UPPER BOUNDS FOR THE SPECTRAL RADIUS IN TERMS OF THE (p,q)-EXTENDED GENERALIZED ALUTHGE TRANSFORM OF BOUNDED OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a *contraction* $V\in\mathcal{B}(H)$, i.e. $0\le V^{*}V\le I$, and an operator $T\in\mathcal{B}(H)$ we define the (p,q)-*extended generalized Aluthge transform* by

$$\Delta_{p,q,V}(T):=|T|^{p}V|T|^{q},$$

where $p,q\ge 0$. In this paper we provide some upper bounds for the spectral radius in terms of the (p,q)-extended generalized Aluthge transform $\Delta_{p,q,V}(T)$. We show among others that for V a contraction and $T\in\mathcal{B}(H)$, then for $q\ge r\ge 0$ and $p\ge s\ge 0$,

$$r\left(\Delta_{p,q,V}(T)\right)\le\frac{1}{2}\max\left\{\omega\left(\Delta_{p,q,V}(T)\right),\omega\left(\Delta_{p+r,q-r,V}(T)\right)\right\}+\frac{1}{2}\|T\|^{r/2}\left\|\Delta_{p,q-r,V}(T)\right\|^{1/2}\left\|\Delta_{p+r,q-r,V}(T)\right\|^{1/2}$$

and

$$r\left(\Delta_{p,q,V}(T)\right)\le\frac{1}{2}\max\left\{\omega\left(\Delta_{p,q,V}(T)\right),\omega\left(\Delta_{s,q+p-s,V}(T)\right)\right\}+\frac{1}{2}\|T\|^{(p-s)/2}\left\|\Delta_{s,q,V}(T)\right\|^{1/2}\left\|\Delta_{p,q,V}(T)\right\|^{1/2}.$$

Several particular cases of interest are also presented.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T)=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x\in H$ one has

$$|\langle Tx,x\rangle|\le w(T)\|x\|^{2}. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T:H\to H$, i.e.,

(i) $\omega(T)\ge 0$ for any $T\in B(H)$ and $\omega(T)=0$ if and only if $T=0$;
(ii) $\omega(\lambda T)=|\lambda|\,\omega(T)$ for any $\lambda\in\mathbb{C}$ and $T\in B(H)$;
(iii) $\omega(T+V)\le\omega(T)+\omega(V)$ for any $T,V\in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T)\le\|T\|\le 2\omega(T) \tag{1.3}$$

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for any $T \in B(H)$.

F. Kittaneh, in 2003 [11], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [12] improved the inequality (1.3) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{1.5}$$

for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [10]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{1.6}$$

and

$$\omega^{2r}(T) \leq \left\|\alpha |T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|, \tag{1.7}$$

where $\alpha \in (0,1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (1.6) that

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{1.8}$$

and from (1.7) that

$$\omega^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [8] and [5].

We denote by $r(T)$ the spectral radius of the operator T. It is well known that $r(T) \leq \omega(T)$ for any T and

$$r(AB) = r(BA) \text{ for every } A,\ B \in B(H). \tag{1.10}$$

It is known that if $AB = BA$, then also

$$r(A+B) \leq r(A) + r(B) \text{ and } r(AB) \leq r(A)\, r(B).$$

Let $T = U|T|$ be the *polar decomposition* of the bounded linear operator T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T} := |T|^{1/2} U |T|^{1/2}$, see [1].

The following properties of $\widetilde{T}$ are as follows:

(i) $\left\|\widetilde{T}\right\| \leq \|T\|$,
(ii) $w\left(\widetilde{T}\right) \leq \omega(T)$,
(iii) $r\left(\widetilde{T}\right) = \omega(T)$,
(iv) $\omega\left(\widetilde{T}\right) \leq \left\|T^2\right\|^{1/2} (\leq \|T\|)$, [14].

Utilizing this transform T. Yamazaki, [14] obtained in 2007 the following refinement of Kittaneh's inequality (1.4):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \omega\left(\widetilde{T}\right)\right) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right) \tag{1.11}$$

for any operator $T \in B(H)$.

We remark that if $\widetilde{T} = 0$, then obviously $w(T) = \frac{1}{2}\|T\|$.

Abu-Omar and Kittaneh [2] improved on inequality (1.11) using generalized Aluthge transform $\Delta_t(T) := |T|^t U |T|^{1-t}$ to prove that

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \min_{t\in[0,1]} \omega(\Delta_t(T))\right).$$

For $t = 1$ this also gives the following result for the *Dougal transform* $\widehat{T} := |T| U$,

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \omega\left(\widehat{T}\right)\right). \tag{1.12}$$

In [4] Bunia et al. also proved that

$$\omega(T) \leq \min_{t\in[0,1]}\left\{\frac{1}{2}\omega(\Delta_t(T)) + \frac{1}{4}\left(\|T\|^{2t} + \|T\|^{2(1-t)}\right)\right\},$$

which for $t = 1/2$ gives (1.11) as well.

Motivated by the above results, in this paper we provide some inequalities for the spectral radius in terms of the (p,q)-extended generalized Aluthge transform $\Delta_{p,q,V}(T)$ introduced below. Several particular cases of interest are also presented.

2. Some Preliminary Facts

We define, for V a contraction, *i.e.*, $V^*V \leq I$ and $T \in \mathcal{B}(H)$, the (p,q)-*extended generalized Aluthge transform* by

$$\Delta_{p,q,V}(T) := |T|^p V |T|^q,$$

where p, $q \geq 0$. We also assume in what follows that $|T|^0 := I$.

We denote

$$T_{q,V} := \Delta_{0,q,V}(T) := V|T|^q$$

and

$$T_V := \Delta_{0,1,V}(T) := V|T|.$$

The *p-extended generalized Dougal transform* is defined by

$$\widehat{T}_{p,V} := \Delta_{p,0,V}(T) := |T|^p V,$$

the *extended generalized Dougal transform* by

$$\widehat{T}_V := \Delta_{1,0,V}(T) := V|T|,$$

the *p-extended generalized Aluthge transform* by

$$\widetilde{T}_{p,V} := \Delta_{p,p,V}(T) := |T|^p V |T|^p$$

and the *extended Aluthge transform* by

$$\widetilde{T}_V := \Delta_{1/2,1/2,V}(T) := |T|^{1/2} V |T|^{1/2}.$$

For $p = t$, $q = 1 - t$, where $t \in [0,1]$ we have

$$\Delta_{t,V}(T) := \Delta_{t,1-t,V}(T) = |T|^t V |T|^{1-t}.$$

The transform $\Delta_{t,V}(T)$, called the *extended generalized Aluthge transform*, was introduced and studied in [9].

An operator $U \in \mathcal{B}(H)$ is called a *partial isometry* if $\|Ux\| = \|x\|$ for all $x \in \mathcal{N}^{\perp}(U)$.

Now, let $x \in H$, then there exists a unique $x_1 \in \mathcal{N}(U)$ and a unique $x_2 \in \mathcal{N}^{\perp}(U)$ such that $x = x_1 + x_2$. Then

$$0 \leq \langle U^*Ux, x\rangle = \|Ux\|^2 = \|Ux_1 + Ux_2\|^2 = \|Ux_2\|^2 = \|x_2\|^2.$$

By the fact that $x_1 \perp x_2$, $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$. Therefore $0 \leq \langle U^*Ux, x\rangle \leq \|x\|^2$, which shows that U is a contraction on H.

If the operator T has the polar decomposition $T = U|T|$ with U a partial isometry, then by taking $V = U$, we define

$$\Delta_{p,q}(T) := |T|^p U |T|^q,$$

and all the previous transform by omitting in the notations the V, which will give the concepts of the *p-generalized Dougal transform*

$$\widehat{T}_p := \Delta_{p,0,U}(T) := |T|^p U,$$

the *Dougal transform*

$$\widehat{T} := \Delta_{1,0,U}(T) := |T| U,$$

and the *p-generalized Aluthge transform*

$$\widetilde{T}_p := \Delta_{p,p,U}(T) := |T|^p U |T|^p,$$

which for $p = 1/2$ gives the usual Aluthge transform

$$\widetilde{T} := \Delta_{1/2,1/2,U}(T) := |T|^{1/2} U |T|^{1/2},$$

Also

$$T_q := \Delta_{0,q}(T) := U|T|^q,$$

which gives for $q = 1$ the usual polar decomposition $T = U|T|$.

For $p = t$, $q = 1 - t$, where $t \in [0,1]$ we have

$$\Delta_t(T) := \Delta_{t,1-t}(T) = |T|^t V |T|^{1-t}.$$

The transform $\Delta_t(T)$ was introduced and studied in [7].

If V is a contraction, then $\|V\| \leq 1$ and since $\|V^*\| = \|V\|$, hence V^* is also a contraction. Observe that

$$\Delta^*_{p,q,V}(T) := (|T|^p V |T|^q)^* = |T|^q V^* |T|^p = \Delta_{q,p,V^*}(T)$$

for all $p, q \geq 0$. Therefore

$$(T_{q,V})^* = \widehat{T}_{q,V^*}, \quad \left(\widehat{T}_{p,V}\right)^* = T_{p,V^*}$$

and

$$\left(\widetilde{T}_{p,V}\right)^* = \widetilde{T}_{p,V^*}.$$

Since $\|V^*V\| = \|VV^*\| = \|V\|^2$ and V is a contraction, then $\left\|\frac{V^*V \pm VV^*}{2}\right\| \leq \|V\|^2 \leq 1$ showing that $W := \frac{V^*V \pm VV^*}{2}$ is a contraction and we can consider the transform

$$\Delta_{p,q,\frac{V^*V \pm VV^*}{2}}(T) := |T|^p \left(\frac{V^*V \pm VV^*}{2}\right) |T|^q$$

for $p, q \geq 0$.

