Approximation of Multiple Time Separating Random Functions by Neural Networks

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Abstract

Here we study the multivariate approximation of time separating random functions over a box or all of $\mathbb{R}^N, N \in \mathbb{N}$, by quasi-interpolation neural network operators. These approximations are derived by establishing Jackson type inequalities involving the multivariate modulus of continuity of the engaged random function or its high order partial derivatives. Our operators are defined by using density functions induced by the logistic and hyperbolic tangent activation sigmoid functions. The approximations are pointwise and with respect to the uniform norm. The feed-forward neural networks are with one hidden layer. We finish with a lot of interesting applications.

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1 Introduction

The first author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact

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support. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there. The first author inspired by [10], continued his studies on neural networks approximation by introducing and using the proper quasi interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3]-[7], by treating both the univariate and multivariate cases. In this article we are also inspired by the related works [11], [12]. The authors here use the logistic and hyperbolic tangent sigmoid functions based neural network quantitative approximations to continuous functions over a box all of $\mathbb{R}^N, N \in \mathbb{N}$ with values in $\mathbb{R}$. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order partial derivative and given by very tight Jackson type inequalities. More precisely, here we perform quantitative approximations of time separating random functions by neural networks. We give plenty of varied and interesting applications. Specific motivations came by:

1. Stationary Gaussian processes with an explicit representation such as

$$X_t = \cos(\alpha t)\,\xi_1 + \sin(\alpha t)\,\xi_2, \alpha \in \mathbb{R},$$

   where ξ_1, ξ_2 are independent random variables with the standard normal distribution, see [15],

2. by the "Fourier model" of a stationary process, see [16].

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental neural network models, the activation function is given by various specific sigmoid functions. Here we use the logistic and hyperbolic tangent activation functions. About neural networks in general read [13], [17],[18]. See also [8] for a complete study of real valued approximation by neural network operators.

2 Background

2.1 About the Logistic Sigmoid Activation Function

Here we follow [6]. We consider here the sigmoidal function of logarithmic type

$$s_i(x_i) = \frac{1}{1+e^{-x_i}}, \quad x_i \in \mathbb{R}, \ i = 1, ..., N; \ x := (x_1, ..., x_N) \in \mathbb{R}^N.$$

each has the properties $\lim_{x_i \to +\infty} s_i(x_i) = 1$ and $\lim_{x_i \to -\infty} s_i(x_i) = 0$, $i = 1, ..., N$.

These functions play the role of activation functions in the hidden layer of neural networks, also have applications in biology, demography, etc ([9, 14]).

As in [10], we consider

$$\Phi_i(x_i) := \frac{1}{2}\left(s_i(x_i+1) - s_i(x_i-1)\right), \quad x_i \in \mathbb{R},\ i = 1, ..., N.$$

We notice the following properties:

i) $\Phi_i(x_i) > 0, \ \forall\, x_i \in \mathbb{R}$,

ii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(x_i - k_i) = 1, \ \forall\, x_i \in \mathbb{R}$,

iii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(nx_i - k_i) = 1, \ \forall\, x_i \in \mathbb{R};\ n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi_i(x_i)\, dx_i = 1$,

v) Φ_i is a density function,

vi) Φ_i is even: $\Phi_i(-x_i) = \Phi_i(x_i), \ x_i \geq 0$, for $i = 1, ..., N$.

We see that

$$\Phi_i(x_i) = \left(\frac{e^2-1}{2e^2}\right) \frac{1}{(1+e^{x_i-1})(1+e^{-x_i-1})}, \quad i = 1, ..., N. \tag{1}$$

vii) Φ_i is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$, $i = 1, ..., N$.

