INEQUALITIES FOR THE (p,q)-EXTENDED GENERALIZED ALUTHGE TRANSFORM OF BOUNDED OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a *contraction* $V \in \mathcal{B}(H)$, i.e. $0 \leq V^*V \leq I$, and an operator $T \in \mathcal{B}(H)$ we define the (p,q)-*extended generalized Aluthge transform* by

$$\Delta_{p,q,V}(T) := |T|^p V |T|^q ,$$

where p, $q \geq 0$. In this paper we provide some upper bounds for the (p,q)-extended generalized Aluthge transform $\Delta_{p,q,V}(T)$. Several particular cases of interest are also presented.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leq w(T) \|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T + V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{1.3}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [11], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [12] improved the inequality (1.3) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{1.5}$$

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for any operator $T\in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [10]:

If for an operator $T\in B(H)$ we denote $|T|:=(T^*T)^{1/2}$, then

$$\omega^r(T)\le\frac{1}{2}\left\||T|^{2\alpha r}+|T^*|^{2(1-\alpha)r}\right\| \tag{1.6}$$

and

$$\omega^{2r}(T)\le\left\|\alpha|T|^{2r}+(1-\alpha)|T^*|^{2r}\right\|, \tag{1.7}$$

where $\alpha\in(0,1)$ and $r\ge1$.

If we take $\alpha=\frac{1}{2}$ and $r=1$ we get from (1.6) that

$$\omega(T)\le\frac{1}{2}\||T|+|T^*|\| \tag{1.8}$$

and from (1.7) that

$$\omega^2(T)\le\frac{1}{2}\left\||T|^2+|T^*|^2\right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [8] and [5].

Let $T=U|T|$ be the *polar decomposition* of the bounded linear operator T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T}:=|T|^{1/2}U|T|^{1/2}$, see [1].

The following properties of $\widetilde{T}$ are as follows:

(i) $\left\|\widetilde{T}\right\|\le\|T\|$,
(ii) $w\left(\widetilde{T}\right)\le\omega(T)$,
(iii) $r\left(\widetilde{T}\right)=\omega(T)$,
(iv) $\omega\left(\widetilde{T}\right)\le\left\|T^2\right\|^{1/2}(\le\|T\|)$, [14].

Utilizing this transform T. Yamazaki, [14] obtained in 2007 the following refinement of Kittaneh's inequality (1.4):

$$\omega(T)\le\frac{1}{2}\left(\|T\|+\omega\left(\widetilde{T}\right)\right)\le\frac{1}{2}\left(\|T\|+\left\|T^2\right\|^{1/2}\right) \tag{1.10}$$

for any operator $T\in B(H)$.

We remark that if $\widetilde{T}=0$, then obviously $w(T)=\frac{1}{2}\|T\|$.

Abu-Omar and Kittaneh [2] improved on inequality (1.10) using generalized Aluthge transform $\Delta_t(T):=|T|^tU|T|^{1-t}$ to prove that

$$\omega(T)\le\frac{1}{2}\left(\|T\|+\min_{t\in[0,1]}\omega(\Delta_t(T))\right).$$

For $t=1$ this also gives the following result for the *Dougal transform* $\widehat{T}:=|T|U$,

$$\omega(T)\le\frac{1}{2}\left(\|T\|+\omega\left(\widehat{T}\right)\right). \tag{1.11}$$

In [4] Bunia et al. also proved that

$$\omega(T)\le\min_{t\in[0,1]}\left\{\frac{1}{2}\omega(\Delta_t(T))+\frac{1}{4}\left(\|T\|^{2t}+\|T\|^{2(1-t)}\right)\right\},$$

which for $t=1/2$ gives (1.10) as well.

Motivated by the above results, in this paper we provide some upper bounds for the (p,q)-extended generalized Aluthge transform $\Delta_{p,q,V}(T)$ introduced below. Several particular cases of interest are also presented.

2. Some Preliminary Facts

We define, for V a contraction, *i.e.*, $V^*V \le I$ and $T \in \mathcal{B}(H)$, the (p,q)-*extended generalized Aluthge transform* by

$$\Delta_{p,q,V}(T) := |T|^p V |T|^q ,$$

where p, $q \ge 0$. We also assume in what follows that $|T|^0 := I$.

We denote

$$T_{q,V} := \Delta_{0,q,V}(T) := V|T|^q$$

and

$$T_V := \Delta_{0,1,V}(T) := V|T| .$$

The p-*extended generalized Dougal transform* is defined by

$$\widehat{T}_{p,V} := \Delta_{p,0,V}(T) := |T|^p V,$$

the *extended generalized Dougal transform* by

$$\widehat{T}_V := \Delta_{1,0,V}(T) := V|T| ,$$

the p-*extended generalized Aluthge transform* by

$$\widetilde{T}_{p,V} := \Delta_{p,p,V}(T) := |T|^p V |T|^p$$

and the *extended Aluthge transform* by

$$\widetilde{T}_V := \Delta_{1/2,1/2,V}(T) := |T|^{1/2} V |T|^{1/2} .$$

For $p = t$, $q = 1-t$, where $t \in [0,1]$ we have

$$\Delta_{t,V}(T) := \Delta_{t,1-t,V}(T) = |T|^t V |T|^{1-t} .$$

The transform $\Delta_{t,V}(T)$, called the *extended generalized Aluthge transform*, was introduced and studied in [9].

An operator $U \in \mathcal{B}(H)$ is called a *partial isometry* if $\|Ux\| = \|x\|$ for all $x \in \mathcal{N}^{\perp}(U)$.

Now, let $x \in H$, then there exists a unique $x_1 \in \mathcal{N}(U)$ and a unique $x_2 \in \mathcal{N}^{\perp}(U)$ such that $x = x_1 + x_2$. Then

$$0 \le \langle U^*Ux, x\rangle = \|Ux\|^2 = \|Ux_1 + Ux_2\|^2 = \|Ux_2\|^2 = \|x_2\|^2 .$$

By the fact that $x_1 \perp x_2$,

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 .$$

Therefore

$$0 \le \langle U^*Ux, x\rangle \le \|x\|^2 ,$$

which shows that U is a contraction on H.

If the operator T has the polar decomposition $T = U|T|$ with U a partial isometry, then by taking $V = U$, we define

$$\Delta_{p,q}(T) := |T|^p U |T|^q ,$$

and all the previous transform by omitting in the notations the V, which will give the concepts of the *p-generalized Dougal transform*

$$\widehat{T}_p := \Delta_{p,0,U}(T) := |T|^p U,$$

the *Dougal transform*

$$\widehat{T} := \Delta_{1,0,U}(T) := |T| U,$$

and the *p-generalized Aluthge transform*

$$\widetilde{T}_p := \Delta_{p,p,U}(T) := |T|^p U |T|^p,$$

which for $p = 1/2$ gives the usual Aluthge transform

$$\widetilde{T} := \Delta_{1/2,1/2,U}(T) := |T|^{1/2} U |T|^{1/2},$$

Also

$$T_q := \Delta_{0,q}(T) := U |T|^q,$$

which gives for $q = 1$ the usual polar decomposition $T = U |T|$.

For $p = t$, $q = 1 - t$, where $t \in [0,1]$ we have

$$\Delta_t(T) := \Delta_{t,1-t}(T) = |T|^t V |T|^{1-t}.$$

The transform $\Delta_t(T)$ was introduced and studied in [7].

