Refinements of Generalised Hermite-Hadamard Inequality

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Abstract. New insights, improvements and refinements of the well known Hermite-Hadamard inequality are established for a general class of convex functions. Inequalities involving products of two ϕ_{h-s} functions are also obtained.

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1 Introduction and Preliminaries

One of the most important inequalities involving convex functions is the Hermite-Hadamard inequality which was named after Charles Hermite and Jacques Hadamard. This inequality can be traced to the period between 1883 and 1893 when C. Hermite [10] showed that a convex function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ satsifies

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

and J. Hadamard [9] independently rediscovered the left hand side of (1). It gives both the upper and lower bounds of the mean value of a convex function. The Hemite-Hadamard inequality has since then enjoyed a

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great deal of attention in literature with regards to several notions of convexity, extensions and refinements of Hermite-Hadamard inequality (interested reader may see [1, 2, 3, 4, 5, 6, 11, 12, 13, 14, 15, 16] and the references therein for details). This is partly due to the application of convexity in several areas of Mathematics, Science and other disciplines such as Economics.

Let I be an interval of $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \qquad \text{for all } x, y \in I \quad \text{and} \quad t \in [0,1]. \tag{2}$$

A function f is *concave* if the inequality (2) is reversed. Clearly f is convex if and only if $-f$ is concave. An *affine function* is both concave and convex, that is, the left and right hand side of (2) are equal. In a geometric sense, the function f is convex if the line segment joining any two points on the graph of f lies above or on the graph. A particular case of (2) is when $t = \frac{1}{2}$, we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y))$$

which also coincides with a particular case of Jensen's inequality. Several authors have introduced different notions of convexity and have obtained several estimates of Hermite-Hadamard's type, Jensen's type, Ostrowski type, etc. We will content ourselves, in this paper, with Hermite-Hadamard's type estimates. We obtain refinements of Hermite-Hadamard's inequality for a general class of convex functions.

2 Main Results

Here, we establish a refinements of some inequalities derived in [14]. We begin with the following representation lemma.

Lemma 2.1. *Let $\phi : [a,b] \to (0,\infty)$. Let $f \in L^1[\phi(a), \phi(b)]$ then for $0 \leq \lambda \leq 1$, we have*

$$\begin{aligned} \int_0^1 f\left(\frac{\phi(a)\phi(b)}{(1-t)\phi(a) + t\phi(b)}\right) dt = (1-\lambda) \int_0^1 f\left(\frac{\phi(a)\phi(b)}{(1-t)((1-\lambda)\phi(a) + \lambda\phi(b)) + t\phi(b)}\right) dt \\ + \lambda \int_0^1 f\left(\frac{\phi(a)\phi(b)}{(1-t)\phi(a) + t((1-\lambda)\phi(a) + \lambda\phi(b))}\right) dt \end{aligned} \tag{3}$$

Proof. It is easy to see that equality (3) holds for $\lambda = 0$ and $\lambda = 1$. The change of variable $u = \lambda(1-t)+t$ yields

$$\int_0^1 f\left(\frac{\phi(a)\phi(b)}{(1-t)((1-\lambda)\phi(a)+\lambda\phi(b))+t\phi(b)}\right)dt = \int_0^1 f\left(\frac{\phi(a)\phi(b)}{(\lambda(1-t)+t)\phi(b)+(1-t)(1-\lambda)\phi(a)}\right)dt$$
$$= \frac{1}{1-\lambda}\int_\lambda^1 f\left(\frac{\phi(a)\phi(b)}{u\phi(b)+(1-u)\phi(a)}\right)du.$$

In a similar fashion, the change of variable $u = \lambda t$ gives

$$\int_0^1 f\left(\frac{\phi(a)\phi(b)}{(1-t)\phi(a)+t((1-\lambda)\phi(a)+\lambda\phi(b))}\right)dt = \frac{1}{\lambda}\int_0^\lambda f\left(\frac{\phi(a)\phi(b)}{u\phi(b)+(1-u)\phi(a)}\right)du.$$

Putting the pieces together gives the announced result. □

Remark 2.2. *Note that Lemma 2.1 gives a splitting of integrals of the form*

$$\int_a^b \frac{f(x)}{x^2}dx = \int_a^{\frac{ab}{\lambda b+(1-\lambda)a}} \frac{f(x)}{x^2}dx + \int_{\frac{ab}{\lambda b+(1-\lambda)a}}^b \frac{f(x)}{x^2}dx.$$