Let $0 < \beta < 1$, $n \in \mathbb{N}$. Then as in [7] we get

viii)

$$\sum_{\substack{k_i=-\infty \\ : |nx_i - k_i| > n^{1-\beta}}}^{\infty} \Phi_i(nx_i - k_i) = \sum_{\substack{k_i=-\infty \\ : |nx_i - k_i| > n^{1-\beta}}}^{\infty} \Phi_i(|nx_i - k_i|)$$

$$\leq 3.1992 e^{-n^{(1-\beta)}}, \quad i = 1, ..., N. \tag{2}$$

Denote by $\lceil \cdot \rceil$ the ceiling of a number, and by $\lfloor \cdot \rfloor$ the integral part of a number. Consider here $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right) \subset \mathbb{R}^N$, $N \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$; $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

We obtain

ix)

$$0 < \frac{1}{\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i)} < \frac{1}{\Phi_i(1)} = 5.250312578, \tag{3}$$

$\forall\, x_i \in [a_i, b_i]$, $i = 1, ..., N$.

x) As in [7], we see that

$$\lim_{n\to\infty} \sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i) \neq 1, \tag{4}$$

for at least some $x_i \in [a_i, b_i]$, $i = 1, ..., N$.

We will use here

$$\Phi\left(x_1,...,x_N\right):=\Phi\left(x\right):=\prod_{i=1}^{N}\Phi_i\left(x_i\right)=:T_1(x),\quad x\in\mathbb{R}^N. \tag{5}$$

It has the properties:

(i)' $\Phi\left(x\right)>0,\ \forall\ x\in\mathbb{R}^N$,

We see that

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\Phi\left(x_1-k_1,x_2-k_2,...,x_N-k_N\right)=$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\prod_{i=1}^{N}\Phi_i\left(x_i-k_i\right)=\prod_{i=1}^{N}\left(\sum_{k_i=-\infty}^{\infty}\Phi_i\left(x_i-k_i\right)\right)=1. \tag{6}$$

That is

(ii)'

$$\sum_{k=-\infty}^{\infty}\Phi\left(x-k\right):=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\Phi\left(x_1-k_1,...,x_N-k_N\right)=1, \tag{7}$$

$k:=(k_1,...,k_n)$, $\forall\ x\in\mathbb{R}^N$.

(iii)'

$$\sum_{k=-\infty}^{\infty}\Phi\left(nx-k\right):=$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\Phi\left(nx_1-k_1,...,nx_N-k_N\right)=1, \tag{8}$$

$\forall\ x\in\mathbb{R}^N$; $n\in\mathbb{N}$.

(iv)'

$$\int_{\mathbb{R}^N}\Phi\left(x\right)dx=1, \tag{9}$$

that is Φ is a multivariate density function.

Here $\|x\|_\infty:=\max\left\{|x_1|,...,|x_N|\right\}$, $x\in\mathbb{R}^N$, also set $\infty:=(\infty,...,\infty)$, $-\infty:=(-\infty,...,-\infty)$ upon the multivariate context, and

$$\begin{aligned}\lceil na\rceil\ &:\ =\left(\lceil na_1\rceil,...,\lceil na_N\rceil\right),\\ \lfloor nb\rfloor\ &:\ =\left(\lfloor nb_1\rfloor,...,\lfloor nb_N\rfloor\right).\end{aligned}$$

For $0<\beta<1$ and $n\in\mathbb{N}$, fixed $x\in\mathbb{R}^N$, have that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)=$$

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty \le \frac{1}{n^\beta}}}^{\lfloor nb\rfloor} \Phi(nx-k) + \sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb\rfloor} \Phi(nx-k). \tag{10}$$

In the last two sums the counting is over disjoint vector of k's, because the condition $\left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^\beta}$, $r \in \{1,...,N\}$.

We treat

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb\rfloor} \Phi(nx-k) = \prod_{i=1}^{N} \left(\sum_{\substack{k_i=\lceil na_i\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb_i\rfloor} \Phi_i(nx_i-k_i) \right)$$

$$\le \left(\prod_{\substack{i=1 \\ i\ne r}}^{N} \left(\sum_{k_i=-\infty}^{\infty} \Phi_i(nx_i-k_i) \right) \right) \cdot \left(\sum_{\substack{k_r=\lceil na_r\rceil \\ \left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^\beta}}}^{\lfloor nb_r\rfloor} \Phi_r(nx_r-k_r) \right)$$

$$= \left(\sum_{\substack{k_r=\lceil na_r\rceil \\ \left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^\beta}}}^{\lfloor nb_r\rfloor} \Phi_r(nx_r-k_r) \right)$$

$$\le \sum_{\substack{k_r=-\infty \\ \left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^\beta}}}^{\infty} \Phi_r(nx_r-k_r) \overset{\text{(by (viii))}}{\le} 3.1992e^{-n^{(1-\beta)}}. \tag{11}$$

We have proved that

(v)'

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb\rfloor} \Phi(nx-k) \le 3.1992e^{-n^{(1-\beta)}}, \tag{12}$$

$0<\beta<1$, $n \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N [a_i,b_i]\right)$.