If V is a contraction, then $\|V\| \le 1$ and since $\|V^*\| = \|V\|$, hence V^* is also a contraction. Observe that

$$\Delta^*_{p,q,V}(T) := (|T|^p V |T|^q)^* = |T|^q V^* |T|^p = \Delta_{q,p,V^*}(T)$$

for all $p, q \ge 0$. Therefore

$$(T_{q,V})^* = \widehat{T}_{q,V^*}, \ \left(\widehat{T}_{p,V}\right)^* = T_{p,V^*}$$

and

$$\left(\widetilde{T}_{p,V}\right)^* = \widetilde{T}_{p,V^*}.$$

Since $\|V^*V\| = \|VV^*\| = \|V\|^2$ and V is a contraction, then

$$\left\| \frac{V^*V \pm VV^*}{2} \right\| \le \|V\|^2 \le 1$$

showing that

$$W := \frac{V^*V \pm VV^*}{2}$$

is a contraction and we can consider the transform

$$\Delta_{p,q,\frac{V^*V \pm VV^*}{2}}(T) := |T|^p \left(\frac{V^*V \pm VV^*}{2} \right) |T|^q$$

for $p, q \ge 0$.

For a contraction V, we have

$$\operatorname{Im}(V) := \frac{V - V^*}{2i}, \ \operatorname{Re}(V) := \operatorname{Re}\left(\frac{V + V^*}{2} \right)$$

and since

$$\|\operatorname{Im}(V)\| = \left\| \frac{V - V^*}{2i} \right\| \le \|V\| \le 1 \text{ and } \|\operatorname{Re}(V)\| \le \|V\| \le 1,$$

hence $\operatorname{Im}(V)$ and $\operatorname{Re}(V)$ are contractions as well. We can then consider the transforms

$$\Delta_{p,q,\operatorname{Im}(V)}(T) := |T|^p \operatorname{Im}(V) |T|^q \text{ and } \Delta_{p,q,\operatorname{Re}(V)}(T) := |T|^p \operatorname{Re}(V) |T|^q$$

for $p, q \geq 0$.

For $T \in \mathcal{B}(H)$ we define

$$T_+ := \frac{1}{2}(|T| + T) \text{ and } T_- := \frac{1}{2}(|T| - T).$$

If U is the partial isometry in the polar representation of T, then

$$V := \frac{I \pm U}{2}$$

is a contraction and we can consider

$$\Delta_{p,q,\frac{I \pm U}{2}}(T) := |T|^p \frac{I \pm U}{2} |T|^q = \frac{|T|^{p+q} \pm \Delta_{p,q}(T)}{2}.$$

In particular, we get

$$T_{q,\frac{I\pm U}{2}} = \frac{|T|^q \pm T_q}{2} = T_\pm, \widehat{T}_{p,\frac{I\pm U}{2}} = \frac{|T|^p \pm \widehat{T_p}}{2}$$

and

$$\widetilde{T}_{p,\frac{I\pm U}{2}} = \frac{|T|^p \pm \widetilde{T_p}}{2}$$

for any operator $T \in \mathcal{B}(H)$.

3. Main Results

Our first main result is as follows:

Theorem 1. *Let V be a contraction and $T \in \mathcal{B}(H)$. Then for $q \geq r \geq 0$ and $p \geq s \geq 0$,*

$$\begin{aligned} \text{(3.1)} \quad & \omega(\Delta_{p,q,V}(T)) \\ & \leq \frac{1}{2}\omega(\Delta_{p+r,q-r,V}(T)) + \frac{1}{4}\left(\|\Delta_{p,q,V}(T)\| + \|\Delta_{p,q-r,V}(T)\| \|T\|^r\right) \\ & \leq \frac{1}{2}\omega(\Delta_{p+r,q-r,V}(T)) + \frac{1}{2}\|T\|^r \|\Delta_{p,q-r,V}(T)\| \end{aligned}$$

and

$$\begin{aligned} \text{(3.2)} \quad & \omega(\Delta_{p,q,V}(T)) \\ & \leq \frac{1}{2}\omega(\Delta_{s,q+p-s,V}(T)) + \frac{1}{4}\left(\|\Delta_{p,q,V}(T)\| + \|T\|^{p-s}\|\Delta_{s,q,V}(T)\|\right) \\ & \leq \frac{1}{2}\omega(\Delta_{s,q+p-s,V}(T)) + \frac{1}{2}\|T\|^{p-s}\|\Delta_{s,q,V}(T)\|. \end{aligned}$$

Proof. We use the following inequality obtained in [3] (see also [6] for a generalization):

$$\begin{aligned} \text{(3.3)} \quad \omega(AB) & \leq \frac{1}{2}\omega(BA) + \frac{1}{4}(\|AB\| + \|A\| \|B\|) \\ & \leq \frac{1}{2}\omega(BA) + \frac{1}{2}\|A\| \|B\| \end{aligned}$$

that holds for all A, $B \in \mathcal{B}(H)$.

If we take $A=|T|^{p}V|T|^{q-r}$ and $B=|T|^{r}$ then we get from (3.3) that

$$\begin{aligned}
&\omega\left(\Delta_{p,q,V}(T)\right)\\
&\leq\frac{1}{2}\omega\left(|T|^{p+r}V|T|^{q-r}\right)+\frac{1}{4}\left(\left\|\Delta_{p,q,V}(T)\right\|+\left\||T|^{p}V|T|^{q-r}\right\|\|T\|^{r}\right)\\
&=\frac{1}{2}\omega\left(\Delta_{p+r,q-r,V}(T)\right)+\frac{1}{4}\left(\left\|\Delta_{p,q,V}(T)\right\|+\left\|\Delta_{p,q-r,V}(T)\right\|\|T\|^{r}\right)\\
&\leq\frac{1}{2}\omega\left(\Delta_{p+r,q-r,V}(T)\right)+\frac{1}{2}\left\|\Delta_{p,q-r,V}(T)\right\|\|T\|^{r},
\end{aligned}$$

which proves (3.1)

Now, if we take $A=|T|^{p-s}$ and $B=|T|^{s}V|T|^{q}$ in (3.3), then we get

$$\begin{aligned}
&\omega\left(\Delta_{p,q,V}(T)\right)\\
&\leq\frac{1}{2}\omega\left(|T|^{s}V|T|^{q+p-s}\right)+\frac{1}{4}\left(\left\|\Delta_{p,q,V}(T)\right\|+\|T\|^{p-s}\left\||T|^{s}V|T|^{q}\right\|\right)\\
&=\frac{1}{2}\omega\left(\Delta_{s,q+p-s,V}(T)\right)+\frac{1}{4}\left(\left\|\Delta_{p,q,V}(T)\right\|+\|T\|^{p-s}\left\|\Delta_{s,q,V}(T)\right\|\right)\\
&\leq\frac{1}{2}\omega\left(\Delta_{s,q+p-s,V}(T)\right)+\frac{1}{2}\|T\|^{p-s}\left\|\Delta_{s,q,V}(T)\right\|,
\end{aligned}$$

which proves the inequality (3.2). □

Corollary 1. *Let V be a contraction and $T\in\mathcal{B}(H)$. Then for $p\geq s\geq 0$,*

$$\begin{aligned}
\omega\left(\widetilde{T}_{p,V}\right)&\leq\frac{1}{2}\omega\left(\Delta_{p+s,p-s,V}(T)\right)+\frac{1}{4}\left(\left\|\widetilde{T}_{p,V}\right\|+\left\|\Delta_{p,p-s,V}(T)\right\|\|T\|^{s}\right)\\
&\leq\frac{1}{2}\omega\left(\Delta_{p+s,p-s,V}(T)\right)+\frac{1}{2}\|T\|^{s}\left\|\Delta_{p,p-s,V}(T)\right\|
\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}
\omega\left(\widetilde{T}_{p,V}\right)&\leq\frac{1}{2}\omega\left(\Delta_{s,2p-s,V}(T)\right)+\frac{1}{4}\left(\left\|\widetilde{T}_{p,V}\right\|+\|T\|^{p-s}\left\|\Delta_{s,p,V}(T)\right\|\right)\\
&\leq\frac{1}{2}\omega\left(\Delta_{s,2p-s,V}(T)\right)+\frac{1}{2}\|T\|^{p-s}\left\|\Delta_{s,p,V}(T)\right\|.
\end{aligned}\tag{3.5}$$

Follows by Theorem 1 by taking $p=q\geq 0$.