Definition 2.3 (see also [15] and [14])**.** *Let I be an interval of $\mathbb{R}$ and let $[a,b] \subseteq I$. Let $\phi : [a,b] \to \mathbb{R}$ and $h : [0,1] \to (0,\infty)$.*

1. *A function $f : I \to \mathbb{R}$ is said to be ϕ_{h-s} convex if for all $x, y \in [a,b]$ and $s, t \in [0,1]$*

$$f(t\phi(x)+(1-t)\phi(y)) \le \left(\frac{t}{h(t)}\right)^s f(\phi(x)) + \left(\frac{1-t}{h(1-t)}\right)^s f(\phi(y)).$$

2. *A function $f : I \to \mathbb{R}$ is harmonically ϕ_{h-s} convex if for all $x, y \in [a,b]$ and $s, t \in [0,1]$*

$$f\left(\frac{\phi(x)\phi(y)}{t\phi(x)+(1-t)\phi(y)}\right) \le \left(\frac{t}{h(t)}\right)^s f(\phi(y)) + \left(\frac{1-t}{h(1-t)}\right)^s f(\phi(x)).$$

See [15] for examples and [7, 8] for recent modifications of Definition 2.3 on time scales. For convenience, we will set throughout the paper the notations $\alpha_s(t) := \left(\frac{t}{h(t)}\right)^s$ and $\alpha_{2s}(t) := (\alpha_s(t))^2$. There is an interesting relationship between the class of ϕ_{h-s} convex functions and the class of harmonically ϕ_{h-s} convex functions. The remaining part of this section is devoted to exploring this relationship.

Claim 2.4. *Let $\phi : [a, b] \to (0, \infty)$. Let $f : [\phi(a), \phi(b)] \to \mathbb{R}$ be an harmonically ϕ_{h-s} convex function. Then the real valued function $g(x) = f(\frac{1}{x})$ defined for all $\frac{1}{\phi(b)} \leq x \leq \frac{1}{\phi(a)}$ is ϕ_{h-s} convex.*

Proof. It suffices to observe that for $0 < \lambda < 1$,

$$\begin{aligned} g\left(\lambda \frac{1}{\phi(b)} + (1-\lambda)\frac{1}{\phi(a)}\right) &= f\left(\frac{1}{\lambda\frac{1}{\phi(b)} + (1-\lambda)\frac{1}{\phi(a)}}\right) \\ &= f\left(\frac{\phi(a)\phi(b)}{\lambda\phi(a) + (1-\lambda)\phi(b)}\right) \\ &\leq \alpha_s(\lambda) f(\phi(b)) + \alpha_s(1-\lambda) f(\phi(a)) \\ &= \alpha_s(\lambda) g\left(\frac{1}{\phi(b)}\right) + \alpha_s(1-\lambda) g\left(\frac{1}{\phi(a)}\right). \end{aligned}$$

□

Theorem 2.5. *Let $\phi : [a, b] \to (0, \infty)$ be a C^1 function. Let $f : [\phi(a), \phi(b)] \to \mathbb{R}$ be an harmonically ϕ_{h-s} convex function with $t^{1-s} \geq h^s(t)$ for all $t \in [0, 1]$. Then for $0 \leq \lambda \leq 1$, the following holds*

$$\begin{aligned} 2^{s-1}\left(h\left(\frac{1}{2}\right)\right)^s f\left(\frac{2\phi(a)\phi(b)}{\phi(a)+\phi(b)}\right) &\leq (1-\lambda) f\left(\frac{2\phi(a)\phi(b)}{(1-\lambda)\phi(a) + (\lambda+1)\phi(b)}\right) + \lambda f\left(\frac{2\phi(a)\phi(b)}{(2-\lambda)\phi(a) + \lambda\phi(b)}\right) \\ &\leq \frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)} \int_{\phi(a)}^{\phi(b)} \frac{f(\phi(x))}{\phi^2(x)} \phi'(x)dx \\ &\leq (f(\phi(a)) + f(\phi(b))) \int_0^1 \alpha_s(x)dx. \end{aligned}$$