By (ix) clearly we obtain

$$0 < \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k)} = \frac{1}{\prod_{i=1}^N \left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} \Phi_i(nx_i-k_i)\right)}$$

$$< \frac{1}{\prod_{i=1}^{N} \Phi_i(1)} = (5.250312578)^N . \quad (13)$$

That is,

(vi)' it holds

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} < (5.250312578)^N , \quad (14)$$

$\forall\, x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $n \in \mathbb{N}$.

It is clear also that

(vii)'

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} \Phi(nx-k) \leq 3.1992 e^{-n^{(1-\beta)}}, \quad (15)$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$.

By (x) we obviously see that

(viii)'

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \neq 1 \quad (16)$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

Let $f \in C\left(\prod_{i=1}^{N} [a_i, b_i] \right)$ and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We introduce and define the multivariate positive linear neural network operator ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$)

$$G_n(f, x_1, ..., x_N) := G_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} \quad (17)$$

$$:= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Phi_i(nx_i - k_i) \right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i) \right)}.$$

For large enough n we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We study here the pointwise and uniform convergence of $G_n(f)$ to f with rates.

For convinience we call

$$G_n^*(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k) \quad (18)$$

$$:= \sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor} ... \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right)\left(\prod_{i=1}^{N} \Phi_i\left(nx_i - k_i\right)\right),$$

$\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

That is

$$G_n(f, x) := \frac{G_n^*(f, x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx - k)}, \quad \forall x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right), \ n \in \mathbb{N}. \tag{19}$$

Hence

$$G_n(f, x) - f(x) = \frac{G_n^*(f, x) - f(x) \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx - k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx - k)}. \tag{20}$$

Consequently we derive

$$|G_n(f, x) - f(x)| \leq (5.250312578)^N \left| G_n^*(f, x) - f(x) \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx - k) \right|, \tag{21}$$

$\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

We will estimate the right hand side of (21).

For that we need, for $f \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$ the first multivariate modulus of continuity

$$\omega_1(f, h) := \sup_{\substack{x, y \in \left(\prod_{i=1}^{N} [a_i, b_i]\right) \\ \|x - y\|_\infty \leq h}} |f(x) - f(y)|, \quad h > 0. \tag{22}$$

Similarly it is defined for $f \in C_B\left(\mathbb{R}^N\right)$ (continuous and bounded functions on $\mathbb{R}^N$). We have that $\lim_{h \to 0} \omega_1(f, h) = 0$.

When $f \in C_B\left(\mathbb{R}^N\right)$ we define

$$\overline{G}_n(f, x) := \overline{G}_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx - k) \tag{23}$$

$$:= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right)\left(\prod_{i=1}^{N} \Phi_i\left(nx_i - k_i\right)\right),$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$, $N \geq 1$, the multivariate quasi-interpolation neural network operator.

Notice here that for large enough $n \in \mathbb{N}$ we get that

$$e^{-n^{(1-\beta)}} < n^{-\beta j}, \quad j = 1, ..., m \in \mathbb{N}, \quad 0 < \beta < 1. \tag{24}$$

Thus be given fixed $A, B > 0$, for the linear combination $\left(An^{-\beta j} + Be^{-n^{(1-\beta)}}\right)$ the (dominant) rate of convergence to zero is $n^{-\beta j}$. The closer β is to 1 we get faster and better rate of convergence to zero.

Let $f \in C^m\left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, and $|\alpha| := \sum_{i=1}^{N} \alpha_i = l$, where $l = 0, 1, ..., m$. We write also $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is of order l.