For a contraction V, we have

$$\operatorname{Im}(V) := \frac{V - V^*}{2i}, \quad \operatorname{Re}(V) := \operatorname{Re}\left(\frac{V + V^*}{2}\right)$$

and since

$$\|\operatorname{Im}(V)\| = \left\|\frac{V - V^*}{2i}\right\| \le \|V\| \le 1 \text{ and } \|\operatorname{Re}(V)\| \le \|V\| \le 1,$$

hence $\operatorname{Im}(V)$ and $\operatorname{Re}(V)$ are contractions as well. We can then consider the transforms

$$\Delta_{p,q,\operatorname{Im}(V)}(T) := |T|^p \operatorname{Im}(V) |T|^q \text{ and } \Delta_{p,q,\operatorname{Re}(V)}(T) := |T|^p \operatorname{Re}(V) |T|^q$$

for $p, q \ge 0$.

For $T \in \mathcal{B}(H)$ we define $T_+ := \frac{1}{2}(|T| + T)$ and $T_- := \frac{1}{2}(|T| - T)$. If U is the partial isometry in the polar representation of T, then $V := \frac{I \pm U}{2}$ is a contraction and we can consider

$$\Delta_{p,q,\frac{I \pm U}{2}}(T) := |T|^p \frac{I \pm U}{2} |T|^q = \frac{|T|^{p+q} \pm \Delta_{p,q}(T)}{2}.$$

In particular, we get

$$T_{q,\frac{I \pm U}{2}} = \frac{|T|^q \pm T_q}{2} = T_{\pm}, \widehat{T}_{p,\frac{I \pm U}{2}} = \frac{|T|^p \pm \widehat{T_p}}{2} \text{ and } \widetilde{T}_{p,\frac{I \pm U}{2}} = \frac{|T|^p \pm \widetilde{T_p}}{2}$$

for any operator $T \in \mathcal{B}(H)$.

3. Main Results

We use the following result:

Lemma 1. *Let A, $B \in \mathcal{B}(H)$, then*

$$\begin{aligned}
(3.1) \quad r(AB) &\le \frac{1}{4}(\omega(AB) + \omega(BA)) \\
&+ \frac{1}{4}\left[(\omega(AB) - \omega(BA))^2 + 4\min\{\|A\|\|BAB\|, \|B\|\|ABA\|\}\right]^{1/2} \\
&\le \frac{1}{2}\max\{\omega(AB), \omega(BA)\} \\
&+ \frac{1}{2}\min\left\{\|A\|^{1/2}\|BAB\|^{1/2}, \|B\|^{1/2}\|ABA\|^{1/2}\right\}.
\end{aligned}$$

Proof. The first part was obtained by Abu-Omar and Kittaneh in [2].

Using the elementary inequality $(c + d)^{1/2} \le c^{1/2} + d^{1/2}$ for $c, d \ge 0$, we have

$$\begin{aligned}
&\left[(\omega(AB) - \omega(BA))^2 + 4\min\{\|A\|\|BAB\|, \|B\|\|ABA\|\}\right]^{1/2} \\
&\le |\omega(AB) - \omega(BA)| + 2\min\left\{\|A\|^{1/2}\|BAB\|^{1/2}, \|B\|^{1/2}\|ABA\|^{1/2}\right\}.
\end{aligned}$$

Since

$$\max\{a, b\} = \frac{1}{2}(a + b + |a - b|),$$

hence

$$\begin{aligned}
&\frac{1}{4}\left(\omega\left(AB\right)+\omega\left(BA\right)\right)+\frac{1}{4}\left|\omega\left(AB\right)-\omega\left(BA\right)\right| \\
&+\frac{1}{2}\min\left\{\left\|A\right\|^{1/2}\left\|BAB\right\|^{1/2},\left\|B\right\|^{1/2}\left\|ABA\right\|^{1/2}\right\} \\
&=\frac{1}{2}\max\left\{\omega\left(AB\right),\omega\left(BA\right)\right\} \\
&+\frac{1}{2}\min\left\{\left\|A\right\|^{1/2}\left\|BAB\right\|^{1/2},\left\|B\right\|^{1/2}\left\|ABA\right\|^{1/2}\right\}
\end{aligned}$$

and the last part is also proved. □

Using the property of the spectral radius we have that

$$r\left(\Delta_{p,q,V}\left(T\right)\right)=r\left(T_{p+q,V}\right)=r\left(\widehat{T}_{p+q,V}\right)\text{ for all }p,\ q\geq 0.$$

Obviously, if $p+q=1$, p, $q\geq 0$ then

$$r\left(\Delta_{p,1-p,V}\left(T\right)\right)=r\left(T_V\right)=r\left(\widehat{T}_V\right)=r\left(\widetilde{T}_V\right)\text{ for all }p\in\left[0,1\right].$$

Our first main result is as follows:

Theorem 1. *Let V be a contraction and $T\in\mathcal{B}\left(H\right)$. Then for $q\geq r\geq 0$ and $p\geq s\geq 0$,*

$$\begin{aligned}
r\left(\Delta_{p,q,V}\left(T\right)\right)&\leq\frac{1}{4}\left(\omega\left(\Delta_{p,q,V}\left(T\right)\right)+\omega\left(\Delta_{p+r,q-r,V}\left(T\right)\right)\right) \\
&+\frac{1}{4}\left[\left(\omega\left(\Delta_{p,q,V}\left(T\right)\right)-\omega\left(\Delta_{p+r,q-r,V}\left(T\right)\right)\right)^2\right. \\
&\left.+4\left\|T\right\|^r\left\|\Delta_{p,q-r,V}\left(T\right)\right\|\left\|\Delta_{p+r,q-r,V}\left(T\right)\right\|\right]^{1/2} \\
&\leq\frac{1}{2}\max\left\{\omega\left(\Delta_{p,q,V}\left(T\right)\right),\omega\left(\Delta_{p+r,q-r,V}\left(T\right)\right)\right\} \\
&+\frac{1}{2}\left\|T\right\|^{r/2}\left\|\Delta_{p,q-r,V}\left(T\right)\right\|^{1/2}\left\|\Delta_{p+r,q-r,V}\left(T\right)\right\|^{1/2}
\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}
r\left(\Delta_{p,q,V}\left(T\right)\right)&\leq\frac{1}{4}\left(\omega\left(\Delta_{p,q,V}\left(T\right)\right)+\omega\left(\Delta_{s,q+p-s,V}\left(T\right)\right)\right) \\
&+\frac{1}{4}\left[\left(\omega\left(\Delta_{p,q,V}\left(T\right)\right)-\omega\left(\Delta_{s,q+p-s,V}\left(T\right)\right)\right)^2\right. \\
&\left.+4\left\|T\right\|^{p-s}\left\|\Delta_{s,q,V}\left(T\right)\right\|\left\|\Delta_{p,q,V}\left(T\right)\right\|\right]^{1/2} \\
&\leq\frac{1}{2}\max\left\{\omega\left(\Delta_{p,q,V}\left(T\right)\right),\omega\left(\Delta_{s,q+p-s,V}\left(T\right)\right)\right\} \\
&+\frac{1}{2}\left\|T\right\|^{(p-s)/2}\left\|\Delta_{s,q,V}\left(T\right)\right\|^{1/2}\left\|\Delta_{p,q,V}\left(T\right)\right\|^{1/2}.
\end{aligned}\tag{3.3}$$

Proof. If we take $A = |T|^p V |T|^{q-r}$ and $B = |T|^r$ then we get from (3.1) that

$$
\begin{aligned}
(3.4)\quad & r\left(|T|^p V |T|^q\right) \\
& \leq \frac{1}{4}\left(\omega\left(|T|^p V |T|^q\right) + \omega\left(|T|^{p+r} V |T|^{q-r}\right)\right) \\
& + \frac{1}{4}\left\{\left(\omega\left(|T|^p V |T|^q\right) - \omega\left(|T|^{p+r} V |T|^{q-r}\right)\right)^2\right. \\
& \left. +4 \min\left\{|T|^p V |T|^{q-r} \left\| |T|^{p+r} V |T|^q \right\|, \|T\|^r \left\| |T|^p V |T|^{q+p} V |T|^{q-r} \right\|\right\}\right\}^{1/2} \\
& \leq \frac{1}{2} \max\left\{\omega\left(|T|^p V |T|^q\right), \omega\left(|T|^{p+r} V |T|^{q-r}\right)\right\} \\
& + \frac{1}{2} \min\left\{\left\| |T|^p V |T|^{q-r} \right\|^{1/2} \left\| |T|^{p+r} V |T|^q \right\|^{1/2},\right. \\
& \left. \|T\|^{r/2} \left\| |T|^p V |T|^{q+p} V |T|^{q-r} \right\|^{1/2}\right\}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\left\| |T|^p V |T|^{q+p} V |T|^{q-r} \right\| &= \left\| |T|^p V |T|^{q-r+p-r} V |T|^{q-r} \right\| \\
&= \left\| |T|^p V |T|^{q-r} |T|^{p+r} V |T|^{q-r} \right\| \\
&\leq \left\| |T|^p V |T|^{q-r} \right\| \left\| |T|^{p+r} V |T|^{q-r} \right\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \min\left\{\left\| |T|^p V |T|^{q-r} \right\|^{1/2} \left\| |T|^{p+r} V |T|^q \right\|^{1/2},\right. \\
& \left. \|T\|^{r/2} \left\| |T|^p V |T|^{q+p} V |T|^{q-r} \right\|^{1/2}\right\} \\
& \leq \min\left\{\left\| |T|^p V |T|^{q-r} \right\|^{1/2} \left\| |T|^{p+r} V |T|^q \right\|^{1/2},\right. \\
& \left. \|T\|^{r/2} \left\| |T|^p V |T|^{q-r} \right\|^{1/2} \left\| |T|^{p+r} V |T|^{q-r} \right\|^{1/2}\right\} \\
& = \left\| |T|^p V |T|^{q-r} \right\|^{1/2} \\
& \times \min\left\{\left\| |T|^{p+r} V |T|^q \right\|^{1/2}, \|T\|^{r/2} \left\| |T|^{p+r} V |T|^{q-r} \right\|^{1/2}\right\}.
\end{aligned}
$$