Now, if we take $p=1/2\geq s\geq 0$ in Corollary 1, then we get

$$\begin{aligned}
&\omega\left(\widetilde{T}_{V}\right)\\
&\leq\frac{1}{2}\omega\left(\Delta_{1/2+s,1/2-s,V}(T)\right)+\frac{1}{4}\left(\left\|\widetilde{T}_{V}\right\|+\left\|\Delta_{1/2,1/2-s,V}(T)\right\|\|T\|^{s}\right)\\
&\leq\frac{1}{2}\omega\left(\Delta_{1/2+s,1/2-s,V}(T)\right)+\frac{1}{2}\|T\|^{s}\left\|\Delta_{1/2,1/2-s,V}(T)\right\|
\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}
\omega\left(\widetilde{T}_{V}\right)&\leq\frac{1}{2}\omega\left(\Delta_{s,1-s,V}(T)\right)+\frac{1}{4}\left(\left\|\widetilde{T}_{V}\right\|+\|T\|^{1/2-s}\left\|\Delta_{s,1/2,V}(T)\right\|\right)\\
&\leq\frac{1}{2}\omega\left(\Delta_{s,1-s,V}(T)\right)+\frac{1}{2}\|T\|^{1/2-s}\left\|\Delta_{s,1/2,V}(T)\right\|.
\end{aligned}\tag{3.7}$$

If we choose $s=1/2$ in (3.6), then we obtain

$$\omega\left(\widetilde{T}_V\right)\leq\frac{1}{2}\omega\left(\widehat{T}_V\right)+\frac{1}{4}\left(\left\|\widetilde{T}_V\right\|+\left\|\widehat{T}_{1/2,V}\right\|\|T\|^{1/2}\right)\leq\frac{1}{2}\omega\left(\widehat{T}_V\right)+\frac{1}{2}\left\|\widehat{T}_{1/2,V}\right\|\|T\|^{1/2}. \tag{3.8}$$

If we take $s=0$ in (3.7), then we get

$$\omega\left(\widetilde{T}_V\right)\leq\frac{1}{2}\omega\left(T_V\right)+\frac{1}{4}\left(\left\|\widetilde{T}_V\right\|+\|T\|^{1/2}\left\|T_{1/2,V}\right\|\right)\leq\frac{1}{2}\omega\left(T_V\right)+\frac{1}{2}\|T\|^{1/2}\left\|T_{1/2,V}\right\|. \tag{3.9}$$

Corollary 2. *Let V be a contraction and $T\in\mathcal{B}(H)$. Then for $q\geq r\geq 0$,*

$$\omega\left(T_{q,V}\right)\leq\frac{1}{2}\omega\left(\Delta_{r,q-r,V}(T)\right)+\frac{1}{4}\left(\left\|T_{q,V}\right\|+\left\|T_{q-r,V}\right\|\|T\|^{r}\right)\leq\frac{1}{2}\omega\left(\Delta_{r,q-r,V}(T)\right)+\frac{1}{2}\|T\|^{r}\left\|T_{q-r,V}\right\|. \tag{3.10}$$

It follows by Theorem 1 by taking $p=0$.

Now, if we take $q=1$ in (3.10) then we get

$$\omega\left(T_V\right)\leq\frac{1}{2}\omega\left(\Delta_{r,1-r,V}(T)\right)+\frac{1}{4}\left(\left\|T_V\right\|+\left\|T_{1-r,V}\right\|\|T\|^{r}\right)\leq\frac{1}{2}\omega\left(\Delta_{r,1-r,V}(T)\right)+\frac{1}{2}\|T\|^{r}\left\|T_{1-r,V}\right\| \tag{3.11}$$

for all $r\in[0,1]$.

If we consider $r=1$ in the inequality (3.11), then we obtain

$$\omega\left(T_V\right)\leq\frac{1}{2}\omega\left(\widehat{T}_V\right)+\frac{1}{4}\left(\left\|T_V\right\|+\|V\|\|T\|\right)\leq\frac{1}{2}\omega\left(\widehat{T}_V\right)+\frac{1}{2}\|V\|\|T\|. \tag{3.12}$$

From the same inequality (3.11) for $r=1/2$, we derive

$$\omega\left(T_V\right)\leq\frac{1}{2}\omega\left(\widetilde{T}_V\right)+\frac{1}{4}\left(\left\|T_V\right\|+\left\|T_{1/2,V}\right\|\|T\|^{1/2}\right)\leq\frac{1}{2}\omega\left(\widetilde{T}_V\right)+\frac{1}{2}\left\|T_{1/2,V}\right\|\|T\|^{1/2}. \tag{3.13}$$

We also have:

Corollary 3. *Let V be a contraction and $T\in\mathcal{B}(H)$. Then for $p\geq s\geq 0$,*

$$\omega\left(\widehat{T}_{p,V}\right)\leq\frac{1}{2}\omega\left(\Delta_{s,p-s,V}(T)\right)+\frac{1}{4}\left(\left\|\widehat{T}_{p,V}\right\|+\|T\|^{p-s}\left\|\widehat{T}_{s,V}\right\|\right)\leq\frac{1}{2}\omega\left(\Delta_{s,p-s,V}(T)\right)+\frac{1}{2}\|T\|^{p-s}\left\|\widehat{T}_{s,V}\right\|. \tag{3.14}$$

It follows by Theorem 1 by taking $q=0$.

Further, if we take $p=1$ in (3.14), then we get

$$\omega\left(\widehat{T}_V\right)\leq\frac{1}{2}\omega\left(\Delta_{s,1-s,V}(T)\right)+\frac{1}{4}\left(\left\|\widehat{T}_V\right\|+\|T\|^{1-s}\left\|\widehat{T}_{s,V}\right\|\right)\leq\frac{1}{2}\omega\left(\Delta_{s,1-s,V}(T)\right)+\frac{1}{2}\|T\|^{1-s}\left\|\widehat{T}_{s,V}\right\| \tag{3.15}$$

for all $s\in[0,1]$.