We denote

$$\omega_{1,m}^{\max}\left(f_{\alpha},h\right):=\max_{\alpha:|\alpha|=m}\omega_{1}\left(f_{\alpha},h\right). \tag{25}$$

Call also

$$\|f_{\alpha}\|_{\infty,m}^{\max}:=\max_{|\alpha|=m}\left\{\|f_{\alpha}\|_{\infty}\right\}, \tag{26}$$

$\|\cdot\|_{\infty}$ is the supremum norm.

Next we present a series of multivariate neural network approximations to a function given with rates.

We first give

Theorem 1. *Let* $f\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta<1$, $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n,N\in\mathbb{N}$. *Then*

i)

$$\left|G_{n}(f,x)-f(x)\right|\leq(5.250312578)^{N}\cdot$$
$$\left\{\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)+(6.3984)\|f\|_{\infty}e^{-n^{(1-\beta)}}\right\}=:\lambda_{1}, \tag{27}$$

ii)

$$\|G_{n}(f)-f\|_{\infty}\leq\lambda_{1}. \tag{28}$$

Next we present

Theorem 2. *([6]) Let* $f\in C_{B}\left(\mathbb{R}^{N}\right)$, $0<\beta<1$, $x\in\mathbb{R}^{N}$, $n,N\in\mathbb{N}$. *Then*

i)

$$\left|\overline{G}_{n}(f,x)-f(x)\right|\leq\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)+(6.3984)\|f\|_{\infty}e^{-n^{(1-\beta)}}=:\lambda_{2}, \tag{29}$$

ii)

$$\left\|\overline{G}_{n}(f)-f\right\|_{\infty}\leq\lambda_{2}. \tag{30}$$

In the next we discuss high order of approximation by using the smoothness of f.

We give

Theorem 3. *([6]) Let* $f\in C^{m}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta<1$, $n,m,N\in\mathbb{N}$, $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$. *Then*

i)

$$\left|G_{n}(f,x)-f(x)-\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{f_{\alpha}(x)}{\prod_{i=1}^{N}\alpha_{i}!}\right)G_{n}\left(\prod_{i=1}^{N}(\cdot-x_{i})^{\alpha_{i}},x\right)\right)\right|\leq \tag{31}$$
$$(5.250312578)^{N}\cdot\left\{\frac{N^{m}}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)+\right.$$
$$\left.\left(\frac{(6.3984)\|b-a\|_{\infty}^{m}\|f_{\alpha}\|_{\infty,m}^{\max}N^{m}}{m!}\right)e^{-n^{(1-\beta)}}\right\},$$

ii)

$$\left|G_{n}(f,x)-f(x)\right|\leq(5.250312578)^{N}\cdot \tag{32}$$
$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{|f_{\alpha}(x)|}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^{N}(b_{i}-a_{i})^{\alpha_{i}}\right)\cdot(3.1992)e^{-n^{(1-\beta)}}\right]\right)+\right.$$

$$\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left(\frac{(6.3984)\,\|b-a\|_\infty^m \,\|f_\alpha\|_{\infty,m}^{\max} N^m}{m!}\right) e^{-n^{(1-\beta)}}\Bigg\},$$

iii)

$$\|G_n(f) - f\|_\infty \leq (5.250312578)^N \cdot \tag{33}$$

$$\Bigg\{\sum_{j=1}^N \left(\sum_{|\alpha|=j}\left(\frac{\|f_\alpha\|_\infty}{\prod_{i=1}^N \alpha_i!}\right)\left[\frac{1}{n^{\beta j}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i}\right)(3.1992)\, e^{-n^{(1-\beta)}}\right]\right) +$$

$$\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left(\frac{(6.3984)\,\|b-a\|_\infty^m \,\|f_\alpha\|_{\infty,m}^{\max} N^m}{m!}\right) e^{-n^{(1-\beta)}}\Bigg\},$$

iv) Assume $f_\alpha(x_0) = 0$, for all $\alpha : |\alpha| = 1, ..., m$; $x_0 \in \left(\prod_{i=1}^N [a_i, b_i]\right)$. Then

$$|G_n(f, x_0) - f(x_0)| \leq (5.250312578)^N \cdot \tag{34}$$

$$\Bigg\{\frac{N^m}{m!n^{m\beta}}\omega_1^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left(\frac{(6.3984)\,\|b-a\|_\infty^m \,\|f_\alpha\|_{\infty,m}^{\max} N^m}{m!}\right) e^{-n^{(1-\beta)}}\Bigg\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

2.2 About the Hyperbolic Tangent Sigmoid Activation Function

We consider now the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\, x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$.