By using (3.4) we get

$$r\left(|T|^{p}V|T|^{q}\right)\leq\frac{1}{4}\left(\omega\left(|T|^{p}V|T|^{q}\right)+\omega\left(|T|^{p+r}V|T|^{q-r}\right)\right)+\frac{1}{4}\left\{\left(\omega\left(|T|^{p}V|T|^{q}\right)-\omega\left(|T|^{p+r}V|T|^{q-r}\right)\right)^{2}+4\left\||T|^{p}V|T|^{q-r}\right\|\times\min\left\{\left\||T|^{p+r}V|T|^{q}\right\|,\|T\|^{r}\left\||T|^{p+r}V|T|^{q-r}\right\|\right\}\right\}^{1/2}$$
$$\leq\frac{1}{2}\max\left\{\omega\left(|T|^{p}V|T|^{q}\right),\omega\left(|T|^{p+r}V|T|^{q-r}\right)\right\}+\frac{1}{2}\left\||T|^{p}V|T|^{q-r}\right\|^{1/2}\times\min\left\{\left\||T|^{p+r}V|T|^{q}\right\|^{1/2},\|T\|^{r/2}\left\||T|^{p+r}V|T|^{q-r}\right\|^{1/2}\right\}. \tag{3.5}$$

Since

$$\left\||T|^{p+r}V|T|^{q}\right\|=\left\||T|^{p+r}V|T|^{q-r}|T|^{r}\right\|\leq\left\||T|^{p+r}V|T|^{q-r}\right\|\|T\|^{r}$$

hence by (3.5) we deduce the desired inequality (3.2).

Now, if we take $A=|T|^{p-s}$ and $B=|T|^{s}V|T|^{q}$ in (3.1), then we get

$$r\left(\Delta_{p,q,V}(T)\right)\leq\frac{1}{4}\left(\omega\left(\Delta_{p,q,V}(T)\right)+\omega\left(\Delta_{s,q+p-s,V}(T)\right)\right)+\frac{1}{4}\left[\left(\omega\left(\Delta_{p,q,V}(T)\right)-\omega\left(\Delta_{s,q+p-s,V}(T)\right)\right)^{2}+4\min\left\{\|T\|^{p-s}\left\||T|^{s}V|T|^{q+p}V|T|^{q}\right\|,\|\Delta_{s,q,V}(T)\|\left\||T|^{p}V|T|^{q+p-s}\right\|\right\}\right]^{1/2} \tag{3.6}$$

$$\leq\frac{1}{2}\max\left\{\omega\left(\Delta_{p,q,V}(T)\right),\omega\left(\Delta_{s,q+p-s,V}(T)\right)\right\}+\frac{1}{2}\min\left\{\|T\|^{(p-s)/2}\left\||T|^{s}V|T|^{q+p}V|T|^{q}\right\|^{1/2},\|\Delta_{s,q,V}(T)\|^{1/2}\left\||T|^{p}V|T|^{q+p-s}\right\|^{1/2}\right\}.$$

Now, observe that

$$\begin{aligned}\left\||T|^{s}V|T|^{q+p}V|T|^{q}\right\| &= \left\||T|^{s}V|T|^{q}|T|^{p}V|T|^{q}\right\|\\ &\leq \left\||T|^{s}V|T|^{q}\right\|\left\||T|^{p}V|T|^{q}\right\|\\ &= \|\Delta_{s,q,V}(T)\|\|\Delta_{p,q,V}(T)\|\end{aligned}$$

also

$$\left\||T|^{p}V|T|^{q+p-s}\right\|\leq\left\||T|^{p}V|T|^{q}|T|^{p-s}\right\|\leq\|\Delta_{p,q,V}(T)\|\|T\|^{p-s}.$$

Therefore

$$\begin{aligned}
&\min\left\{\|T\|^{p-s}\left\||T|^{s}V|T|^{q+p}V|T|^{q}\right\|,\|\Delta_{s,q,V}(T)\|\left\||T|^{p}V|T|^{q+p-s}\right\|\right\}\\
&\leq\min\left\{\|T\|^{p-s}\|\Delta_{s,q,V}(T)\|\|\Delta_{p,q,V}(T)\|,\|\Delta_{s,q,V}(T)\|\|\Delta_{p,q,V}(T)\|\|T\|^{p-s}\right\}\\
&=\|T\|^{p-s}\|\Delta_{s,q,V}(T)\|\|\Delta_{p,q,V}(T)\|
\end{aligned}$$

and by (3.6) we get (3.3). □

Corollary 1. *Let V be a contraction and $T\in\mathcal{B}(H)$. Then for $p\geq s\geq 0$,*

$$\begin{aligned}
r\left(\widetilde{T}_{p,V}\right)&\leq\frac{1}{4}\left(\omega\left(\widetilde{T}_{p,V}\right)+\omega\left(\Delta_{p+s,p-s,V}(T)\right)\right)\\
&+\frac{1}{4}\left[\left(\omega\left(\widetilde{T}_{p,V}\right)-\omega\left(\Delta_{p+s,p-s,V}(T)\right)\right)^{2}\right.\\
&\left.+4\|T\|^{s}\|\Delta_{p,p-s,V}(T)\|\|\Delta_{p+s,p-s,V}(T)\|\right]^{1/2}\\
&\leq\frac{1}{2}\max\left\{\omega\left(\widetilde{T}_{p,V}\right),\omega\left(\Delta_{p+s,p-s,V}(T)\right)\right\}\\
&+\frac{1}{2}\|T\|^{s/2}\|\Delta_{p,p-s,V}(T)\|^{1/2}\|\Delta_{p+s,p-s,V}(T)\|^{1/2}
\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}
r\left(\widetilde{T}_{p,V}\right)&\leq\frac{1}{4}\left(\omega\left(\widetilde{T}_{p,V}\right)+\omega\left(\Delta_{s,2p-s,V}(T)\right)\right)\\
&+\frac{1}{4}\left[\left(\omega\left(\widetilde{T}_{p,V}\right)-\omega\left(\Delta_{s,2p-s,V}(T)\right)\right)^{2}\right.\\
&\left.+4\|T\|^{p-s}\|\Delta_{s,p,V}(T)\|\left\|\widetilde{T}_{p,V}\right\|\right]^{1/2}\\
&\leq\frac{1}{2}\max\left\{\omega\left(\Delta\widetilde{T}_{p,V}\right),\omega\left(\Delta_{s,2p-s,V}(T)\right)\right\}\\
&+\frac{1}{2}\|T\|^{(p-s)/2}\|\Delta_{s,p,V}(T)\|^{1/2}\left\|\widetilde{T}_{p,V}\right\|^{1/2}.
\end{aligned}\tag{3.8}$$

Follows by Theorem 1 by taking $p=q\geq 0$ and $r=s$.

