\}\right]^{1/2} \le \frac{1}{2}\max\{\omega(AB), \omega(BA)\} + \frac{1}{2}\|A\|^{1/2}\|B\|^{1/2}\min\left\{\|AB\|^{1/2}, \|BA\|^{1/2}\right\}. \tag{2.4}$$

By taking $A = V|T|^{1-t}$ and $B = |T|^t$, we observe that

$$\begin{aligned} &\|A\|\|B\|\min\{\|AB\|, \|BA\|\} \\ &= \left\|V|T|^{1-t}\right\|\left\||T|^t\right\|\min\left\{\left\|V|T|^{1-t}|T|^t\right\|, \left\||T|^t V|T|^{1-t}\right\|\right\} \\ &\le \|V\|\|T\|^{1-t}\|T\|^t\min\left\{\left\|V|T|^{1-t}|T|^t\right\|, \left\||T|^t V|T|^{1-t}\right\|\right\} \\ &= \|V\|\|T\|\min\left\{\|V|T|\|, \left\||T|^t V|T|^{1-t}\right\|\right\} \\ &= \|V\|\|T\|\min\{\|T_V\|, \|\Delta_{t,V}(T)\|\}. \end{aligned}$$

Since

$$r(AB) = r(T_V), \ \omega(AB) = \omega(T_V) \text{ and } \omega(BA) = \omega(\Delta_{t,V}(T)),$$

hence by (2.4) we derive (2.2).

If we take in Lemma 1 $A = |T|^{1-t}$, $B = |T|^t V$, then

$$\begin{aligned} &r(AB) = r\left(\widehat{T}_V\right) = r(T_V), \ \omega(AB) = \omega\left(\widehat{T}_V\right), \\ &\omega(BA) = \omega(\Delta_{t,V}(T)) \end{aligned}$$

and

$$\begin{aligned}&\|A\|\,\|B\|\min\{\|AB\|,\|BA\|\}\\&=\left\||T|^{1-t}\right\|\left\||T|^{t}V\right\|\min\left\{\left\|\widehat{T}_V\right\|,\|\Delta_{t,V}(T)\|\right\}\\&\le\|V\|\,\|T\|\min\left\{\left\|\widehat{T}_V\right\|,\|\Delta_{t,V}(T)\|\right\}\end{aligned}$$

and by (2.4) we obtain (2.3). □

In particular, from (2.2) we get

$$\begin{aligned}r(T_V)&\le\frac{1}{4}\left(\omega(T_V)+\omega\left(\widetilde{T}_V\right)\right)\\&\quad+\frac{1}{4}\left[\left(\omega(T_V)-\omega\left(\widetilde{T}_V\right)\right)^2+4\|V\|\,\|T\|\min\left\{\|T_V\|,\left\|\widetilde{T}_V\right\|\right\}\right]^{1/2}\\&\le\frac{1}{2}\max\left\{\omega(T_V),\omega\left(\widetilde{T}_V\right)\right\}\\&\quad+\frac{1}{2}\|V\|^{1/2}\|T\|^{1/2}\min\left\{\|T_V\|^{1/2},\left\|\widetilde{T}_V\right\|^{1/2}\right\}\end{aligned}$$

and

$$\begin{aligned}r(T_V)&\le\frac{1}{4}\left(\omega(T_V)+\omega\left(\widehat{T}_V\right)\right)\\&\quad+\frac{1}{4}\left[\left(\omega(T_V)-\omega\left(\widehat{T}_V\right)\right)^2+4\|V\|\,\|T\|\min\left\{\|T_V\|,\left\|\widehat{T}_V\right\|\right\}\right]^{1/2}\\&\le\frac{1}{2}\max\left\{\omega(T_V),\omega\left(\widehat{T}_V\right)\right\}\\&\quad+\frac{1}{2}\|V\|^{1/2}\|T\|^{1/2}\min\left\{\|T_V\|^{1/2},\left\|\widehat{T}_V\right\|^{1/2}\right\}.\end{aligned}$$