Proof. Consider the function $g(x) = f(\frac{1}{x})$ on the interval $\frac{1}{\phi(b)} \leq x \leq \frac{1}{\phi(a)}$. The ϕ_{h-s} convexity of the function g (thanks to Claim 2.4) implies

$$\begin{aligned} g\left(\frac{(1-\lambda)\phi(a) + (\lambda+1)\phi(b))}{2\phi(a)\phi(b)}\right) &= g\left(\frac{(1-\lambda)\frac{1}{\phi(b)} + \lambda\frac{1}{\phi(a)} + \frac{1}{\phi(a)}}{2}\right) \\ &\leq 2\left(2h\left(\frac{1}{2}\right)\right)^{-s} \int_0^1 g\left[(1-t)\left((1-\lambda)\frac{1}{\phi(b)} + \lambda\frac{1}{\phi(a)}\right) + t\frac{1}{\phi(a)}\right] dt \\ &\leq 2\left(2h\left(\frac{1}{2}\right)\right)^{-s} \left[\frac{g\left(\frac{(1-\lambda)\phi(a)+\lambda\phi(b)}{\phi(a)\phi(b)}\right) + g\left(\frac{1}{\phi(a)}\right)}{2}\right] \int_0^1 \alpha_s(t)\, dt. \qquad (4) \end{aligned}$$

In a similar fashion, standard ϕ_{h-s} convexity estimates (c.f [14, 15, 13]) guarantees that

$$
\begin{aligned}
g\left(\frac{(2-\lambda)\phi(a)+\lambda\phi(b)}{2\phi(a)\phi(b)}\right) &= g\left(\frac{\frac{1}{\phi(b)}+(1-\lambda)\frac{1}{\phi(b)}+\lambda\frac{1}{\phi(a)}}{2}\right) \\
&\leq 2\left(2h\left(\frac{1}{2}\right)\right)^{-s}\int_0^1 g\left[(1-t)\frac{1}{\phi(b)}+t\left((1-\lambda)\frac{1}{\phi(b)}+\lambda\frac{1}{\phi(a)}\right)\right]dt \\
&\leq 2\left(2h\left(\frac{1}{2}\right)\right)^{-s}\left[\frac{g\left(\frac{(1-\lambda)\phi(a)+\lambda\phi(b)}{\phi(a)\phi(b)}\right)+g\left(\frac{1}{\phi(b)}\right)}{2}\right]\int_0^1 \alpha_s(t)dt. \qquad (5)
\end{aligned}
$$

So far, we have been able to bound g from above on the interval $\left[\frac{1}{\phi(b)}, \frac{1}{\phi(a)}\right]$. We need to update this information onto the function f via its harmonic ϕ_{h-s} convexity. To do this, we multiply (4) by $(1-\lambda)$ and (5) by λ. The resulting inequalities are then added to obtain

$$
\begin{aligned}
&(1-\lambda)f\left(\frac{2\phi(a)\phi(b)}{(1-\lambda)\phi(a)+(\lambda+1)\phi(b)}\right)+\lambda f\left(\frac{2\phi(a)\phi(b)}{(2-\lambda)\phi(a)+\lambda\phi(b)}\right) \\
&\leq 2\left(2h\left(\frac{1}{2}\right)\right)^{-s}\int_0^1 f\left(\frac{\phi(a)\phi(b)}{(1-t)\phi(a)+t\phi(b)}\right)dt \\
&\leq \left(2h\left(\frac{1}{2}\right)\right)^{-s}\left[f\left(\frac{\phi(a)\phi(b)}{(1-\lambda)\phi(a)+\lambda\phi(b)}\right)+(1-\lambda)f(\phi(a))+\lambda f(\phi(b))\right]\int_0^1 \alpha_s(t)dt
\end{aligned}
$$

where we have used Lemma 2.1 in the second line above. Next, we use the ϕ_{h-s} convexity of g to write