Now, if we take $p=1/2\geq s\geq 0$ in Corollary 1, then we get

$$\begin{aligned}
r(T_V)&\leq\frac{1}{4}\left(\omega\left(\widetilde{T}_{V}\right)+\omega\left(\Delta_{1/2+s,1/2-s,V}(T)\right)\right)\\
&+\frac{1}{4}\left[\left(\omega\left(\widetilde{T}_{V}\right)-\omega\left(\Delta_{1/2+s,1/2-s,V}(T)\right)\right)^{2}\right.\\
&\left.+4\|T\|^{s}\left\|\Delta_{1/2,1/2-s,V}(T)\right\|\left\|\Delta_{1/2+s,1/2-s,V}(T)\right\|\right]^{1/2}\\
&\leq\frac{1}{2}\max\left\{\omega\left(\widetilde{T}_{V}\right),\omega\left(\Delta_{1/2+s,1/2-s,V}(T)\right)\right\}\\
&+\frac{1}{2}\|T\|^{s/2}\left\|\Delta_{1/2,1/2-s,V}(T)\right\|^{1/2}\left\|\Delta_{1/2+s,1/2-s,V}(T)\right\|^{1/2}
\end{aligned}\tag{3.9}$$

and

$$r(T_V) \le \frac{1}{4}\left(\omega\left(\widetilde{T}_V\right) + \omega\left(\Delta_{s,1-s,V}(T)\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widetilde{T}_V\right) - \omega\left(\Delta_{s,1-s,V}(T)\right)\right)^2 + 4\|T\|^{1/2-s}\left\|\Delta_{s,1/2,V}(T)\right\|\left\|\widetilde{T}_V\right\|\right]^{1/2} \le \frac{1}{2}\max\left\{\omega\left(\widetilde{T}_V\right), \omega\left(\Delta_{s,1-s,V}(T)\right)\right\} + \frac{1}{2}\|T\|^{(1/2-s)/2}\left\|\Delta_{s,1/2,V}(T)\right\|^{1/2}\left\|\widetilde{T}_V\right\|^{1/2}. \tag{3.10}$$

If we choose $s = 1/2$ in (3.9), then we get

$$r(T_V) \le \frac{1}{4}\left(\omega\left(\widetilde{T}_V\right) + \omega\left(\widehat{T}_V\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widetilde{T}_V\right) - \omega\left(\widehat{T}_V\right)\right)^2 + 4\|T\|^{1/2}\left\|\widehat{T}_{1/2,V}\right\|\left\|\widehat{T}_V\right\|\right]^{1/2} \le \frac{1}{2}\max\left\{\omega\left(\widetilde{T}_V\right), \omega\left(\widehat{T}_V\right)\right\} + \frac{1}{2}\|T\|^{1/4}\left\|\widehat{T}_{1/2,V}\right\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2} \tag{3.11}$$

and from (3.10) for $s = 0$, that

$$r(T_V) \le \frac{1}{4}\left(\omega\left(\widetilde{T}_V\right) + \omega(T_V)\right) + \frac{1}{4}\left[\left(\omega\left(\widetilde{T}_V\right) - \omega(T_V)\right)^2 + 4\|T\|^{1/2}\left\|T_{1/2,V}\right\|\left\|\widetilde{T}_V\right\|\right]^{1/2} \le \frac{1}{2}\max\left\{\omega\left(\widetilde{T}_V\right), \omega(T_V)\right\} + \frac{1}{2}\|T\|^{1/4}\left\|T_{1/2,V}\right\|^{1/2}\left\|\widetilde{T}_V\right\|^{1/2}. \tag{3.12}$$

Corollary 2. *Let V be a contraction and $T \in \mathcal{B}(H)$. Then for $q \ge r \ge 0$,*

$$r(T_{q,V}) \le \frac{1}{4}\left(\omega(T_{q,V}) + \omega\left(\Delta_{r,q-r,V}(T)\right)\right) + \frac{1}{4}\left[\left(\omega(T_{q,V}) - \omega\left(\Delta_{r,q-r,V}(T)\right)\right)^2 + 4\|T\|^r\left\|T_{q-r,V}\right\|\left\|\Delta_{r,q-r,V}(T)\right\|\right]^{1/2} \le \frac{1}{2}\max\left\{\omega(T_{q,V}), \omega\left(\Delta_{r,q-r,V}(T)\right)\right\} + \frac{1}{2}\|T\|^{r/2}\left\|T_{q-r,V}\right\|^{1/2}\left\|\Delta_{r,q-r,V}(T)\right\|^{1/2}. \tag{3.13}$$

It follows by Theorem 1 by taking $p = 0$.

Now, if we take $q=1$ in (3.13) then we get

$$
\begin{aligned}
(3.14)\quad & r\left(T_V\right) \\
& \leq \frac{1}{4}\left(\omega\left(T_V\right)+\omega\left(\Delta_{r,1-r,V}(T)\right)\right) \\
& +\frac{1}{4}\left[\left(\omega\left(T_V\right)-\omega\left(\Delta_{r,1-r,V}(T)\right)\right)^2+4\|T\|^r\left\|T_{1-r,V}\right\|\left\|\Delta_{r,1-r,V}(T)\right\|\right]^{1/2} \\
& \leq \frac{1}{2}\max\left\{\omega\left(T_V\right),\omega\left(\Delta_{r,1-r,V}(T)\right)\right\} \\
& +\frac{1}{2}\|T\|^{r/2}\left\|T_{1-r,V}\right\|^{1/2}\left\|\Delta_{r,1-r,V}(T)\right\|^{1/2}.
\end{aligned}
$$

for all $r\in[0,1]$.

If we consider $r=1$ in the inequality (3.14), then we obtain

$$
\begin{aligned}
(3.15)\quad r\left(T_V\right) & \leq \frac{1}{4}\left(\omega\left(T_V\right)+\omega\left(\widehat{T}_V\right)\right) \\
& +\frac{1}{4}\left[\left(\omega\left(T_V\right)-\omega\left(\widehat{T}_V\right)\right)^2+4\|T\|\left\|T_V\right\|\left\|\widehat{T}_V\right\|\right]^{1/2} \\
& \leq \frac{1}{2}\max\left\{\omega\left(T_V\right),\omega\left(\widehat{T}_V\right)\right\}+\frac{1}{2}\|T\|^{1/2}\left\|T_V\right\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2}.
\end{aligned}
$$

From the same inequality (3.14) for $r=1/2$, we derive

$$
\begin{aligned}
(3.16)\quad r\left(T_V\right) & \leq \frac{1}{4}\left(\omega\left(T_V\right)+\omega\left(\widetilde{T}_V\right)\right) \\
& +\frac{1}{4}\left[\left(\omega\left(T_V\right)-\omega\left(\widetilde{T}_V\right)\right)^2+4\|T\|^{1/2}\left\|T_{1/2,V}\right\|\left\|\widetilde{T}_V\right\|\right]^{1/2} \\
& \leq \frac{1}{2}\max\left\{\omega\left(T_V\right),\omega\left(\widetilde{T}_V\right)\right\}+\frac{1}{2}\|T\|^{1/4}\left\|T_{1/2,V}\right\|^{1/2}\left\|\widetilde{T}_V\right\|^{1/2}.
\end{aligned}
$$

We also have:

Corollary 3. *Let V be a contraction and $T\in\mathcal{B}(H)$. Then for $p\geq s\geq 0$,*

$$
\begin{aligned}
(3.17)\quad & r\left(\widehat{T}_{p,V}\right) \\
& \leq \frac{1}{4}\left(\omega\left(\widehat{T}_{p,V}\right)+\omega\left(\Delta_{s,p-s,V}(T)\right)\right) \\
& +\frac{1}{4}\left[\left(\omega\left(\widehat{T}_{p,V}\right)-\omega\left(\Delta_{s,p-s,V}(T)\right)\right)^2+4\|T\|^{p-s}\left\|\widehat{T}_{s,V}\right\|\left\|\widehat{T}_{p,V}\right\|\right]^{1/2} \\
& \leq \frac{1}{2}\max\left\{\omega\left(\widehat{T}_{p,V}\right),\omega\left(\Delta_{s,p-s,V}(T)\right)\right\} \\
& +\frac{1}{2}\|T\|^{(p-s)/2}\left\|\widehat{T}_{s,V}\right\|^{1/2}\left\|\widehat{T}_{p,V}\right\|^{1/2}.
\end{aligned}
$$

It follows by Theorem 1 by taking $q=0$.

Further, if we take $p=1$ in (3.17), then we get

$$r\left(T_V\right) \leq \frac{1}{4}\left(\omega\left(\widehat{T}_V\right)+\omega\left(\Delta_{s,1-s,V}\left(T\right)\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widehat{T}_V\right)-\omega\left(\Delta_{s,1-s,V}\left(T\right)\right)\right)^2+4\left\|T\right\|^{1-s}\left\|\widehat{T}_{s,V}\right\|\left\|\widehat{T}_V\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(\widehat{T}_V\right),\omega\left(\Delta_{s,1-s,V}\left(T\right)\right)\right\} + \frac{1}{2}\left\|T\right\|^{(1-s)/2}\left\|\widehat{T}_{s,V}\right\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2} \tag{3.18}$$

for all $s\in[0,1]$.

Now, for $s=0$ in (3.18) we obtain

$$r\left(T_V\right) \leq \frac{1}{4}\left(\omega\left(\widehat{T}_V\right)+\omega\left(T_V\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widehat{T}_V\right)-\omega\left(T_V\right)\right)^2+4\left\|T\right\|\left\|V\right\|\left\|\widehat{T}_V\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(\widehat{T}_V\right),\omega\left(T_V\right)\right\}+\frac{1}{2}\left\|T\right\|^{1/2}\left\|V\right\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2}, \tag{3.19}$$

while for $s=1/2$ we derive

$$r\left(T_V\right) \leq \frac{1}{4}\left(\omega\left(\widehat{T}_V\right)+\omega\left(\widetilde{T}_V\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widehat{T}_V\right)-\omega\left(\widetilde{T}_V\right)\right)^2+4\left\|T\right\|^{1/2}\left\|\widehat{T}_{1/2,V}\right\|\left\|\widehat{T}_V\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(\widehat{T}_V\right),\omega\left(\widetilde{T}_V\right)\right\}+\frac{1}{2}\left\|T\right\|^{1/4}\left\|\widehat{T}_{1/2,V}\right\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2}. \tag{3.20}$$

Remark 1. *If we take $r=q$ in (3.2), then we get*

$$r\left(\Delta_{p,q,V}\left(T\right)\right) \leq \frac{1}{4}\left(\omega\left(\Delta_{p,q,V}\left(T\right)\right)+\omega\left(\widehat{T}_{p+q,V}\right)\right) + \frac{1}{4}\left[\left(\omega\left(\Delta_{p,q,V}\left(T\right)\right)-\omega\left(\widehat{T}_{p+q,V}\right)\right)^2+4\left\|T\right\|^{q}\left\|\widehat{T}_{p,V}\right\|\left\|\widehat{T}_{p+q,V}\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(\Delta_{p,q,V}\left(T\right)\right),\omega\left(\widehat{T}_{p+q,V}\right)\right\}+\frac{1}{2}\left\|T\right\|^{q/2}\left\|\widehat{T}_{p,V}\right\|^{1/2}\left\|\widehat{T}_{p+q,V}\right\|^{1/2} \tag{3.21}$$

for all $p,q\geq 0$.