Also, from (2.3) we obtain

$$\begin{aligned}r(T_V)&\le\frac{1}{4}\left(\omega\left(\widehat{T}_V\right)+\omega\left(\widetilde{T}_V\right)\right)\\&\quad+\frac{1}{4}\left[\left(\omega\left(\widehat{T}_V\right)-\omega\left(\widetilde{T}_V\right)\right)^2+4\|V\|\,\|T\|\min\left\{\left\|\widehat{T}_V\right\|,\left\|\widetilde{T}_V\right\|\right\}\right]^{1/2}\\&\le\frac{1}{2}\max\left\{\omega\left(\widehat{T}_V\right),\omega\left(\widetilde{T}_V\right)\right\}\\&\quad+\frac{1}{2}\|V\|^{1/2}\|T\|^{1/2}\min\left\{\left\|\widehat{T}_V\right\|^{1/2},\left\|\widetilde{T}_V\right\|^{1/2}\right\}.\end{aligned}$$

Remark 1. *Let $T\in\mathcal{B}(H)$ and $T=U|T|$ the polar decomposition of T with U a partial isometry. Then for $V=U$ we have $T_V=T$, $\widehat{T}_V=\widehat{T}$ the Dougal transform, $\widetilde{T}_V=\widetilde{T}$ the Aluthge transform and $\Delta_{t,V}(T)=\Delta_t(T)$ the generalized Aluthge transform.*

Observe that

$$\min\{\|T\|,\|\Delta_t(T)\|\}=\|\Delta_t(T)\|,\ t\in[0,1]$$

and by (2.2) we get

$$r(T) \le \frac{1}{4}\left(\omega(T) + \omega(\Delta_t(T))\right) + \frac{1}{4}\left[\left(\omega(T) - \omega(\Delta_t(T))\right)^2 + 4\|T\|\|\Delta_t(T)\|\right]^{1/2} \le \frac{1}{2}\max\{\omega(T), \omega(\Delta_t(T))\} + \frac{1}{2}\|T\|^{1/2}\|\Delta_t(T)\|^{1/2}. \tag{2.5}$$

In particular,

$$r(T) \le \frac{1}{4}\left(\omega(T) + \omega\left(\widetilde{T}\right)\right) + \frac{1}{4}\left[\left(\omega(T) - \omega\left(\widetilde{T}\right)\right)^2 + 4\|T\|\left\|\widetilde{T}\right\|\right]^{1/2} \le \frac{1}{2}\omega(T) + \frac{1}{2}\|T\|^{1/2}\left\|\widetilde{T}\right\|^{1/2}$$

and

$$r(T) \le \frac{1}{4}\left(\omega(T) + \omega\left(\widehat{T}\right)\right) + \frac{1}{4}\left[\left(\omega(T) - \omega\left(\widehat{T}\right)\right)^2 + 4\|T\|\left\|\widehat{T}\right\|\right]^{1/2} \le \frac{1}{2}\max\left\{\omega(T), \omega\left(\widehat{T}\right)\right\} + \frac{1}{2}\|T\|^{1/2}\left\|\widehat{T}\right\|^{1/2}.$$

From (2.3) we also obtain

$$r(T) \le \frac{1}{4}\left(\omega\left(\widehat{T}\right) + \omega(\Delta_t(T))\right) + \frac{1}{4}\left[\left(\omega\left(\widehat{T}\right) - \omega(\Delta_t(T))\right)^2 + 4\|T\|\min\left\{\left\|\widehat{T}\right\|, \|\Delta_t(T)\|\right\}\right]^{1/2} \le \frac{1}{2}\max\left\{\omega\left(\widehat{T}\right), \omega(\Delta_t(T))\right\} + \frac{1}{2}\min\left\{\left\|\widehat{T}\right\|^{1/2}, \|\Delta_t(T)\|^{1/2}\right\} \tag{2.6}$$

for all $t \in [0,1]$.

In particular,

$$r(T) \le \frac{1}{4}\left(\omega\left(\widehat{T}\right) + \omega\left(\widetilde{T}\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widehat{T}\right) - \omega\left(\widetilde{T}\right)\right)^2 + 4\|T\|\min\left\{\left\|\widehat{T}\right\|, \left\|\widetilde{T}\right\|\right\}\right]^{1/2} \le \frac{1}{2}\max\left\{\omega\left(\widehat{T}\right), \omega\left(\widetilde{T}\right)\right\} + \frac{1}{2}\|T\|^{1/2}\min\left\{\left\|\widehat{T}\right\|^{1/2}, \left\|\widetilde{T}\right\|^{1/2}\right\}.$$