Now, for $s=0$ in (3.15) we obtain

$$\omega\left(\widehat{T}_V\right)\leq\frac{1}{2}\omega\left(T_V\right)+\frac{1}{4}\left(\left\|\widehat{T}_V\right\|+\|T\|\|V\|\right)\leq\frac{1}{2}\omega\left(T_V\right)+\frac{1}{2}\|T\|\|V\|, \tag{3.16}$$

while for $s=1/2$ we derive

$$\begin{aligned}\omega\left(\widehat{T}_V\right)&\leq\frac{1}{2}\omega\left(\widetilde{T}_V\right)+\frac{1}{4}\left(\left\|\widehat{T}_V\right\|+\|T\|^{1/2}\left\|\widehat{T}_{1/2,V}\right\|\right)\\&\leq\frac{1}{2}\omega\left(\widetilde{T}_V\right)+\frac{1}{2}\|T\|^{1/2}\left\|\widehat{T}_{1/2,V}\right\|.\end{aligned}\tag{3.17}$$

Remark 1. *If we take $r=q$ in (3.1), then we get*

$$\begin{aligned}\omega\left(\Delta_{p,q,V}(T)\right)&\leq\frac{1}{2}\omega\left(\widehat{T}_{p+q,V}\right)+\frac{1}{4}\left(\left\|\Delta_{p,q,V}(T)\right\|+\left\|\widehat{T}_{p,V}\right\|\|T\|^{q}\right)\\&\leq\frac{1}{2}\omega\left(\widehat{T}_{p+q,V}\right)+\frac{1}{2}\|T\|^{q}\left\|\widehat{T}_{p,V}\right\|\end{aligned}\tag{3.18}$$

for all $p,q\geq0$.

If we choose in (3.18) $p=0$, then we obtain

$$\begin{aligned}\omega\left(T_{q,V}\right)&\leq\frac{1}{2}\omega\left(\widehat{T}_{q,V}\right)+\frac{1}{4}\left(\left\|T_{q,V}\right\|+\|V\|\|T\|^{q}\right)\\&\leq\frac{1}{2}\omega\left(\widehat{T}_{q,V}\right)+\frac{1}{2}\|T\|^{q}\|V\|\end{aligned}\tag{3.19}$$

for all $q\geq0$.

If we take $q=1$ in (3.19), then we get

$$\begin{aligned}\omega\left(T_V\right)&\leq\frac{1}{2}\omega\left(\widehat{T}_V\right)+\frac{1}{4}\left(\|T_V\|+\|V\|\|T\|\right)\\&\leq\frac{1}{2}\omega\left(\widehat{T}_V\right)+\frac{1}{2}\|T\|\|V\|.\end{aligned}\tag{3.20}$$

Also, if we take in (3.18) $p=q$, then we get

$$\begin{aligned}\omega\left(\widetilde{T}_{p,V}\right)&\leq\frac{1}{2}\omega\left(\widehat{T}_{2p,V}\right)+\frac{1}{4}\left(\left\|\widetilde{T}_{p,V}\right\|+\left\|\widehat{T}_{p,V}\right\|\|T\|^{p}\right)\\&\leq\frac{1}{2}\omega\left(\widehat{T}_{2p,V}\right)+\frac{1}{2}\|T\|^{p}\left\|\widehat{T}_{p,V}\right\|,\end{aligned}\tag{3.21}$$

which, for $p=1/2$, gives

$$\begin{aligned}\omega\left(\widetilde{T}_V\right)&\leq\frac{1}{2}\omega\left(\widehat{T}_V\right)+\frac{1}{4}\left(\left\|\widetilde{T}_V\right\|+\left\|\widehat{T}_{1/2,V}\right\|\|T\|^{1/2}\right)\\&\leq\frac{1}{2}\omega\left(\widehat{T}_V\right)+\frac{1}{2}\left\|\widehat{T}_{1/2,V}\right\|\|T\|^{1/2}.\end{aligned}\tag{3.22}$$

If we take $s=0$ in (3.2), then we get

$$\begin{aligned}\omega\left(\Delta_{p,q,V}(T)\right)&\leq\frac{1}{2}\omega\left(T_{q+p,V}\right)+\frac{1}{4}\left(\left\|\Delta_{p,q,V}(T)\right\|+\|T\|^{p}\left\|T_{q,V}\right\|\right)\\&\leq\frac{1}{2}\omega\left(T_{q+p,V}\right)+\frac{1}{2}\|T\|^{p}\left\|T_{q,V}\right\|.\end{aligned}\tag{3.23}$$

for all $p,q\geq0$.

If we choose $q=0$ in (3.23), then we obtain

$$\omega\left(\widehat{T}_{p,V}\right) \leq \frac{1}{2}\omega\left(T_{p,V}\right)+\frac{1}{4}\left(\left\|\widehat{T}_{p,V}\right\|+\|T\|^{p}\|V\|\right) \leq \frac{1}{2}\omega\left(T_{p,V}\right)+\frac{1}{2}\|T\|^{p}\|V\| \tag{3.24}$$

for all $p\geq 0$. If in this inequality we take $p=1$, then we get

$$\omega\left(\widehat{T}_{V}\right) \leq \frac{1}{2}\omega\left(T_{V}\right)+\frac{1}{4}\left(\left\|\widehat{T}_{V}\right\|+\|T\|\|V\|\right) \leq \frac{1}{2}\omega\left(T_{V}\right)+\frac{1}{2}\|T\|\|V\|. \tag{3.25}$$

Moreover, if we take in (3.23) $q=p$, then we derive

$$\omega\left(\widetilde{T}_{p,V}\right) \leq \frac{1}{2}\omega\left(T_{2p,V}\right)+\frac{1}{4}\left(\left\|\widetilde{T}_{p,V}\right\|+\|T\|^{p}\|T_{p,V}\|\right) \leq \frac{1}{2}\omega\left(T_{2p,V}\right)+\frac{1}{2}\|T\|^{p}\|T_{p,V}\| \tag{3.26}$$

for all $p\geq 0$, which for $p=1/2$ produces

$$\omega\left(\widetilde{T}_{V}\right) \leq \frac{1}{2}\omega\left(T_{V}\right)+\frac{1}{4}\left(\left\|\widetilde{T}_{V}\right\|+\|T\|^{1/2}\left\|T_{1/2,V}\right\|\right) \leq \frac{1}{2}\omega\left(T_{V}\right)+\frac{1}{2}\|T\|^{1/2}\left\|T_{1/2,V}\right\|. \tag{3.27}$$

We also have:

Theorem 2. *Let V be a contraction and $T\in\mathcal{B}(H)$. Then for $q\geq r\geq 0$ and $p\geq s\geq 0$,*

$$\omega\left(\Delta_{p,q,V}(T)\right) \leq \frac{1}{2}\omega\left(\Delta_{p+r,q-r,V}(T)\right)+\frac{1}{4}\left\||T|^{2r}+\|T\|^{2p}\widetilde{T}_{q-r,|V|^{2}}\right\| \leq \frac{1}{2}\omega\left(\Delta_{p+r,q-r,V}(T)\right)+\frac{1}{4}\left\||T|^{2r}+\|T\|^{2p}|T|^{2(q-r)}\right\| \tag{3.28}$$

and

$$\omega\left(\Delta_{p,q,V}(T)\right) \leq \frac{1}{2}\omega\left(\Delta_{s,q+p-s,V}(T)\right)+\frac{1}{4}\left\||T|^{2(p-s)}+\|T\|^{2q}\widetilde{T}_{s,|V^{*}|^{2}}\right\| \leq \frac{1}{2}\omega\left(\Delta_{s,q+p-s,V}(T)\right)+\frac{1}{4}\left\||T|^{2(p-s)}+\|T\|^{2q}|T|^{2s}\right\|. \tag{3.29}$$

Proof. We use the following inequality obtained in [13]

$$\omega(AB)\leq\frac{1}{2}\omega(BA)+\frac{1}{4}\|BB^{*}+A^{*}A\| \tag{3.30}$$

for all $A,\ B\in\mathcal{B}(H)$.