$$
\begin{aligned}
&(1-\lambda)f\left(\frac{2\phi(a)\phi(b)}{(1-\lambda)\phi(a)+(\lambda+1)\phi(b)}\right)+\lambda f\left(\frac{2\phi(a)\phi(b)}{(2-\lambda)\phi(a)+\lambda\phi(b)}\right) \\
&\geq \alpha_s(1-\lambda)g\left(\frac{(1-\lambda)\phi(a)+(\lambda+1)\phi(b)}{2\phi(a)\phi(b)}\right)+\alpha_s(\lambda)g\left(\frac{(2-\lambda)\phi(a)+\lambda\phi(b)}{2\phi(a)\phi(b)}\right) \\
&\geq g\left(\frac{(1-\lambda)^2\phi(a)+(1-\lambda^2)\phi(b)}{2\phi(a)\phi(b)}+\frac{(2\lambda-\lambda^2)\phi(a)+\lambda^2\phi(b)}{2\phi(a)\phi(b)}\right) \\
&= f\left(\frac{2\phi(a)\phi(b)}{\phi(a)+\phi(b)}\right).
\end{aligned}
$$

By using a suitable change of variable, we observe that

$$
\int_0^1 f\left(\frac{\phi(a)\phi(b)}{(1-t)\phi(a)+t\phi(b)}\right)dt = \frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)}\frac{f(\phi(x))}{\phi^2(x)}\phi'(x)dx.
$$

Finally, we use the ϕ_{h-s} convexity of g again to obtain the estimates

$$\begin{aligned}
&f\left(\frac{\phi(a)\phi(b)}{(1-\lambda)\phi(a)+\lambda\phi(b)}\right)+(1-\lambda)f(\phi(a))+\lambda f(\phi(b))\\
&\le \alpha_s(1-\lambda)f(\phi(b))+\alpha_s(\lambda)f(\phi(a))+(1-\lambda)f(\phi(a))+\lambda f(\phi(b))\\
&\le (1-\lambda)f(\phi(b))+\lambda f(\phi(a))+(1-\lambda)f(\phi(a))+\lambda f(\phi(b))\\
&= f(\phi(a))+f(\phi(a)).
\end{aligned}$$

□

Corollary 2.6. *Under the assumptions of Theorem 2.5, we have*

$$2^{s-1}f\left(\frac{2ab}{a+b}\right)\le\frac{1}{2}f\left(\frac{4ab}{a+3b}\right)+\frac{1}{2}f\left(\frac{4ab}{3a+b}\right)\le\frac{1}{s+1}(f(a)+f(b)).$$

Proof. It suffices to set $\lambda=\frac{1}{2}$, $h(t)=1$ for all $t\in[0,1]$ and $\phi(x)=x$ for all $x\in[a,b]$ in Theorem 2.5. □

Hemite-Hadamard inequalities for products of functions have been widely studied for different classes of convex functions and under different conditions on the functions involved. In 2003, Pachpatte [16] proved an inequality for products of two convex functions. Later in [17] established some results of Pachpatte's type involving sum of products of convex functions. In [18, 12], these results were extended to the class of MT-convex functions and λ-MT convex functions. We now focus our attention on obtaining inequalities involving product of two harmonically ϕ_{h-s} convex functions. We begin with the original result due to Pachpatte [16] which is as follows.

Theorem 2.7. *Let f and g be two nonnegative real valued convex functions on the interval $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx\le\frac{1}{3}[f(a)g(a)+f(b)g(b)]+\frac{1}{6}[f(a)g(b)+f(b)g(a)].$$

We state without proof the following result similar to Theorem 3 in [14].

Theorem 2.8. *Let $\phi:[a,b]\to(0,\infty)$ be a C^1 function and let $f,\tilde{f}:[\phi(a),\phi(b)]\to(0,\infty)$ be harmonically ϕ_{h-s} convex functions. Then*

$$\frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)}\frac{f(\phi(x))\tilde{f}(\phi(x))}{(\phi(x))^2}\phi'(x)dx\le M(a,b)\int_0^1\alpha_{2s}(t)\;dt+N(a,b)\int_0^1\alpha_s(t)\alpha_s(1-t)\;dt$$

where

$$M(a,b) := f(\phi(a))\tilde{f}(\phi(a)) + f(\phi(b)\tilde{f}(\phi(b)) \qquad \textit{and} \qquad N(a,b) := f(\phi(a))\tilde{f}(\phi(b)) + f(\phi(b)\tilde{f}(\phi(a)).$$

For the sake of readability, we set the following notations. Let $A := \frac{t\phi(a)+(1-t)\phi(b)}{\phi(a)\phi(b)}$ and $B := \frac{(1-t)\phi(a)+t\phi(b)}{\phi(a)\phi(b)}$. We recall the following elementary inequality which will prove useful in estimating products of sums.