3. Related Results

Recall the following result obtained in [2]:

Lemma 2. *Let* A, $B \in \mathcal{B}(H)$, *then*

$$r(AB \pm BA) \le \omega(AB) + \min\left\{\|A\|^{1/2}\left\|AB^2\right\|^{1/2}, \|B\|^{1/2}\left\|A^2B\right\|^{1/2}\right\} \tag{3.1}$$

and

$$r(AB \pm BA) \le \omega(BA) + \min\left\{\|A\|^{1/2}\left\|B^2A\right\|^{1/2}, \|B\|^{1/2}\left\|BA^2\right\|^{1/2}\right\}. \tag{3.2}$$

We can state the following result:

Theorem 2. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ we have that*

$$r\left(T_V \pm \Delta_{t,V}(T)\right) \le \omega(T_V) + \|V\|^{1/2} \|T\|^{1/2} \|T_V\|^{1/2} \tag{3.3}$$

and

$$r\left(T_V \pm \Delta_{t,V}(T)\right) \le \omega\left(\Delta_{t,V}(T)\right) + \|V\|^{1/2} \|T\|^{1/2} \|\Delta_{t,V}(T)\|^{1/2} \tag{3.4}$$

for all $t \in [0,1]$

Proof. Take $A = V|T|^{1-t}$ and $B = |T|^t$. We observe that

$$r(AB \pm BA) = r\left(T_V \pm \Delta_{t,V}(T)\right),\ \omega(AB) = \omega(T_V),$$

$$\begin{aligned}\|A\|^{1/2} \left\|AB^2\right\|^{1/2} &= \left\|V|T|^{1-t}\right\|^{1/2} \left\|V|T|^{1-t}|T|^{2t}\right\|^{1/2} \\ &= \left\|V|T|^{1-t}\right\|^{1/2} \left\|V|T|\,|T|^{t}\right\|^{1/2} \\ &\le \|V\|^{1/2} \|T\|^{(1-t)/2} \|V|T|\|^{1/2} \|T\|^{t/2} \\ &= \|V\|^{1/2} \|T\|^{1/2} \|T_V\|^{1/2}\end{aligned}$$

and

$$\begin{aligned}\|B\|^{1/2} \left\|A^2B\right\|^{1/2} &= \left\||T|^t\right\|^{1/2} \left\|V|T|^{1-t}V|T|^{1-t}|T|^{t}\right\|^{1/2} \\ &= \|T\|^{t/2} \left\|V|T|^{1-t}V|T|\right\|^{1/2} \\ &\le \|T\|^{t/2} \left\|V|T|^{1-t}\right\|^{1/2} \|V|T|\|^{1/2} \\ &\le \|T\|^{t/2} \|V\|^{1/2} \|T\|^{(1-t)/2} \|V|T|\|^{1/2} \\ &= \|V\|^{1/2} \|T\|^{1/2} \|T_V\|^{1/2}\end{aligned}$$

and by (3.1) we get (3.3).

Also, for the same choices of A and B, we have $\omega(BA) = \omega(\Delta_t(T))$,

$$\begin{aligned}\|A\|^{1/2} \left\|B^2A\right\|^{1/2} &= \left\|V|T|^{1-t}\right\|^{1/2} \left\||T|^t|T|^tV|T|^{1-t}\right\|^{1/2} \\ &\le \|V\|^{1/2} \|T\|^{(1-t)/2} \|T\|^{t/2} \left\||T|^tV|T|^{1-t}\right\|^{1/2} \\ &= \|V\|^{1/2} \|T\|^{1/2} \|\Delta_{t,V}(T)\|^{1/2}\end{aligned}$$

and

$$\begin{aligned}\|B\|^{1/2} \left\|BA^2\right\|^{1/2} &= \left\||T|^t\right\|^{1/2} \left\||T|^tV|T|^{1-t}V|T|^{1-t}\right\|^{1/2} \\ &\le \|T\|^{t/2} \left\||T|^tV|T|^{1-t}\right\|^{1/2} \left\|V|T|^{1-t}\right\| \\ &\le \|T\|^{t/2} \left\||T|^tV|T|^{1-t}\right\|^{1/2} \|V\|^{1/2} \|T\|^{(1-t)/2} \\ &= \|V\|^{1/2} \|T\|^{1/2} \|\Delta_{t,V}(T)\|^{1/2}\end{aligned}$$