If we take $A=|T|^{p}V|T|^{q-r}$ and $B=|T|^{r}$ then we get from (3.30) that

$$\omega\left(\Delta_{p,q,V}(T)\right) \leq \frac{1}{2}\omega\left(\Delta_{p+r,q-r,V}(T)\right)+\frac{1}{4}\left\||T|^{2r}+|T|^{q-r}V^{*}|T|^{2p}V|T|^{q-r}\right\|. \tag{3.31}$$

Observe that

$$0\leq|T|^{2p}\leq\left\||T|^{2p}\right\|I=\|T\|^{2p}I$$

and if we multiply this inequality to the left with V^* and to the right with V, then we get

$$V^* |T|^{2p} V \le \|T\|^{2p} V^*V = \|T\|^{2p} |V|^2 \le \|T\|^{2p}.$$

Moreover, if we multiply both sides with $|T|^{q-r} \ge 0$, then we further obtain

$$\begin{aligned} |T|^{q-r} V^* |T|^{2p} V |T|^{q-r} &\le \|T\|^{2p} |T|^{q-r} |V|^2 |T|^{q-r} = \|T\|^{2p} \widetilde{T}_{q-r,|V|^2} \\ &\le \|T\|^{2p} |T|^{2(q-r)} \end{aligned}$$

and by (3.31) we derive the desired result (3.28).

Now, if we take $A = |T|^{p-s}$ and $B = |T|^s V |T|^q$ in (3.30), then we get

$$\begin{aligned} &\omega\left(\Delta_{p,q,V}(T)\right) \\ &\le \frac{1}{2}\omega\left(\Delta_{s,q+p-s,V}(T)\right) + \frac{1}{4}\left\| |T|^s V |T|^{2q} V^* |T|^s + |T|^{2(p-s)} \right\|. \end{aligned} \tag{3.32}$$

Since, as above

$$\begin{aligned} |T|^s V |T|^{2q} V^* |T|^s &\le \|T\|^{2q} |T|^s VV^* |T|^s = \|T\|^{2q} |T|^s |V^*|^2 |T|^s \\ &= \|T\|^{2q} \widetilde{T}_{s,|V^*|^2} \le \|T\|^{2q} |T|^{2s}, \end{aligned}$$

hence by (3.32) we derive (3.29). □

Corollary 4. *Let V be a contraction and $T \in \mathcal{B}(H)$. Then for $p \ge s \ge 0$,*

$$\begin{aligned} \omega\left(\widetilde{T}_{p,V}\right) &\le \frac{1}{2}\omega\left(\Delta_{p+s,p-s,V}(T)\right) + \frac{1}{4}\left\| |T|^{2s} + \|T\|^{2p} \widetilde{T}_{p-s,|V|^2} \right\| \\ &\le \frac{1}{2}\omega\left(\Delta_{p+s,p-s,V}(T)\right) + \frac{1}{4}\left\| |T|^{2s} + \|T\|^{2p} |T|^{2(p-s)} \right\| \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} \omega\left(\widetilde{T}_{p,V}\right) &\le \frac{1}{2}\omega\left(\Delta_{s,2p-s,V}(T)\right) + \frac{1}{4}\left\| |T|^{2(p-s)} + \|T\|^{2p} \widetilde{T}_{s,|V^*|^2} \right\| \\ &\le \frac{1}{2}\omega\left(\Delta_{s,2p-s,V}(T)\right) + \frac{1}{4}\left\| |T|^{2(p-s)} + \|T\|^{2p} |T|^{2s} \right\|. \end{aligned} \tag{3.34}$$

Follows by Theorem 2 for $p = q \ge 0$.

Now, if we take $p = 1/2 \ge s \ge 0$ in Corollary 4, then we get

$$\begin{aligned} \omega\left(\widetilde{T}_V\right) &\le \frac{1}{2}\omega\left(\Delta_{1/2+s,1/2-s,V}(T)\right) + \frac{1}{4}\left\| |T|^{2s} + \|T\| \widetilde{T}_{1/2-s,|V|^2} \right\| \\ &\le \frac{1}{2}\omega\left(\Delta_{1/2+s,1/2-s,V}(T)\right) + \frac{1}{4}\left\| |T|^{2s} + \|T\| |T|^{2(1/2-s)} \right\| \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} \omega\left(\widetilde{T}_V\right) &\le \frac{1}{2}\omega\left(\Delta_{s,1-s,V}(T)\right) + \frac{1}{4}\left\| |T|^{2(1/2-s)} + \|T\| \widetilde{T}_{s,|V^*|^2} \right\| \\ &\le \frac{1}{2}\omega\left(\Delta_{s,1-s,V}(T)\right) + \frac{1}{4}\left\| |T|^{2(1/2-s)} + \|T\| |T|^{2s} \right\|. \end{aligned} \tag{3.36}$$

If we choose $s = 1/2$ in (3.35), then we get

$$\omega\left(\widetilde{T}_V\right) \le \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left\| |T| + \|T\| |V|^2 \right\| \le \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left\| |T| + \|T\| I \right\|. \tag{3.37}$$

If we take $s = 0$ in (3.36), then we obtain

$$\omega\left(\widetilde{T}_V\right) \le \frac{1}{2}\omega\left(T_V\right) + \frac{1}{4}\left\| |T| + \|T\| |V^*|^2 \right\| \le \frac{1}{2}\omega\left(T_V\right) + \frac{1}{4}\left\| |T| + \|T\| I \right\|. \tag{3.38}$$

Corollary 5. *Let V be a contraction and $T\in\mathcal{B}(H)$. Then for $q\geq r\geq 0$,*

$$\begin{aligned}\omega\left(T_{q,V}\right) &\leq \frac{1}{2}\omega\left(\Delta_{r,q-r,V}\left(T\right)\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2r}+\widetilde{T}_{q-r,\left\vert V\right\vert^{2}}\right\Vert \\ &\leq \frac{1}{2}\omega\left(\Delta_{r,q-r,V}\left(T\right)\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2r}+\left\vert T\right\vert^{2(q-r)}\right\Vert \end{aligned} \tag{3.39}$$

It follows by Theorem 2 for $p=0$.

Now, if we take $q=1$ in (3.39) then we get

$$\begin{aligned}\omega\left(T_{V}\right) &\leq \frac{1}{2}\omega\left(\Delta_{r,1-r,V}\left(T\right)\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2r}+\widetilde{T}_{1-r,\left\vert V\right\vert^{2}}\right\Vert \\ &\leq \frac{1}{2}\omega\left(\Delta_{r,1-r,V}\left(T\right)\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2r}+\left\vert T\right\vert^{2(1-r)}\right\Vert \end{aligned} \tag{3.40}$$

for all $r\in[0,1]$.