Lemma 2.9 (Chebyshev's inequality). *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. Then*

$$\left(\sum_i^n a_i\right)\left(\sum_i^n b_i\right) \leq n\left(\sum_i^n a_i b_i\right). \tag{6}$$

Inequality (6) also holds if $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$.

Theorem 2.10. *Let $\phi : [a,b] \to (0,\infty)$ be a C^1 function and let $f, \tilde{f} : [\phi(a), \phi(b)] \to (0,\infty)$ be harmonically ϕ_{h-s} convex functions satisfying the ratio bound*

$$\frac{t^s}{h^s(t)}\frac{h^s(1-t)}{(1-t)^s} \leq \frac{f(\phi(a))}{f(\phi(b))} \leq \frac{(1-t)^s}{h^s(1-t)}\frac{h^s(t)}{t^s} \qquad \textit{for all} \quad t \in (0,1]. \tag{7}$$

Suppose that $\tilde{f}$ also satisfies the ratio bound (7). Then we have

$$\begin{aligned} f\left(\frac{2\phi(a)\phi(b)}{\phi(a)+\phi(b)}\right)\tilde{f}\left(\frac{2\phi(a)\phi(b)}{\phi(a)+\phi(b)}\right) &\leq \alpha_{2s}\left(\frac{1}{2}\right)\frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} [F]\ \frac{\phi'(x)}{\phi^2(x)}\,dx \\ &\leq \left[f(\phi(a))f(\phi(b)) + \tilde{f}(\phi(a))\tilde{f}(\phi(b))\right]\int_0^1 H_s(t)\ dt \end{aligned}$$

where we have set $H_s(t) := 8\alpha_{2s}\left(\frac{1}{2}\right)\alpha_s(t)\,\alpha_s(1-t)\left[\alpha_{2s}(t) + \alpha_{2s}(1-t)\right]$ and

$$[F] := f^2(\phi(x)) + \tilde{f}^2(\phi(x)) + 2f(\phi(x))f\left(\frac{1}{\frac{1}{\phi(a)}+\frac{1}{\phi(b)}-\frac{1}{\phi(x)}}\right) + 2\tilde{f}(\phi(x))\tilde{f}\left(\frac{1}{\frac{1}{\phi(a)}+\frac{1}{\phi(b)}-\frac{1}{\phi(x)}}\right).$$

Proof. Consider the functions $g, \tilde{g} : \left[\frac{1}{\phi(b)}, \frac{1}{\phi(a)}\right] \to (0,\infty)$ defined by $g(x) = f\left(\frac{1}{x}\right)$ and $\tilde{g}(x) = \tilde{f}\left(\frac{1}{x}\right)$ respectively. With the help of Young's inequality and the fact that $\phi(a)+\phi(b) = (A+B)\phi(a)\phi(b)$, we have the estimate

$$
\begin{aligned}
2g\left(\frac{\phi(a)+\phi(b)}{2\phi(a)\phi(b)}\right)\tilde{g}\left(\frac{\phi(a)+\phi(b)}{2\phi(a)\phi(b)}\right) &\leq g^2\left(\frac{\phi(a)+\phi(b)}{2\phi(a)\phi(b)}\right)+\tilde{g}^2\left(\frac{\phi(a)+\phi(b)}{2\phi(a)\phi(b)}\right) \\
&= g^2\left(\frac{A}{2}+\frac{B}{2}\right)+\tilde{g}^2\left(\frac{A}{2}+\frac{B}{2}\right) \\
&\leq \alpha_{2s}\left(\frac{1}{2}\right)\left[g^2(A)+g^2(B)+2g(A)g(B)\right. \\
&\qquad \left. +\tilde{g}^2(A)+\tilde{g}^2(B)+2\tilde{g}(A)\tilde{g}(B)\right]. \qquad (8)
\end{aligned}
$$