and by (3.2) we obtain (3.4). □

From (3.3) we derive that

$$r\left(T_V \pm \widehat{T}_V\right) \leq \omega\left(T_V\right) + \|V\|^{1/2}\|T\|^{1/2}\|T_V\|^{1/2}$$

and

$$r\left(T_V \pm \widetilde{T}_V\right) \leq \omega\left(T_V\right) + \|V\|^{1/2}\|T\|^{1/2}\|T_V\|^{1/2},$$

while from (3.4) we obtain that

$$r\left(T_V \pm \widehat{T}_V\right) \leq \omega\left(\widehat{T}_V\right) + \|V\|^{1/2}\|T\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2}$$

and

$$r\left(T_V \pm \widetilde{T}_V\right) \leq \omega\left(\widetilde{T}_V\right) + \|V\|^{1/2}\|T\|^{1/2}\left\|\widetilde{T}_V\right\|^{1/2}.$$

Remark 2. *Let $T \in \mathcal{B}(H)$ and $T = U|T|$ the polar decomposition of T with U a partial isometry. Then for $V = U$ we get*

$$r\left(T \pm \Delta_t(T)\right) \leq \omega(T) + \|T\|$$

and

$$r\left(T \pm \Delta_t(T)\right) \leq \omega\left(\Delta_t(T)\right) + \|T\|^{1/2}\|\Delta_t(T)\|^{1/2}$$

for all $t \in [0,1]$.

In particular,

$$r\left(T \pm \widehat{T}\right) \leq \omega(T) + \|T\|$$

and

$$r\left(T \pm \widetilde{T}\right) \leq \omega(T) + \|T\|.$$

Also

$$r\left(T \pm \widehat{T}\right) \leq \omega\left(\widehat{T}\right) + \|T\|^{1/2}\left\|\widehat{T}\right\|^{1/2}$$

and

$$r\left(T \pm \widetilde{T}\right) \leq \omega\left(\widetilde{T}\right) + \|T\|^{1/2}\left\|\widetilde{T}\right\|^{1/2}.$$

We also have:

Theorem 3. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ we have that*

$$r\left(\Delta_{t,V}(T) \pm \widehat{T}_V\right) \leq \omega\left(\Delta_{t,V}(T)\right) + \|V\|^{1/2}\|T\|^{1/2}\|\Delta_{t,V}(T)\|^{1/2} \tag{3.5}$$

and

$$r\left(\Delta_{t,V}(T) \pm \widehat{T}_V\right) \leq \omega\left(\widehat{T}_V\right) + \|T\|^{1/2}\|V\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2} \tag{3.6}$$

for all $t \in [0,1]$.

Proof. If we take $A = |T|^t V$ and $B = |T|^{1-t}$ in (3.1) we get

$$\begin{aligned}
& r\left(\Delta_{t,V}(T) \pm \widehat{T}_V\right) \\
& \leq \omega\left(\Delta_{t,V}(T)\right) + \min\left\{\left\||T|^t V\right\|^{1/2}\left\||T|^t V|T|^{1-t}|T|^{1-t}\right\|^{1/2},\right. \\
& \left.\left\||T|^{1-t}\right\|^{1/2}\left\||T|^t V|T|^t V|T|^{1-t}\right\|^{1/2}\right\} \\
& = \omega\left(\Delta_{t,V}(T)\right) + \min\left\{\left\||T|^t V\right\|^{1/2}\left\||T|^t V|T|^{1-t}|T|^{1-t}\right\|^{1/2},\right. \\
& \left.\|T\|^{(1-t)/2}\left\||T|^t V|T|^t V|T|^{1-t}\right\|^{1/2}\right\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
& \left\||T|^t V\right\|^{1/2}\left\||T|^t V|T|^{1-t}|T|^{1-t}\right\|^{1/2} \\
& \leq \|V\|^{1/2}\|T\|^{t/2}\left\||T|^t V|T|^{1-t}\right\|^{1/2}\|T\|^{(1-t)/2} \\
& = \|V\|^{1/2}\|T\|^{1/2}\left\|\Delta_{t,V}(T)\right\|^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
& \|T\|^{(1-t)/2}\left\||T|^t V|T|^t V|T|^{1-t}\right\|^{1/2} \\
& \leq \|T\|^{(1-t)/2}\left\||T|^t V\right\|^{1/2}\left\||T|^t V|T|^{1-t}\right\|^{1/2} \\
& \leq \|T\|^{(1-t)/2}\|T\|^{t/2}\|V\|^{1/2}\left\||T|^t V|T|^{1-t}\right\|^{1/2} \\
& = \|V\|^{1/2}\|T\|^{1/2}\left\|\Delta_{t,V}(T)\right\|^{1/2}
\end{aligned}$$

and the inequality (3.5) is proved.