If we choose in (3.21) $p=0$, then we obtain

$$r\left(T_{q,V}\right) \leq \frac{1}{4}\left(\omega\left(T_{q,V}\right)+\omega\left(\widehat{T}_{q,V}\right)\right) + \frac{1}{4}\left[\left(\omega\left(T_{q,V}\right)-\omega\left(\widehat{T}_{q,V}\right)\right)^2+4\left\|T\right\|^{q}\left\|V\right\|\left\|\widehat{T}_{q,V}\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(T_{q,V}\right),\omega\left(\widehat{T}_{q,V}\right)\right\}+\frac{1}{2}\left\|T\right\|^{q/2}\left\|V\right\|^{1/2}\left\|\widehat{T}_{q,V}\right\|^{1/2} \tag{3.22}$$

for all $q\geq 0$.

If we take $q=1$ in (3.10), then we get

$$r\left(T_V\right) \leq \frac{1}{4}\left(\omega\left(T_V\right)+\omega\left(\widehat{T}_V\right)\right) + \frac{1}{4}\left[\left(\omega\left(T_V\right)-\omega\left(\widehat{T}_V\right)\right)^2+4\|T\|\|V\|\left\|\widehat{T}_V\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(T_V\right),\omega\left(\widehat{T}_V\right)\right\}+\frac{1}{2}\|T\|^{1/2}\|V\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2}. \tag{3.23}$$

If we take in (3.21) $p=q$, then we get

$$r\left(\widetilde{T}_{p,V}\right) \leq \frac{1}{4}\left(\omega\left(\widetilde{T}_{p,V}\right)+\omega\left(\widehat{T}_{2p,V}\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widetilde{T}_{p,V}\right)-\omega\left(\widehat{T}_{2p,V}\right)\right)^2+4\|T\|^p\left\|\widehat{T}_{p,V}\right\|\left\|\widehat{T}_{2p,V}\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(\widetilde{T}_{p,V}\right),\omega\left(\widehat{T}_{2p,V}\right)\right\}+\frac{1}{2}\|T\|^{p/2}\left\|\widehat{T}_{p,V}\right\|^{1/2}\left\|\widehat{T}_{2p,V}\right\|^{1/2}, \tag{3.24}$$

which, for $p=1/2$, gives

$$r\left(T_V\right) \leq \frac{1}{4}\left(\omega\left(\widetilde{T}_V\right)+\omega\left(\widehat{T}_V\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widetilde{T}_V\right)-\omega\left(\widehat{T}_V\right)\right)^2+4\|T\|^{1/2}\left\|\widehat{T}_{1/2,V}\right\|\left\|\widehat{T}_V\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(\widetilde{T}_V\right),\omega\left(\widehat{T}_V\right)\right\}+\frac{1}{2}\|T\|^{1/4}\left\|\widehat{T}_{1/2,V}\right\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2}. \tag{3.25}$$

If we take $s=0$ in (3.3), then we get

$$r\left(\Delta_{p,q,V}(T)\right) \leq \frac{1}{4}\left(\omega\left(\Delta_{p,q,V}(T)\right)+\omega\left(T_{p+q,V}\right)\right) + \frac{1}{4}\left[\left(\omega\left(\Delta_{p,q,V}(T)\right)-\omega\left(T_{p+q,V}\right)\right)^2+4\|T\|^p\left\|T_{q,V}\right\|\left\|\Delta_{p,q,V}(T)\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(\Delta_{p,q,V}(T)\right),\omega\left(T_{p+q,V}\right)\right\} + \frac{1}{2}\|T\|^{p/2}\left\|T_{q,V}\right\|^{1/2}\left\|\Delta_{p,q,V}(T)\right\|^{1/2} \tag{3.26}$$

for all p, $q\geq 0$.

If we choose $q=0$ in (3.26), then we obtain

$$r\left(\widehat{T}_{p,V}\right) \leq \frac{1}{4}\left(\omega\left(\widehat{T}_{p,V}\right)+\omega\left(T_{p,V}\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widehat{T}_{p,V}\right)-\omega\left(T_{p,V}\right)\right)^2+4\|T\|^p\|V\|\left\|\widehat{T}_{p,V}\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(\widehat{T}_{p,V}\right),\omega\left(T_{p,V}\right)\right\}+\frac{1}{2}\|T\|^{p/2}\|V\|^{1/2}\left\|\widehat{T_{p,V}}\right\|^{1/2} \tag{3.27}$$

for all $p \geq 0$. If in this inequality we take $p=1$, then we get

$$r\left(\widehat{T}_V\right) \leq \frac{1}{4}\left(\omega\left(\widehat{T}_V\right)+\omega\left(T_V\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widehat{T}_V\right)-\omega\left(T_V\right)\right)^2+4\|T\|^p\|V\|\left\|\widehat{T}_V\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(\widehat{T}_V\right),\omega\left(T_V\right)\right\}+\frac{1}{2}\|T\|^{p/2}\|V\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2}. \tag{3.28}$$

Moreover, if we take in (3.26) $q=p$, then we derive

$$r\left(\widetilde{T}_{p,V}\right) \leq \frac{1}{4}\left(\omega\left(\widetilde{T}_{p,V}\right)+\omega\left(T_{2p,V}\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widetilde{T}_{p,V}\right)-\omega\left(T_{2p,V}\right)\right)^2+4\|T\|^p\|T_{p,V}\|\left\|\widetilde{T}_{p,V}\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(\widetilde{T}_{p,V}\right),\omega\left(T_{2p,V}\right)\right\}+\frac{1}{2}\|T\|^{p/2}\|T_{p,V}\|^{1/2}\left\|\widetilde{T}_{p,V}\right\|^{1/2} \tag{3.29}$$

for all $p \geq 0$, which for $p=1/2$ produces

$$r\left(T_V\right) \leq \frac{1}{4}\left(\omega\left(\widetilde{T}_V\right)+\omega\left(T_V\right)\right) + \frac{1}{4}\left[\left(\omega\left(\widetilde{T}_V\right)-\omega\left(T_V\right)\right)^2+4\|T\|^{1/2}\|T_{1/2,V}\|\left\|\widetilde{T}_V\right\|\right]^{1/2} \leq \frac{1}{2}\max\left\{\omega\left(\widetilde{T}_V\right),\omega\left(T_V\right)\right\}+\frac{1}{2}\|T\|^{1/4}\|T_{1/2,V}\|^{1/2}\left\|\widetilde{T}_V\right\|^{1/2}. \tag{3.30}$$

4. Related Results

Recall the following result obtained in [2]:

Lemma 2. *Let A, $B \in \mathcal{B}(H)$, then*

$$r(AB \pm BA) \leq \omega(AB)+\min\left\{\|A\|^{1/2}\left\|AB^2\right\|^{1/2},\|B\|^{1/2}\left\|A^2B\right\|^{1/2}\right\} \tag{4.1}$$

and

$$r(AB \pm BA) \leq \omega(BA)+\min\left\{\|A\|^{1/2}\left\|B^2A\right\|^{1/2},\|B\|^{1/2}\left\|BA^2\right\|^{1/2}\right\}. \tag{4.2}$$

We can state the following result:

Theorem 2. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ we have for $p \geq 0$ and $q \geq r \geq 0$, that*

$$r\left(\Delta_{p,q,V}(T) \pm \Delta_{p+r,q-r,V}(T)\right) \leq \omega\left(\Delta_{p,q,V}(T)\right)+\|\Delta_{p,q-r,V}(T)\|^{1/2}\|\Delta_{p,q+r,V}(T)\|^{1/2} \leq \omega\left(\Delta_{p,q,V}(T)\right)+\|T\|^{r/2}\|\Delta_{p,q-r,V}(T)\|^{1/2}\|\Delta_{p,q,V}(T)\|^{1/2} \tag{4.3}$$

and

$$r\left(\Delta_{p,q,V}(T) \pm \Delta_{p+r,q-r,V}(T)\right) \leq \omega\left(\Delta_{p+r,q-r,V}(T)\right)+\|\Delta_{p,q-r,V}(T)\|^{1/2}\|\Delta_{p+2r,q-r,V}(T)\|^{1/2} \leq \omega\left(\Delta_{p+r,q-r,V}(T)\right)+\|T\|^{r/2}\|\Delta_{p,q-r,V}(T)\|^{1/2}\|\Delta_{p+r,q-r,V}(T)\|^{1/2}. \tag{4.4}$$