If we choose $r=1$ in the inequality (3.40), then we obtain

$$\omega\left(T_{V}\right)\leq\frac{1}{2}\omega\left(\widehat{T}_{V}\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2}+\left\vert V\right\vert^{2}\right\Vert \leq\frac{1}{2}\omega\left(\widehat{T}_{V}\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2}+I\right\Vert . \tag{3.41}$$

From the same inequality(3.40) for $r=1/2$, we derive

$$\omega\left(T_{V}\right)\leq\frac{1}{2}\omega\left(\widetilde{T}_{V}\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert+\widetilde{T}_{1/2,\left\vert V\right\vert^{2}}\right\Vert \leq\frac{1}{2}\omega\left(\widetilde{T}_{V}\right)+\frac{1}{2}\left\Vert T\right\Vert . \tag{3.42}$$

We also have:

Corollary 6. *Let V be a contraction and $T\in\mathcal{B}(H)$. Then for $p\geq s\geq 0$,*

$$\begin{aligned}\omega\left(\widehat{T}_{p,V}\right) &\leq \frac{1}{2}\omega\left(\Delta_{s,p-s,V}\left(T\right)\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2(p-s)}+\widetilde{T}_{s,\left\vert V^{*}\right\vert^{2}}\right\Vert \\ &\leq \frac{1}{2}\omega\left(\Delta_{s,p-s,V}\left(T\right)\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2(p-s)}+\left\vert T\right\vert^{2s}\right\Vert . \end{aligned} \tag{3.43}$$

It follows by Theorem 1 for $q=0$.

Further, if we take $p=1$ in (3.43), then we get

$$\begin{aligned}\omega\left(\widehat{T}_{V}\right) &\leq \frac{1}{2}\omega\left(\Delta_{s,1-s,V}\left(T\right)\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2(1-s)}+\widetilde{T}_{s,\left\vert V^{*}\right\vert^{2}}\right\Vert \\ &\leq \frac{1}{2}\omega\left(\Delta_{s,1-s,V}\left(T\right)\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2(1-s)}+\left\vert T\right\vert^{2s}\right\Vert \end{aligned} \tag{3.44}$$

for all $s\in[0,1]$.

For $s=0$ we obtain from (3.44) that

$$\omega\left(\widehat{T}_{V}\right)\leq\frac{1}{2}\omega\left(T_{V}\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2}+\left\vert V^{*}\right\vert^{2}\right\Vert \leq\frac{1}{2}\omega\left(T_{V}\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2}+I\right\Vert ,$$

while for $s=1/2$ we derive

$$\omega\left(\widehat{T}_{V}\right)\leq\frac{1}{2}\omega\left(\widetilde{T}_{V}\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert+\widetilde{T}_{1/2,\left\vert V^{*}\right\vert^{2}}\right\Vert \leq\frac{1}{2}\omega\left(\widetilde{T}_{V}\right)+\frac{1}{2}\left\Vert T\right\Vert .$$

Remark 2. *If we take $r=q$ in (3.28), then we get*

$$\begin{aligned}\omega\left(\Delta_{p,q,V}\left(T\right)\right) &\leq \frac{1}{2}\omega\left(\widehat{T}_{p+q,V}\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2q}+\left\Vert T\right\Vert^{2p}\left\vert V\right\vert^{2}\right\Vert \\ &\leq \frac{1}{2}\omega\left(\widehat{T}_{p+q,V}\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2q}+\left\Vert T\right\Vert^{2p}I\right\Vert . \end{aligned} \tag{3.45}$$

Further, if we choose $p=0$ in (3.45) then we obtain

$$\omega\left(T_{q,V}\right)\leq\frac{1}{2}\omega\left(\widehat{T}_{q,V}\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2q}+\left\vert V\right\vert^{2}\right\Vert \leq\frac{1}{2}\omega\left(\widehat{T}_{q,V}\right)+\frac{1}{4}\left\Vert \left\vert T\right\vert^{2q}+I\right\Vert ,$$

for all $q \geq 0$, while $q = 1$ produces

$$\omega\left(T_V\right) \leq \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left\|\left|T\right|^2 + \left|V\right|^2\right\| \leq \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left\|\left|T\right|^2 + I\right\|.$$

Also, if we take in (3.45) $p = q$, then we get

$$\begin{aligned}\omega\left(\widetilde{T}_{p,V}\right) &\leq \frac{1}{2}\omega\left(\widehat{T}_{2p,V}\right) + \frac{1}{4}\left\|\left|T\right|^{2p} + \left\|T\right\|^{2p}\left|V\right|^2\right\| \\ &\leq \frac{1}{2}\omega\left(\widehat{T}_{2p,V}\right) + \frac{1}{4}\left\|\left|T\right|^{2p} + \left\|T\right\|^{2p} I\right\|,\end{aligned}$$

which for $p = 1/2$ gives that

$$\omega\left(\widetilde{T}_V\right) \leq \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left\|\left|T\right| + \left\|T\right\|\left|V\right|^2\right\| \leq \frac{1}{2}\omega\left(\widehat{T}_V\right) + \frac{1}{4}\left\|\left|T\right| + \left\|T\right\| I\right\|.$$

If we take $s = 0$ in (3.29), then we get

$$\begin{aligned}\omega\left(\Delta_{p,q,V}\left(T\right)\right) &\leq \frac{1}{2}\omega\left(T_{q+p,V}\right) + \frac{1}{4}\left\|\left|T\right|^{2p} + \left\|T\right\|^{2q}\left|V^*\right|^2\right\| \\ &\leq \frac{1}{2}\omega\left(T_{q+p,V}\right) + \frac{1}{4}\left\|\left|T\right|^{2p} + \left\|T\right\|^{2q} I\right\|\end{aligned} \tag{3.46}$$

for all $p, q \geq 0$.

If we choose $q = 0$ in (3.46), then we obtain

$$\omega\left(\widehat{T}_{p,V}\right) \leq \frac{1}{2}\omega\left(T_{p,V}\right) + \frac{1}{4}\left\|\left|T\right|^{2p} + \left|V^*\right|^2\right\| \leq \frac{1}{2}\omega\left(T_{p,V}\right) + \frac{1}{4}\left\|\left|T\right|^{2p} + I\right\|$$

for all $p \geq 0$, which for $p = 1$ gives

$$\omega\left(\widehat{T}_V\right) \leq \frac{1}{2}\omega\left(T_V\right) + \frac{1}{4}\left\|\left|T\right|^2 + \left|V^*\right|^2\right\| \leq \frac{1}{2}\omega\left(T_V\right) + \frac{1}{4}\left\|\left|T\right|^2 + I\right\|.$$

Moreover, if we take in Theorem 2 $q = p$, then we get

$$\begin{aligned}\omega\left(\widetilde{T}_{p,V}\right) &\leq \frac{1}{2}\omega\left(\Delta_{p+s,p-s,V}\left(T\right)\right) + \frac{1}{4}\left\|\left|T\right|^{2s} + \left\|T\right\|^{2p}\widetilde{T}_{p-s,\left|V\right|^2}\right\| \\ &\leq \frac{1}{2}\omega\left(\Delta_{p+s,p-s,V}\left(T\right)\right) + \frac{1}{4}\left\|\left|T\right|^{2s} + \left\|T\right\|^{2p}\left|T\right|^{2(p-s)}\right\|\end{aligned}$$

and

$$\begin{aligned}\omega\left(\widetilde{T}_V\right) &\leq \frac{1}{2}\omega\left(\Delta_{s,2p-s,V}\left(T\right)\right) + \frac{1}{4}\left\|\left|T\right|^{2(p-s)} + \left\|T\right\|^{2p}\widetilde{T}_{s,\left|V^*\right|^2}\right\| \\ &\leq \frac{1}{2}\omega\left(\Delta_{s,2p-s,V}\left(T\right)\right) + \frac{1}{4}\left\|\left|T\right|^{2(p-s)} + \left\|T\right\|^{2p}\left|T\right|^{2s}\right\|\end{aligned}$$

for all $0 \leq s \leq p$.