With the same choice for A and B in (3.2), we get

$$\begin{aligned}
& r\left(\Delta_{t,V}(T) \pm \widehat{T}_V\right) \\
& \leq \omega\left(\widehat{T}_V\right) + \min\left\{\left\||T|^t V\right\|^{1/2}\left\||T|^{1-t}|T|^{1-t}|T|^t V\right\|^{1/2},\right. \\
& \left.\left\||T|^{1-t}\right\|^{1/2}\left\||T|^{1-t}|T|^t V|T|^t V\right\|^{1/2}\right\} \\
& = \omega\left(\widehat{T}_V\right) + \min\left\{\left\||T|^t V\right\|^{1/2}\left\||T|^{1-t}|T| V\right\|^{1/2},\right. \\
& \left.\left\||T|^{1-t}\right\|^{1/2}\left\||T| V|T|^t V\right\|^{1/2}\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
\left\||T|^t V\right\|^{1/2}\left\||T|^{1-t}|T| V\right\|^{1/2} & \leq \|T\|^{t/2}\|V\|^{1/2}\|T\|^{(1-t)/2}\||T| V\|^{1/2} \\
& = \|T\|^{1/2}\|V\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2}
\end{aligned}$$

and

$$\left\| |T|^{1-t} \right\|^{1/2} \left\| |T| V |T|^t V \right\|^{1/2} \leq \|T\|^{(1-t)/2} \| |T| V \|^{1/2} \|T\|^{t/2} \|V\|^{1/2}$$
$$= \|T\|^{1/2} \|V\|^{1/2} \left\| \widehat{T}_V \right\|^{1/2},$$

and the inequality (3.6). □

From (3.5) we get

$$r\left(T_V \pm \widehat{T}_V\right) \leq \omega(T_V) + \|V\|^{1/2} \|T\|^{1/2} \|T_V\|^{1/2}$$

and

$$r\left(\widetilde{T}_V \pm \widehat{T}_V\right) \leq \omega\left(\widetilde{T}_V\right) + \|V\|^{1/2} \|T\|^{1/2} \left\|\widetilde{T}_V\right\|^{1/2}.$$

From (3.6)

$$r\left(T_V \pm \widehat{T}_V\right) \leq \omega\left(\widehat{T}_V\right) + \|T\|^{1/2} \|V\|^{1/2} \left\|\widehat{T}_V\right\|^{1/2}$$

and

$$r\left(\widetilde{T}_V \pm \widehat{T}_V\right) \leq \omega\left(\widehat{T}_V\right) + \|T\|^{1/2} \|V\|^{1/2} \left\|\widehat{T}_V\right\|^{1/2}.$$

Remark 3. *Let $T \in \mathcal{B}(H)$ and $T = U|T|$ the polar decomposition of T with U a partial isometry. Then for $V = U$ we get*

$$r\left(\Delta_t(T) \pm \widehat{T}\right) \leq \omega(\Delta_t(T)) + \|T\|^{1/2} \|\Delta_t(T)\|^{1/2}$$

and

$$r\left(\Delta_t(T) \pm \widehat{T}\right) \leq \omega\left(\widehat{T}_V\right) + \|T\|^{1/2} \left\|\widehat{T}_V\right\|^{1/2}$$

for all $t \in [0,1]$.

In particular,

$$r\left(\widetilde{T} \pm \widehat{T}\right) \leq \omega\left(\widetilde{T}\right) + \|T\|^{1/2} \left\|\widetilde{T}\right\|^{1/2}.$$

Also

$$r\left(\widetilde{T} \pm \widehat{T}\right) \leq \omega\left(\widehat{T}\right) + \|T\|^{1/2} \left\|\widehat{T}\right\|^{1/2}.$$

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