Proof. If we take $A = |T|^p V |T|^{q-r}$ and $B = |T|^r$ then we get from (4.1) that

$$\begin{aligned}
& r\left(|T|^p V |T|^q \pm |T|^{p+r} V |T|^{q-r}\right) \\
& \leq \omega\left(|T|^p V |T|^q\right) \\
& + \min\left\{\left\||T|^p V |T|^{q-r}\right\|^{1/2} \left\||T|^p V |T|^q |T|^r\right\|^{1/2},\right. \\
& \left.\|T\|^{r/2} \left\||T|^p V |T|^{q-r} |T|^p V |T|^q\right\|^{1/2}\right\} \\
& \leq \omega\left(|T|^p V |T|^q\right) \\
& + \min\left\{\left\||T|^p V |T|^{q-r}\right\|^{1/2} \left\||T|^p V |T|^{q+r}\right\|^{1/2},\right. \\
& \left.\|T\|^{r/2} \left\||T|^p V |T|^{q-r}\right\|^{1/2} \left\||T|^p V |T|^q\right\|^{1/2}\right\} \\
& = \omega\left(|T|^p V |T|^q\right) \\
& + \left\||T|^p V |T|^{q-r}\right\|^{1/2} \min\left\{\left\||T|^p V |T|^{q+r}\right\|^{1/2}, \|T\|^{r/2} \left\||T|^p V |T|^q\right\|^{1/2}\right\} \\
& = \omega\left(|T|^p V |T|^q\right) + \left\||T|^p V |T|^{q-r}\right\|^{1/2} \left\||T|^p V |T|^{q+r}\right\|^{1/2} \\
& \leq \omega\left(|T|^p V |T|^q\right) + \|T\|^{r/2} \left\||T|^p V |T|^{q-r}\right\|^{1/2} \left\||T|^p V |T|^q\right\|^{1/2}
\end{aligned}$$

since

$$\left\||T|^p V |T|^{q+r}\right\|^{1/2} \leq \|T\|^{r/2} \left\||T|^p V |T|^q\right\|^{1/2}$$

and the inequality (4.3) is proved.

If we use the same choice of A and B in the inequality (4.2), then we also have

$$\begin{aligned}
& r\left(|T|^p V |T|^q \pm |T|^{p+r} V |T|^{q-r}\right) \\
& \leq \omega\left(|T|^{p+r} V |T|^{q-r}\right) \\
& + \min\left\{\left\||T|^p V |T|^{q-r}\right\|^{1/2} \left\||T|^r |T|^{p+r} V |T|^{q-r}\right\|^{1/2},\right. \\
& \left.\|T\|^{r/2} \left\||T|^{p+r} V |T|^{q-r} |T|^p V |T|^{q-r}\right\|^{1/2}\right\} \\
& \leq \omega\left(|T|^{p+r} V |T|^{q-r}\right) \\
& + \min\left\{\left\||T|^p V |T|^{q-r}\right\|^{1/2} \left\||T|^r |T|^{p+r} V |T|^{q-r}\right\|^{1/2},\right. \\
& \left.\|T\|^{r/2} \left\||T|^{p+r} V |T|^{q-r}\right\|^{1/2} \left\||T|^p V |T|^{q-r}\right\|^{1/2}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \omega\left(|T|^{p+r} V |T|^{q-r}\right) \\
&+ \left\| |T|^{p} V |T|^{q-r} \right\|^{1/2} \\
&\times \min\left\{ \left\| |T|^{r} |T|^{p+r} V |T|^{q-r} \right\|^{1/2}, \|T\|^{r/2} \left\| |T|^{p+r} V |T|^{q-r} \right\|^{1/2} \right\} \\
&= \omega\left(|T|^{p+r} V |T|^{q-r}\right) + \left\| |T|^{p} V |T|^{q-r} \right\|^{1/2} \left\| |T|^{p+2r} V |T|^{q-r} \right\|^{1/2} \\
&\leq \omega\left(|T|^{p+r} V |T|^{q-r}\right) + \left\| |T|^{p} V |T|^{q-r} \right\|^{1/2} \|T\|^{r/2} \left\| |T|^{p+r} V |T|^{q-r} \right\|^{1/2}
\end{aligned}
$$

since

$$
\left\| |T|^{r} |T|^{p+r} V |T|^{q-r} \right\|^{1/2} \leq \|T\|^{r/2} \left\| |T|^{p+r} V |T|^{q-r} \right\|^{1/2},
$$

which proves (4.4). □

Corollary 4. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ we have for $q \geq r \geq 0$, that*

$$
\begin{aligned}
r\left(T_{q,V} \pm \Delta_{r,q-r,V}(T)\right) &\leq \omega\left(T_{q,V}\right) + \left\|T_{q-r,V}\right\|^{1/2} \left\|\Delta T_{q+r,V}\right\|^{1/2} \\
&\leq \omega\left(T_{q,V}\right) + \|T\|^{r/2} \left\|T_{q-r,V}\right\|^{1/2} \left\|T_{q,V}\right\|^{1/2}
\end{aligned} \tag{4.5}
$$

and

$$
\begin{aligned}
& r\left(T_{q,V} \pm \Delta_{r,q-r,V}(T)\right) \\
&\leq \omega\left(\Delta_{r,q-r,V}(T)\right) + \left\|T_{q-r,V}\right\|^{1/2} \left\|\Delta_{2r,q-r,V}(T)\right\|^{1/2} \\
&\leq \omega\left(\Delta_{r,q-r,V}(T)\right) + \|T\|^{r/2} \left\|T_{q-r,V}\right\|^{1/2} \left\|\Delta_{r,q-r,V}(T)\right\|^{1/2}.
\end{aligned} \tag{4.6}
$$

It follows by Theorem 2 by taking $p = 0$.

If we take in Corollary 4 $q = 1$, then we get

$$
\begin{aligned}
r\left(T_{V} \pm \Delta_{r,1-r,V}(T)\right) &\leq \omega\left(T_{V}\right) + \left\|T_{1-r,V}\right\|^{1/2} \left\|\Delta T_{1+r,V}\right\|^{1/2} \\
&\leq \omega\left(T_{V}\right) + \|T\|^{r/2} \left\|T_{1-r,V}\right\|^{1/2} \left\|T_{V}\right\|^{1/2}
\end{aligned} \tag{4.7}
$$

and

$$
\begin{aligned}
& r\left(T_{V} \pm \Delta_{r,1-r,V}(T)\right) \\
&\leq \omega\left(\Delta_{r,1-r,V}(T)\right) + \left\|T_{1-r,V}\right\|^{1/2} \left\|\Delta_{2r,1-r,V}(T)\right\|^{1/2} \\
&\leq \omega\left(\Delta_{r,1-r,V}(T)\right) + \|T\|^{r/2} \left\|T_{1-r,V}\right\|^{1/2} \left\|\Delta_{r,1-r,V}(T)\right\|^{1/2}
\end{aligned} \tag{4.8}
$$

for $r \in [0,1]$.

If in (4.7) we get for $r = 1/2$ that

$$
\begin{aligned}
r\left(T_{V} \pm \widetilde{T}_{V}\right) &\leq \omega\left(T_{V}\right) + \left\|T_{1/2,V}\right\|^{1/2} \left\|\Delta T_{3/2,V}\right\|^{1/2} \\
&\leq \omega\left(T_{V}\right) + \|T\|^{1/4} \left\|T_{1/2,V}\right\|^{1/2} \left\|T_{V}\right\|^{1/2}
\end{aligned} \tag{4.9}
$$

while from (4.8)

$$
\begin{aligned}
r\left(T_{V} \pm \widetilde{T}_{V}\right) &\leq \omega\left(\widetilde{T}_{V}\right) + \left\|T_{1/2,V}\right\|^{1/2} \left\|\Delta_{1,1/2,V}(T)\right\|^{1/2} \\
&\leq \omega\left(\widetilde{T}_{V}\right) + \|T\|^{1/4} \left\|T_{1/2,V}\right\|^{1/2} \left\|\widetilde{T}_{V}\right\|^{1/2}.
\end{aligned} \tag{4.10}
$$

Corollary 5. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ we have for $q \geq r \geq 0$, that*

$$\begin{aligned} r\left(\widetilde{T}_{q,V} \pm \Delta_{q+r,q-r,V}(T)\right) &\leq \omega\left(\widetilde{T}_{q,V}\right) + \|\Delta_{q,q-r,V}(T)\|^{1/2} \|\Delta_{q,q+r,V}(T)\|^{1/2} \\ &\leq \omega\left(\widetilde{T}_{q,V}\right) + \|T\|^{r/2} \|\Delta_{q,q-r,V}(T)\|^{1/2} \left\|\widetilde{T}_{q,V}\right\|^{1/2} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} r\left(\widetilde{T}_{q,V} \pm \Delta_{q+r,q-r,V}(T)\right) &\leq \omega(\Delta_{q+r,q-r,V}(T)) + \|\Delta_{q,q-r,V}(T)\|^{1/2} \|\Delta_{q+2r,q-r,V}(T)\|^{1/2} \\ &\leq \omega(\Delta_{q+r,q-r,V}(T)) + \|T\|^{r/2} \|\Delta_{q,q-r,V}(T)\|^{1/2} \|\Delta_{q+r,q-r,V}(T)\|^{1/2}. \end{aligned} \tag{4.12}$$