More similar inequalities may be stated if one further takes some particular values for p and s. The details are omitted.

4. Related Results

We also have:

Theorem 3. *Let V be a contraction and $T \in \mathcal{B}(H)$. Then for $q \geq 0$ and $p \geq 0$,*

$$\begin{aligned}\omega\left(\Delta_{p,q,V}\left(T\right)\right) &\leq \frac{\sqrt{2}}{2}\left\|\left\|T\right\|^{2p}\left|T\right|^q\left|V\right|^2\left|T\right|^q + \left\|T\right\|^{2q}\left|T\right|^p\left|V^*\right|^2\left|T\right|^p\right\|^{1/2} \\ &\leq \frac{\sqrt{2}}{2}\left\|\left\|T\right\|^{2p}\left|T\right|^{2q} + \left\|T\right\|^{2q}\left|T\right|^{2p}\right\|^{1/2} \leq \left\|T\right\|^{p+q}.\end{aligned} \tag{4.1}$$

In particular,

$$\omega\left(\widetilde{T}_{p,V}\right) \leq \frac{\sqrt{2}}{2}\left\|T\right\|^{p}\left\|\left|T\right|^{p}\left(\left|V\right|^{2}+\left|V^{*}\right|^{2}\right)\left|T\right|^{p}\right\|^{1/2} \leq \left\|T\right\|^{2p}. \tag{4.2}$$

Proof. We use the following inequality obtained by Kittaneh in [12]

$$\omega^{2}\left(A\right) \leq \frac{1}{2}\left\|A^{*}A+AA^{*}\right\| \tag{4.3}$$

for all $A\in\mathcal{B}\left(H\right)$.

If we write (4.3) for $A=\Delta_{p,q,V}\left(T\right)=\left|T\right|^{p}V\left|T\right|^{q}$, then we get

$$\begin{aligned}\omega^{2}\left(\Delta_{p,q,V}\left(T\right)\right) &\leq \frac{1}{2}\left\|\left|T\right|^{q}V^{*}\left|T\right|^{p}\left|T\right|^{p}V\left|T\right|^{q}+\left|T\right|^{p}V\left|T\right|^{q}\left|T\right|^{q}V^{*}\left|T\right|^{p}\right\| \\ &= \frac{1}{2}\left\|\left|T\right|^{q}V^{*}\left|T\right|^{2p}V\left|T\right|^{q}+\left|T\right|^{p}V\left|T\right|^{2q}V^{*}\left|T\right|^{p}\right\|.\end{aligned} \tag{4.4}$$

Observe that

$$\begin{aligned}0 &\leq \left|T\right|^{q}V^{*}\left|T\right|^{2p}V\left|T\right|^{q} \leq \left\|T\right\|^{2p}\left|T\right|^{q}V^{*}V\left|T\right|^{q}=\left\|T\right\|^{2p}\left|T\right|^{q}\left|V\right|^{2}\left|T\right|^{q} \\ &\leq \left\|T\right\|^{2p}\left|T\right|^{2q}\end{aligned}$$

and

$$\begin{aligned}0 &\leq \left|T\right|^{p}V\left|T\right|^{2q}V^{*}\left|T\right|^{p} \leq \left\|T\right\|^{2q}\left|T\right|^{p}VV^{*}\left|T\right|^{p}=\left\|T\right\|^{2q}\left|T\right|^{p}\left|V^{*}\right|^{2}\left|T\right|^{p} \\ &\leq \left\|T\right\|^{2q}\left|T\right|^{2p},\end{aligned}$$

which gives that

$$\begin{aligned}0 &\leq \left|T\right|^{q}V^{*}\left|T\right|^{2p}V\left|T\right|^{q}+\left|T\right|^{p}V\left|T\right|^{2q}V^{*}\left|T\right|^{p} \\ &\leq \left\|T\right\|^{2p}\left|T\right|^{q}\left|V\right|^{2}\left|T\right|^{q}+\left\|T\right\|^{2q}\left|T\right|^{p}\left|V^{*}\right|^{2}\left|T\right|^{p} \\ &\leq \left\|T\right\|^{2p}\left|T\right|^{2q}+\left\|T\right\|^{2q}\left|T\right|^{2p}.\end{aligned}$$

Therefore

$$\begin{aligned}&\left\|\left|T\right|^{q}V^{*}\left|T\right|^{2p}V\left|T\right|^{q}+\left|T\right|^{p}V\left|T\right|^{2q}V^{*}\left|T\right|^{p}\right\| \\ &\leq \left\|\left\|T\right\|^{2p}\left|T\right|^{q}\left|V\right|^{2}\left|T\right|^{q}+\left\|T\right\|^{2q}\left|T\right|^{p}\left|V^{*}\right|^{2}\left|T\right|^{p}\right\| \\ &\leq \left\|\left\|T\right\|^{2p}\left|T\right|^{2q}+\left\|T\right\|^{2q}\left|T\right|^{2p}\right\|\end{aligned}$$

and by (4.4) we get (4.1). □

Corollary 7. *Let V be a contraction and $T\in\mathcal{B}\left(H\right)$. Then,*

$$\begin{aligned}\omega\left(\widehat{T}_{p,V}\right) &\leq \frac{\sqrt{2}}{2}\left\|\left\|T\right\|^{2p}\left|V\right|^{2}+\left|T\right|^{p}\left|V^{*}\right|^{2}\left|T\right|^{p}\right\|^{1/2} \\ &\leq \frac{\sqrt{2}}{2}\left\|\left\|T\right\|^{2p}I+\left|T\right|^{2p}\right\|^{1/2} \leq \left\|T\right\|^{p}\end{aligned} \tag{4.5}$$

for $p\geq 0$ and

$$\begin{aligned}\omega\left(T_{q,V}\right) &\leq \frac{\sqrt{2}}{2}\left\|\left|T\right|^{q}\left|V\right|^{2}\left|T\right|^{q}+\left\|T\right\|^{2q}\left|V^{*}\right|^{2}\right\|^{1/2} \\ &\leq \frac{\sqrt{2}}{2}\left\|\left|T\right|^{2q}+\left\|T\right\|^{2q}I\right\|^{1/2} \leq \left\|T\right\|^{q}\end{aligned} \tag{4.6}$$

for $q\geq 0$.

Remark 3. *If we take $p = 1/2$ in (4.2), then we get*

$$\omega\left(\widetilde{T}_V\right) \leq \frac{\sqrt{2}}{2}\|T\|^{1/2}\left\||T|^{1/2}\left(|V|^2+|V^*|^2\right)|T|^{1/2}\right\|^{1/2} \leq \|T\|. \tag{4.7}$$

If we take $p = 1$ in (4.5), then we obtain

$$\begin{aligned}\omega\left(\widehat{T}_V\right) &\leq \frac{\sqrt{2}}{2}\left\|\|T\|^2|V|^2+|T||V^*|^2|T|\right\|^{1/2} \\ &\leq \frac{\sqrt{2}}{2}\left\||T|^2+\|T\|^2 I\right\|^{1/2} \leq \|T\|,\end{aligned} \tag{4.8}$$

while for $q = 1$ in (4.6), we get

$$\begin{aligned}\omega(T_V) &\leq \frac{\sqrt{2}}{2}\left\||T||V|^2|T|+\|T\|^2|V^*|^2\right\|^{1/2} \\ &\leq \frac{\sqrt{2}}{2}\left\||T|^2+\|T\|^2 I\right\|^{1/2} \leq \|T\|.\end{aligned} \tag{4.9}$$

Theorem 4. *Let V be a contraction and $T \in \mathcal{B}(H)$. Then for $q \geq r \geq 0$ and $p \geq s \geq 0$,*

$$\omega(\Delta_{p,q,V}(T)) \leq \|\Delta_{s,r,V}(T)\|\left\|\alpha|T|^{\frac{(p-s)u}{\alpha}}+(1-\alpha)|T|^{\frac{(q-r)u}{1-\alpha}}\right\|^{1/u} \tag{4.10}$$

for all $u \geq 2$ and $\alpha \in [0,1]$.