It remains to bound each term of (8). The ratio bound condition guarantees the use of Chebyshev's inequality. Using the ϕ_{h-s} convexity of g and Chebyshev's inequality, we have the estimates

$$
\begin{aligned}
g^2(A) &\leq \left[\alpha_s(t)g\left(\frac{1}{\phi(b)}\right)+\alpha_s(1-t)g\left(\frac{1}{\phi(a)}\right)\right]^2 \\
&\leq 2\alpha_{2s}(t)g^2\left(\frac{1}{\phi(b)}\right)+2\alpha_{2s}(1-t)g^2\left(\frac{1}{\phi(a)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
g^2(B) &\leq \left[\alpha_s(1-t)g\left(\frac{1}{\phi(b)}\right)+\alpha_s(t)g\left(\frac{1}{\phi(a)}\right)\right]^2 \\
&\leq 2\alpha_{2s}(1-t)g^2\left(\frac{1}{\phi(b)}\right)+2\alpha_{2s}(t)g^2\left(\frac{1}{\phi(a)}\right).
\end{aligned}
$$

Similar estimates hold for $\tilde{g}^2(A)$ and $\tilde{g}^2(B)$. We next focus on estimating the term involving $g(A)g(B)$. To do this, we first apply the ϕ_{h-s} convexity of g to have

$$
\begin{aligned}
g(A)g(B) \leq\ & \alpha_s(t)\alpha_s(1-t)\left[g^2\left(\frac{1}{\phi(b)}\right)+g^2\left(\frac{1}{\phi(a)}\right)\right] \\
&+\left[\alpha_{2s}(t)+\alpha_{2s}(1-t)\right]g\left(\frac{1}{\phi(a)}\right)g\left(\frac{1}{\phi(b)}\right).
\end{aligned}
$$

Calling upon Young's inequality yields

$$
2g(A)g(B) \leq \left[\alpha_s(t)+\alpha_s(1-t)\right]^2\left[g^2\left(\frac{1}{\phi(b)}\right)+g^2\left(\frac{1}{\phi(a)}\right)\right].
$$

Similar estimates hold for $\tilde{g}(A)\tilde{g}(B)$. We use Young's inequality again to write

$$
g^2(A)+g^2(B)+2g(A)g(B) \leq 4\left[\alpha_{2s}(t)+\alpha_{2s}(1-t)\right]\left[g^2\left(\frac{1}{\phi(b)}\right)+g^2\left(\frac{1}{\phi(a)}\right)\right],
$$

and of course, similar estimate holds for $\tilde{g}^2(A) + \tilde{g}^2(B) + 2\tilde{g}(A)\tilde{g}(B)$. We are now ready to announce the estimate (8). We have

$$
\begin{aligned}
&2f\left(\frac{2\phi(a)\phi(b)}{\phi(a)+\phi(b)}\right)\tilde{f}\left(\frac{2\phi(a)\phi(b)}{\phi(a)+\phi(b)}\right)\\
&\leq 4\alpha_{2s}\left(\frac{1}{2}\right)[\alpha_{2s}(t)+\alpha_{2s}(1-t)]\left[f^2(\phi(b))+f^2(\phi(a))+\tilde{f}^2(\phi(b))+\tilde{f}^2(\phi(a))\right].
\end{aligned}
$$

The ratio bound condition on the functions f and $\tilde{f}$ implies

$$
f\left(\frac{2\phi(a)\phi(b)}{\phi(a)+\phi(b)}\right)\tilde{f}\left(\frac{2\phi(a)\phi(b)}{\phi(a)+\phi(b)}\right) \leq H_s(t)\left[f(\phi(a))f(\phi(b))+\tilde{f}(\phi(a))\tilde{f}(\phi(b))\right]
$$

where

$$
H_s(t) := 8\left(\frac{1/2}{h(1/2)}\right)^{2s}\left[\left(\frac{t}{h(t)}\right)^{-s}\left(\frac{1-t}{h(1-t)}\right)^{s}\right]\left[\left(\frac{t}{h(t)}\right)^{2s}+\left(\frac{1-t}{h(1-t)}\right)^{2s}\right].
$$