If we take $q = r$ in Corollary 5, then we get

$$\begin{aligned} r\left(\widetilde{T}_{q,V} \pm \widehat{T}_{2q,V}\right) &\leq \omega\left(\widetilde{T}_{q,V}\right) + \left\|\widehat{T}_{q,V}\right\|^{1/2} \|\Delta_{q,2q,V}(T)\|^{1/2} \\ &\leq \omega\left(\widetilde{T}_{q,V}\right) + \|T\|^{q/2} \left\|\widehat{T}_{q,V}\right\|^{1/2} \left\|\widetilde{T}_{q,V}\right\|^{1/2} \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} r\left(\widetilde{T}_{q,V} \pm \widehat{T}_{2q,V}\right) &\leq \omega\left(\widehat{T}_{2q,V}\right) + \left\|\widehat{T}_{q,V}\right\|^{1/2} \left\|\widehat{T}_{3q,V}\right\|^{1/2} \\ &\leq \omega\left(\widehat{T}_{2q,V}\right) + \|T\|^{q/2} \left\|\widehat{T}_{q,V}\right\|^{1/2} \left\|\widehat{T}_{2q,V}\right\|^{1/2}. \end{aligned} \tag{4.14}$$

If we take $q = 1/2$ in (4.13) and (4.14), then we get

$$\begin{aligned} r\left(\widetilde{T}_{V} \pm \widehat{T}_{V}\right) &\leq \omega\left(\widetilde{T}_{V}\right) + \left\|\widehat{T}_{1/2,V}\right\|^{1/2} \|\Delta_{1/2,1,V}(T)\|^{1/2} \\ &\leq \omega\left(\widetilde{T}_{V}\right) + \|T\|^{1/4} \left\|\widehat{T}_{1/2,V}\right\|^{1/2} \left\|\widetilde{T}_{V}\right\|^{1/2} \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} r\left(\widetilde{T}_{V} \pm \widehat{T}_{V}\right) &\leq \omega\left(\widehat{T}_{V}\right) + \left\|\widehat{T}_{1/2,V}\right\|^{1/2} \left\|\widehat{T}_{3/2,V}\right\|^{1/2} \\ &\leq \omega\left(\widehat{T}_{V}\right) + \|T\|^{1/4} \left\|\widehat{T}_{1/2,V}\right\|^{1/2} \left\|\widehat{T}_{V}\right\|^{1/2}. \end{aligned} \tag{4.16}$$

We can state the following result as well:

Theorem 3. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ we have for $p \geq s \geq 0$ and $q \geq 0$, that*

$$\begin{aligned} r(\Delta_{p,q,V}(T) \pm \Delta_{p-s,q+s,V}(T)) &\leq \omega(\Delta_{p,q,V}(T)) + \|\Delta_{p-s,q,V}(T)\|^{1/2} \|\Delta_{p+s,q,V}(T)\|^{1/2} \\ &\leq \omega(\Delta_{p,q,V}(T)) + \|T\|^{s/2} \|\Delta_{p-s,q,V}(T)\|^{1/2} \|\Delta_{p,q,V}(T)\|^{1/2} \end{aligned} \tag{4.17}$$

and

$$r\left(\Delta_{p,q,V}\left(T\right)\pm\Delta_{p-s,q+s,V}\left(T\right)\right) \tag{4.18}$$
$$\leq\omega\left(\Delta_{p-s,q+s,V}\left(T\right)\right)+\left\|\Delta_{p-s,q+s,V}\left(T\right)\right\|^{1/2}\left\|\Delta_{p-s,q+2s,V}\left(T\right)\right\|^{1/2}$$
$$\leq\omega\left(\Delta_{p-s,q+s,V}\left(T\right)\right)+\left\|T\right\|^{s/2}\left\|\Delta_{p-s,q+s,V}\left(T\right)\right\|^{1/2}\left\|\Delta_{p-s,q,V}\left(T\right)\right\|^{1/2}.$$

Proof. If we take $A=\left|T\right|^{s}$ and $B=\left|T\right|^{p-s}V\left|T\right|^{q}$, then we get from (4.1) that

$$\begin{aligned}
&r\left(\Delta_{p,q,V}\left(T\right)\pm\Delta_{p-s,q+s,V}\left(T\right)\right)\\
&\leq\omega\left(\Delta_{p,q,V}\left(T\right)\right)\\
&+\min\left\{\left\|T\right\|^{s/2}\left\|\left|T\right|^{p}V\left|T\right|^{q}\left|T\right|^{p-s}V\left|T\right|^{q}\right\|^{1/2},\right.\\
&\left.\left\|\left|T\right|^{p-s}V\left|T\right|^{q}\right\|^{1/2}\left\|\left|T\right|^{p+s}V\left|T\right|^{q}\right\|^{1/2}\right\}\\
&\leq\omega\left(\Delta_{p,q,V}\left(T\right)\right)\\
&+\min\left\{\left\|T\right\|^{s/2}\left\|\left|T\right|^{p}V\left|T\right|^{q}\right\|^{1/2}\left\|\left|T\right|^{p-s}V\left|T\right|^{q}\right\|^{1/2},\left\|\left|T\right|^{p+s}V\left|T\right|^{q}\right\|^{1/2}\right\}\\
&=\omega\left(\Delta_{p,q,V}\left(T\right)\right)\\
&+\left\|\left|T\right|^{p-s}V\left|T\right|^{q}\right\|^{1/2}\min\left\{\left\|T\right\|^{s/2}\left\|\left|T\right|^{p}V\left|T\right|^{q}\right\|^{1/2},\left\|\left|T\right|^{p+s}V\left|T\right|^{q}\right\|^{1/2}\right\}\\
&=\omega\left(\Delta_{p,q,V}\left(T\right)\right)+\left\|\left|T\right|^{p-s}V\left|T\right|^{q}\right\|^{1/2}\left\|\left|T\right|^{p+s}V\left|T\right|^{q}\right\|^{1/2}
\end{aligned}$$

and since

$$\left\|\left|T\right|^{p+s}V\left|T\right|^{q}\right\|^{1/2}\leq\left\|T\right\|^{s/2}\left\|\left|T\right|^{p}V\left|T\right|^{q}\right\|^{1/2},$$

then the inequality (4.17) is proved.

Further on, if we use the same choice of A and B in (4.2), then we also have

$$\begin{aligned}
&r\left(\Delta_{p,q,V}\left(T\right)\pm\Delta_{p-s,q+s,V}\left(T\right)\right)\\
&\leq\omega\left(\Delta_{p-s,q+s,V}\left(T\right)\right)\\
&+\min\left\{\left\|\left|T\right|^{s}\right\|^{1/2}\left\|\left|T\right|^{p-s}V\left|T\right|^{q}\left|T\right|^{p-s}V\left|T\right|^{q}\left|T\right|^{s}\right\|^{1/2},\right.\\
&\left.\left\|\left|T\right|^{p-s}V\left|T\right|^{q}\right\|^{1/2}\left\|\left|T\right|^{p-s}V\left|T\right|^{q}\left|T\right|^{2s}\right\|^{1/2}\right\}\\
&\leq\omega\left(\Delta_{p-s,q+s,V}\left(T\right)\right)\\
&+\min\left\{\left\|T\right\|^{s/2}\left\|\left|T\right|^{p-s}V\left|T\right|^{q}\right\|^{1/2}\left\|\left|T\right|^{p-s}V\left|T\right|^{q+s}\right\|^{1/2},\right.\\
&\left.\left\|\left|T\right|^{p-s}V\left|T\right|^{q}\right\|^{1/2}\left\|\left|T\right|^{p-s}V\left|T\right|^{q+2s}\right\|^{1/2}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \omega\left(\Delta_{p-s,q+s,V}(T)\right) \\
&+ \left\| |T|^{p-s} V |T|^{q+s} \right\|^{1/2} \\
&\times \min\left\{ \|T\|^{s/2} \left\| |T|^{p-s} V |T|^{q} \right\|^{1/2}, \left\| |T|^{p-s} V |T|^{q+2s} \right\|^{1/2} \right\} \\
&= \omega\left(\Delta_{p-s,q+s,V}(T)\right) + \left\| |T|^{p-s} V |T|^{q+s} \right\|^{1/2} \left\| |T|^{p-s} V |T|^{q+2s} \right\|^{1/2}
\end{aligned}
$$

and since

$$\left\| |T|^{p-s} V |T|^{q+2s} \right\|^{1/2} \le \|T\|^{s/2} \left\| |T|^{p-s} V |T|^{q} \right\|^{1/2},$$

hence the desired inequality (4.18) is thus proved. □

Corollary 6. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ we have for $p \ge s \ge 0$, that*

$$
\begin{aligned}
r\left(\widehat{T}_{p,V} \pm \Delta_{p-s,s,V}(T)\right) &\le \omega\left(\widehat{T}_{p,V}\right) + \left\|\widehat{T}_{p-s,V}\right\|^{1/2} \left\|\widehat{T}_{p+s,V}\right\|^{1/2} \\
&\le \omega\left(\widehat{T}_{p,V}\right) + \|T\|^{s/2} \left\|\widehat{T}_{p-s,V}\right\|^{1/2} \left\|\widehat{T}_{p,V}\right\|^{1/2}
\end{aligned} \tag{4.19}
$$

and

$$
\begin{aligned}
&r\left(\widehat{T}_{p,V} \pm \Delta_{p-s,s,V}(T)\right) \\
&\le \omega\left(\Delta_{p-s,s,V}(T)\right) + \left\|\Delta_{p-s,s,V}(T)\right\|^{1/2} \left\|\Delta_{p-s,2s,V}(T)\right\|^{1/2} \\
&\le \omega\left(\Delta_{p-s,s,V}(T)\right) + \|T\|^{s/2} \left\|\Delta_{p-s,s,V}(T)\right\|^{1/2} \left\|\widehat{T}_{p-s,V}\right\|^{1/2}.
\end{aligned} \tag{4.20}
$$

It follows by taking $q = 0$ in Theorem 3.