In particular, for $p \geq s \geq 0$,

$$\omega\left(\widetilde{T}_{p,V}\right) \leq \left\|\widetilde{T}_{s,V}\right\|\left\|\alpha|T|^{\frac{(p-s)u}{\alpha}}+(1-\alpha)|T|^{\frac{(p-s)u}{1-\alpha}}\right\|^{1/u}. \tag{4.11}$$

Proof. We use the following inequality obtained in [13]

$$\omega\left(A^\alpha X B^{1-\alpha}\right) \leq \|X\|\|\alpha A^u+(1-\alpha)B^u\|^{1/u}, \tag{4.12}$$

where $A, B \geq 0$, $X \in \mathbb{B}(\mathcal{H})$, $u \geq 2$ and $\alpha \in (0,1)$.

Observe that

$$\begin{aligned}\omega(\Delta_{p,q,V}(T)) &= \omega\left(|T|^{p-s}|T|^s V|T|^r|T|^{q-r}\right) \\ &= \omega\left(\left(|T|^{\frac{p-s}{\alpha}}\right)^\alpha|T|^s V|T|^r\left(|T|^{\frac{q-r}{1-\alpha}}\right)^{1-\alpha}\right)\end{aligned}$$

and by using the inequality in (4.12) for $A = |T|^{\frac{p-s}{\alpha}}$, $X = |T|^s V|T|^r$ and $B = |T|^{\frac{q-r}{1-\alpha}}$, we get

$$\begin{aligned}\omega(\Delta_{p,q,V}(T)) &\leq \||T|^s V|T|^r\|\left\|\alpha\left(|T|^{\frac{p-s}{\alpha}}\right)^u+(1-\alpha)\left(|T|^{\frac{q-r}{1-\alpha}}\right)^u\right\|^{1/u} \\ &= \|\Delta_{s,r,V}(T)\|\left\|\alpha|T|^{\frac{(p-s)u}{\alpha}}+(1-\alpha)|T|^{\frac{(q-r)u}{1-\alpha}}\right\|^{1/u}\end{aligned}$$

and the inequality (4.10) is proved. □

Corollary 8. *Let V be a contraction and $T \in \mathcal{B}(H)$ and $u \geq 2$, $\alpha \in [0,1]$. Then for $q \geq r \geq 0$,*

$$\omega(T_{q,V}) \leq \|T_{r,V}\|\left\|\alpha I+(1-\alpha)|T|^{\frac{(q-r)u}{1-\alpha}}\right\|^{1/u}$$

and for $p \geq s \geq 0$,

$$\omega\left(\widehat{T}_{p,V}\right) \leq \left\|\widehat{T}_{s,V}\right\| \left\|\alpha\left|T\right|^{\frac{(p-s)u}{\alpha}} + (1-\alpha) I\right\|^{1/u}.$$

Finally, we can state:

Theorem 5. *Let V be a contraction and $T \in \mathcal{B}(H)$. Then for $w \geq r \geq 0$ and $v \geq s \geq 0$,*

$$\omega\left(\Delta_{v,w,V}(T)\right) \leq \left\|\Delta_{s,r,V}(T)\right\| \left\|\frac{1}{p}\left|T\right|^{\frac{(v-s)pu}{\alpha}} + \frac{1}{q}\left|T\right|^{\frac{(w-r)qu}{\alpha}}\right\|^{\alpha/u} \tag{4.13}$$

for $0 \leq \alpha \leq 1$, $u \geq 0$ *and* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pu,\ qu \geq 2$.

In particular, for $v \geq s \geq 0$

$$\omega\left(\widetilde{T}_{v,V}\right) \leq \left\|\widetilde{T}_{s,V}\right\| \left\|\frac{1}{p}\left|T\right|^{\frac{(v-s)pu}{\alpha}} + \frac{1}{q}\left|T\right|^{\frac{(v-s)qu}{\alpha}}\right\|^{\alpha/u}. \tag{4.14}$$

Proof. We use the following inequality obtained in [13]

$$\omega\left(A^{\alpha} X B^{\alpha}\right) \leq \left\|X\right\| \left\|\frac{1}{p}A^{pu} + \frac{1}{q}B^{qu}\right\|^{\alpha/u} \tag{4.15}$$

for $0 \leq \alpha \leq 1$, $u \geq 0$ and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pu,\ qu \geq 2$.

Observe that

$$\begin{aligned}\omega\left(\Delta_{v,w,V}(T)\right) &= \omega\left(\left|T\right|^{v-s}\left|T\right|^{s} V \left|T\right|^{r}\left|T\right|^{w-r}\right)\\ &= \omega\left(\left(\left|T\right|^{\frac{v-s}{\alpha}}\right)^{\alpha}\left|T\right|^{s} V \left|T\right|^{r}\left(\left|T\right|^{\frac{w-r}{\alpha}}\right)^{\alpha}\right)\end{aligned}$$

and by using the inequality in (4.12) for $A = \left|T\right|^{\frac{v-s}{\alpha}}$, $X = \left|T\right|^{s} V \left|T\right|^{r}$ and $B = \left|T\right|^{\frac{w-r}{\alpha}}$, we get

$$\begin{aligned}\omega\left(\Delta_{v,w,V}(T)\right) &\leq \left\|\Delta_{s,r,V}(T)\right\| \left\|\frac{1}{p}\left(\left|T\right|^{\frac{v-s}{\alpha}}\right)^{pu} + \frac{1}{q}\left(\left|T\right|^{\frac{w-r}{\alpha}}\right)^{qu}\right\|^{\alpha/u}\\ &= \left\|\Delta_{s,r,V}(T)\right\| \left\|\frac{1}{p}\left|T\right|^{\frac{(v-s)pu}{\alpha}} + \frac{1}{q}\left|T\right|^{\frac{(w-r)qu}{\alpha}}\right\|^{\alpha/u},\end{aligned}$$

which proves (4.13). □

Corollary 9. *Let V be a contraction and $T \in \mathcal{B}(H)$. Then for $v \geq s \geq 0$,*

$$\omega\left(\widehat{T}_{v,V}\right) \leq \left\|\widehat{T}_{s,V}\right\| \left\|\frac{1}{p}\left|T\right|^{\frac{(v-s)pu}{\alpha}} + \frac{1}{q}I\right\|^{\alpha/u}$$

for $0 \leq \alpha \leq 1$, $u \geq 0$ *and* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pu,\ qu \geq 2$.

For $w \geq r \geq 0$ *we also have*

$$\omega\left(T_{w,V}\right) \leq \left\|T_{r,V}\right\| \left\|\frac{1}{p}I + \frac{1}{q}\left|T\right|^{\frac{(w-r)qu}{\alpha}}\right\|^{\alpha/u}.$$

More similar inequalities may be stated if one further takes some particular values for v, w, s and r. The details are omitted.

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