Integrating with respect to t on the interval $[0,1]$, we have

$$
\begin{aligned}
&2f\left(\frac{2\phi(a)\phi(b)}{\phi(a)+\phi(b)}\right)\tilde{f}\left(\frac{2\phi(a)\phi(b)}{\phi(a)+\phi(b)}\right)\\
&\leq \left(\frac{1/2}{h(1/2)}\right)^{2s}\int_0^1\left[f^2\left(\frac{1}{A}\right)+f^2\left(\frac{1}{B}\right)+2f\left(\frac{1}{A}\right)f\left(\frac{1}{B}\right)+\tilde{f}^2\left(\frac{1}{A}\right)+\tilde{f}^2\left(\frac{1}{B}\right)+2\tilde{f}\left(\frac{1}{A}\right)\tilde{f}\left(\frac{1}{B}\right)\right]dt\\
&\leq \left[f(\phi(a))f(\phi(b))+\tilde{f}(\phi(a))\tilde{f}(\phi(b))\right]\int_0^1 H_s(t)\,dt
\end{aligned}
$$

whence

$$
\int_0^1 f^2\left(\frac{1}{A}\right)dt = \int_0^1 f^2\left(\frac{1}{B}\right)dt = \frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)}\frac{f^2(\phi(x))}{\phi^2(x)}\phi'(x)\,dx
$$

and

$$
\int_0^1 f\left(\frac{1}{A}\right)f\left(\frac{1}{B}\right)dt = \frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)}\frac{f(\phi(x))}{\phi^2(x)}f\left(\frac{1}{\frac{1}{\phi(a)}+\frac{1}{\phi(b)}-\frac{1}{\phi(x)}}\right)\phi'(x)\,dx.
$$

The same integral equalities hold for the terms involving $\tilde{f}$. This completes the proof.

□

Theorem 2.11. *Let $\phi : [a,b] \to (0,\infty)$ be a C^1 function and let $f, \tilde{f} : [\phi(a), \phi(b)] \to \mathbb{R}$ be harmonically ϕ_{h-s} convex functions. Define the functions $M(\phi(a),\phi(b)) := f(\phi(a))\tilde{f}(\phi(a)) + f(\phi(b))\tilde{f}(\phi(b))$ and $N(\phi(a),\phi(b)) := f(\phi(a))\tilde{f}(\phi(b)) + f(\phi(b))\tilde{f}(\phi(a))$. Then the following inequality holds*

$$
\begin{aligned}
& M(\phi(a),\phi(b)) \int_0^1 \alpha_s(t)\alpha_s(1-t)\,dt + N(\phi(a),\phi(b)) \int_0^1 \alpha_{2s}(t) \\
& - \frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)} \int_{\phi(a)}^{\phi(b)} f(\phi(x))\tilde{f}\left(\frac{1}{\frac{1}{\phi(a)}+\frac{1}{\phi(b)}-\frac{1}{\phi(x)}}\right)\frac{\phi'(x)}{\phi^2(x)}\,dx \\
& \geq \begin{cases} \left|[M(\phi(a),\phi(b)) + N(\phi(a),\phi(b))]\displaystyle\int_0^1 \alpha_{2s}(t)\,dt\right| & \text{if} \quad \alpha_s(t) = \alpha_s(1-t) \\ \\ \left|\displaystyle\int_c^d \frac{|x|}{x'}\,dx - \frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)}\left|f(\phi(x))\tilde{f}\left(\frac{1}{\frac{1}{\phi(a)}+\frac{1}{\phi(b)}-\frac{1}{\phi(x)}}\right)\right|\frac{\phi'(x)}{\phi^2(x)}\,dx\right| & \text{if} \quad \alpha_s(t) \neq \alpha_s(1-t) \end{cases}
\end{aligned} \tag{9}
$$

where $c := \frac{1}{h^{2s}(1)} f(\phi(b))\tilde{f}(\phi(a))$ and $d := \frac{1}{h^{2s}(1)} f(\phi(a))\tilde{f}(\phi(b))$. The direction of inequality (9) can be reversed depending on the sign of f and $\tilde{f}$.

Proof. We use the ϕ_{h-s} convexity of the function $g(x) := f(\frac{1}{x})$ to write

$$
\begin{aligned}
& \alpha_s(t)\alpha_s(1-t)M(\phi(a),\phi(b)) + \alpha_{2s}(1-t)f(\phi(b))\tilde{f}(\phi(a)) + \alpha_{2s}(t)f(\phi(a))\tilde{f}(\phi(b)) - g(A)\tilde{g}(B) \\
& = \left|\alpha_s(t)\alpha_s(1-t)M(\phi(a),\phi(b)) + \alpha_{2s}(1-t)f(\phi(b))\tilde{f}(\phi(a)) + \alpha_{2s}(t)f(\phi(a))\tilde{f}(\phi(b)) - g(A)\tilde{g}(B)\right| \\
& \left|\left|\alpha_s(t)\alpha_s(1-t)M(\phi(a),\phi(b)) + \alpha_{2s}(1-t)f(\phi(b))\tilde{f}(\phi(a)) + \alpha_{2s}(t)f(\phi(a))\tilde{f}(\phi(b))\right| - |g(A)\tilde{g}(B)|\right|
\end{aligned} \tag{10}
$$

where we have used $M(\phi(a),\phi(b)) := f(\phi(a))\tilde{f}(\phi(a)) + f(\phi(b))\tilde{f}(\phi(b))$.