If we take $s = p$ in (4.19), then we get

$$
\begin{aligned}
r\left(\widehat{T}_{p,V} \pm T_{p,V}\right) &\le \omega\left(\widehat{T}_{p,V}\right) + \|V\|^{1/2} \left\|\widehat{T}_{2p,V}\right\|^{1/2} \\
&\le \omega\left(\widehat{T}_{p,V}\right) + \|V\|^{1/2} \|T\|^{p/2} \left\|\widehat{T}_{p,V}\right\|^{1/2},
\end{aligned} \tag{4.21}
$$

while from (4.20)

$$
\begin{aligned}
r\left(\widehat{T}_{p,V} \pm T_{p,V}\right) &\le \omega\left(T_{p,V}\right) + \left\|T_{p,V}\right\|^{1/2} \left\|T_{2p,V}\right\|^{1/2} \\
&\le \omega\left(T_{p,V}\right) + \|V\|^{1/2} \|T\|^{p/2} \left\|T_{p,V}\right\|^{1/2}
\end{aligned} \tag{4.22}
$$

for all $p \ge 0$.

For $p = 1$ in (4.21) and (4.22) we get

$$r\left(\widehat{T}_V \pm T_V\right) \le \omega\left(\widehat{T}_V\right) + \|V\|^{1/2} \left\|\widehat{T}_{2,V}\right\|^{1/2} \le \omega\left(\widehat{T}_V\right) + \|V\|^{1/2} \|T\|^{1/2} \left\|\widehat{T}_V\right\|^{1/2}$$

and

$$r\left(\widehat{T}_V \pm T_V\right) \le \omega\left(T_V\right) + \left\|T_V\right\|^{1/2} \left\|T_{2,V}\right\|^{1/2} \le \omega\left(T_V\right) + \|V\|^{1/2} \|T\|^{1/2} \left\|T_V\right\|^{1/2}.$$

If we take $p=1$ and $s=1/2$ in (4.19), then we get

$$r\left(\widehat{T}_V \pm \widetilde{T}_V\right) \leq \omega\left(\widehat{T}_V\right) + \left\|\widehat{T}_{1/2,V}\right\|^{1/2}\left\|\widehat{T}_{3/2,V}\right\|^{1/2} \leq \omega\left(\widehat{T}_V\right) + \|T\|^{1/4}\left\|\widehat{T}_{1/2,V}\right\|^{1/2}\left\|\widehat{T}_V\right\|^{1/2} \tag{4.23}$$

and

$$r\left(\widehat{T}_V \pm \widetilde{T}_V\right) \leq \omega\left(\widetilde{T}_V\right) + \left\|\widetilde{T}_V\right\|^{1/2}\left\|\Delta_{1/2,1,V}(T)\right\|^{1/2} \leq \omega\left(\widetilde{T}_V\right) + \|T\|^{1/4}\left\|\widetilde{T}_V\right\|^{1/2}\left\|\widehat{T}_{1/2,V}\right\|^{1/2}. \tag{4.24}$$

Corollary 7. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ we have for $p \geq s \geq 0$ that*

$$\begin{aligned} & r\left(\widetilde{T}_{p,V} \pm \Delta_{p-s,p+s,V}(T)\right) \\ & \leq \omega\left(\widetilde{T}_{p,V}\right) + \left\|\Delta_{p-s,p,V}(T)\right\|^{1/2}\left\|\Delta_{p+s,p,V}(T)\right\|^{1/2} \\ & \leq \omega\left(\widetilde{T}_{p,V}\right) + \|T\|^{s/2}\left\|\Delta_{p-s,p,V}(T)\right\|^{1/2}\left\|\widetilde{T}_{p,V}\right\|^{1/2} \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} & r\left(\widetilde{T}_{p,V} \pm \Delta_{p-s,p+s,V}(T)\right) \\ & \leq \omega\left(\Delta_{p-s,p+s,V}(T)\right) + \left\|\Delta_{p-s,p+s,V}(T)\right\|^{1/2}\left\|\Delta_{p-s,p+2s,V}(T)\right\|^{1/2} \\ & \leq \omega\left(\Delta_{p-s,p+s,V}(T)\right) + \|T\|^{s/2}\left\|\Delta_{p-s,p+s,V}(T)\right\|^{1/2}\left\|\Delta_{p-s,p,V}(T)\right\|^{1/2}. \end{aligned} \tag{4.26}$$

It follows by Theorem 3 for $q=p$.

If we take $s=p$ in (4.25), then we get

$$r\left(\widetilde{T}_{p,V} \pm T_{2p,V}\right) \leq \omega\left(\widetilde{T}_{p,V}\right) + \left\|T_{p,V}\right\|^{1/2}\left\|\Delta_{2p,p,V}(T)\right\|^{1/2} \leq \omega\left(\widetilde{T}_{p,V}\right) + \|T\|^{p/2}\left\|T_{p,V}\right\|^{1/2}\left\|\widetilde{T}_{p,V}\right\|^{1/2}, \tag{4.27}$$

while from (4.26)

$$r\left(\widetilde{T}_{p,V} \pm T_{2p,V}\right) \leq \omega\left(T_{2p,V}\right) + \left\|T_{2p,V}\right\|^{1/2}\left\|T_{3p,V}\right\|^{1/2} \leq \omega\left(T_{2p,V}\right) + \|T\|^{p/2}\left\|T_{2p,V}\right\|^{1/2}\left\|T_{p,V}\right\|^{1/2}. \tag{4.28}$$

Moreover, if we take $p=1/2$ in (4.27) and (4.28), then we get

$$r\left(\widetilde{T}_V \pm T_V\right) \leq \omega\left(\widetilde{T}_V\right) + \left\|T_{1/2,V}\right\|^{1/2}\left\|\Delta_{1,1/2,V}(T)\right\|^{1/2} \leq \omega\left(\widetilde{T}_V\right) + \|T\|^{1/4}\left\|T_{1/2,V}\right\|^{1/2}\left\|\widetilde{T}_V\right\|^{1/2}$$

and

$$r\left(\widetilde{T}_V \pm T_V\right) \leq \omega\left(T_V\right) + \left\|T_V\right\|^{1/2}\left\|T_{3/2,V}\right\|^{1/2} \leq \omega\left(T_V\right) + \|T\|^{1/4}\left\|T_V\right\|^{1/2}\left\|T_{1/2,V}\right\|^{1/2}.$$

By taking other particular cases for q, p and s one can derive some similar inequalities. We omit the details.

References

[1] A. Aluthge, Some generalized theorems on p-hyponormal operators, *Integral Equations Operator Theory* **24** (1996), 497-501.

[2] A. Abu-Omar and F. Kittaneh, A numerical radius inequality involving the generalized Aluthge transform, *Studia Math.* **216** (1) (2013) 69-75.

[3] A. Abu-Omar and F. Kittaneh, Numerical radius inequalities for products and commutators of operators, *Houston J. Math.* **41** (2015), no. 4, 1163-1173. 5

[4] P. Bhunia, S. Bag, and K. Paul, Numerical radius inequalities and its applications in estimation of zeros of polynomials, *Linear Algebra and its Applications*, vol. **573** (2019) pp. 166–177.

[5] P. Bhunia , S. S. Dragomir , M. S. Moslehian , K. Paul, *Lectures on Numerical Radius Inequalities*, Springer Cham, 2022. https://doi.org/10.1007/978-3-031-13670-2.

[6] P. Bhunia, K. Feki and K. Paul, Numerical radius inequalities for products and sums of semi-Hilberian space operators, *Filomat* Vol **36** (2022), No. 4.

[7] M. Cho and K. Tanahashi, Spectral relations for Aluthge transform, *Scientiae Mathematicae Japonicae*, **55** (1) (2002), 77-83.

[8] Silvestru Sever Dragomir, Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces, SpringerBriefs in Mathematics, 2013. https://doi.org/10.1007/978-3-319-01448-7.

[9] S. S. Dragomir, Upper bounds for the extended ggeneralized Aluthge transform of bounded operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **26** (2023), Art. 8, 14 pp. [Online https://rgmia.org/papers/v26/v26a08.pdf].

[10] M. El-Haddad and F. Kittaneh, Numerical radius inequalities for Hilbert space operators. II, *Studia Math.* **182** (2007), No. 2, 133-140.

[11] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, *Studia Math.* **158** (2003), No. 1, 11-17.

[12] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, *Studia Math.*, **168** (2005), No. 1, 73-80.

[13] M. Sattari, M. S. Moslehian, T. Yamazaki, Some generalized numerical radius inequalities for Hilbert space operators. *Linear Algebra Appl.* **470**(1), 216–227 (2015)

[14] T. Yamazaki, On upper and lower bounds of the numerical radius and an equality condition, *Studia Math.* **178** (2007), No. 1, 83-89.

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