Integrating (10) over the interval $[0,1]$ yields

$$
\begin{aligned}
& M(\phi(a),\phi(b)) \int_0^1 \alpha_s(t)\alpha_s(1-t)\,dt + N(\phi(a),\phi(b)) \int_0^1 \alpha_{2s}(t) - \int_0^1 g(A)\tilde{g}(B)\,dt \\
& \geq \left|\int_0^1 \left|\alpha_s(t)\alpha_s(1-t)M(\phi(a),\phi(b)) + \alpha_{2s}(1-t)f(\phi(b))\tilde{f}(\phi(a)) + \alpha_{2s}(t)f(\phi(a))\tilde{f}(\phi(b))\right|\,dt \right.\\
& \qquad \left. - \int_0^1 |g(A)\tilde{g}(B)|\,dt\right|.
\end{aligned}
$$

A suitable substitution yields

$$\int_0^1 g(A)\tilde{g}(B)dt = \frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)} \int_{\phi(a)}^{\phi(b)} f(\phi(x))\tilde{f}\left(\frac{1}{\frac{1}{\phi(a)}+\frac{1}{\phi(b)}-\frac{1}{\phi(x)}}\right)\frac{\phi'(x)}{\phi^2(x)}\,dx$$

and

$$\int_0^1 |g(A)\tilde{g}(B)|dt = \frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)} \int_{\phi(a)}^{\phi(b)} \left|f(\phi(x))\tilde{f}\left(\frac{1}{\frac{1}{\phi(a)}+\frac{1}{\phi(b)}-\frac{1}{\phi(x)}}\right)\right|\frac{\phi'(x)}{\phi^2(x)}\,dx$$

and

$$\int_0^1 \left|\alpha_s(t)\alpha_s(1-t)M(\phi(a),\phi(b))+\alpha_{2s}(1-t)f(\phi(b))\tilde{f}(\phi(a))+\alpha_{2s}(t)f(\phi(a))\tilde{f}(\phi(b))\right|\,dt = \int_c^d \frac{|x|}{x'(t)}\,dx$$

where $c := \frac{1}{h^{2s}(1)}f(\phi(b))\tilde{f}(\phi(a))$ and $d := \frac{1}{h^{2s}(1)}f(\phi(a))\tilde{f}(\phi(b))$. □

Corollary 2.12. *Let $\phi : [a,b] \to (0,\infty)$ be a C^1 function and let $f : [\phi(a),\phi(b)] \to \mathbb{R}$ be an harmonically ϕ_{h-s} convex function. Define the function $M(\phi(a),\phi(b)) := f(\phi(a))+f(\phi(b))$. Then the following inequality holds*

$$2M(\phi(a),\phi(b))\int_0^1 [\alpha_s(t)\alpha_s(1-t)+\alpha_{2s}(t)]\,dt - \frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} f(\phi(x))\frac{\phi'(x)}{\phi^2(x)}\,dx$$
$$\geq \begin{cases} \left|2M(\phi(a),\phi(b))\int_0^1 \alpha_{2s}(t)\,dt\right| & \text{if} \quad \alpha_s(t)=\alpha_s(1-t) \\ \left|\int_c^d \frac{|x|}{x'(t)}\,dx - \frac{\phi(a)\phi(b)}{\phi(b)-\phi(a)}\int_{\phi(a)}^{\phi(b)} |f(\phi(x))|\,\frac{\phi'(x)}{\phi^2(x)}\,dx\right| & \text{if} \quad \alpha_s(t)\neq\alpha_s(1-t) \end{cases}$$

where $c := \frac{1}{h^{2s}(1)}f(\phi(b))$ and $d := \frac{1}{h^{2s}(1)}f(\phi(a))$.

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