This function plays the role of an activation function in the hidden layer of neural networks.

We further consider

$$\Psi(x) := \frac{1}{4}(\tanh(x+1) - \tanh(x-1)) > 0, \quad \forall\, x \in \mathbb{R}.$$

We easily see that $\Psi(-x) = \Psi(x)$, that is Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Proposition 4. *([3]) $\Psi(x)$ for $x \geq 0$ is strictly decreasing.*

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x\to-\infty} \Psi(x) = 0 = \lim_{x\to\infty} \Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is zero, $\Psi(0) = 0.3809297$.

Theorem 5. *([3]) We have that $\sum_{i=-\infty}^{\infty} \Psi(x-i) = 1$, $\forall\, x \in \mathbb{R}$.*

Thus

$$\sum_{i=-\infty}^{\infty} \Psi(nx - i) = 1, \quad \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}. \tag{35}$$

Also it holds

$$\sum_{i=-\infty}^{\infty} \Psi(x + i) = 1, \quad \forall x \in \mathbb{R}. \tag{36}$$

Theorem 6. *([3]) It holds*

$$\int_{-\infty}^{\infty} \Psi(x)\,dx = 1. \tag{37}$$

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 7. *([3]) Let $0 < \alpha < 1$ and $n \in \mathbb{N}$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \Psi(nx-k) \leq e^4 \cdot e^{-2n^{(1-\alpha)}}. \tag{38}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

Theorem 8. *([3]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)} < \frac{1}{\Psi(1)} = 4.1488766. \tag{39}$$

Also by [3] we get that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k) \neq 1, \tag{40}$$

for at least some $x \in [a,b]$.

In this article we will use

$$\Theta(x_1, ..., x_N) := \Theta(x) := \prod_{i=1}^{N} \Psi(x_i), \quad x = (x_1, ..., x_N) = T_2(x) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{41}$$

It has the properties:

(i) $\Theta(x) > 0, \ \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} \Theta(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Theta(x_1-k_1, ..., x_N-k_N) = 1, \tag{42}$$

where $k := (k_1, ..., k_n)$, $\forall\, x \in \mathbb{R}^N$.

(iii)

$$\sum_{k=-\infty}^{\infty} \Theta(nx-k) := \tag{43}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Theta(nx_1-k_1, ..., nx_N-k_N) = 1,$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)

$$\int_{\mathbb{R}^N} \Theta(x)\,dx = 1, \tag{44}$$

that is Θ is a multivariate density function.

We obviously see that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta(nx-k)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\prod_{i=1}^{N}\Psi(nx_i-x_i)= \tag{45}$$

$$\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\cdots\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor}\prod_{i=1}^{N}\Psi(nx_i-k_i)=\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\Psi(nx_i-k_i)\right).$$

For $0<\beta<1$ and $n\in\mathbb{N}$, fixed $x\in\mathbb{R}^N$, we have that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta(nx-k)=$$

$$\sum_{\begin{cases}k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}\Theta(nx-k)+\sum_{\begin{cases}k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}\Theta(nx-k). \tag{46}$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}$ implies that there exists at least one $\left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^\beta}$, $r\in\{1,...,N\}$.

We treat

$$\sum_{\begin{cases}k=\lceil na\rceil\\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}\Theta(nx-k)=\prod_{i=1}^{N}\left(\sum_{\begin{cases}k_i=\lceil na_i\rceil\\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\lfloor nb_i\rfloor}\Psi(nx_i-k_i)\right)$$

$$\leq\left(\prod_{\substack{i=1\\ i\neq r}}^{N}\left(\sum_{k_i=-\infty}^{\infty}\Psi(nx_i-k_i)\right)\right)\cdot\left(\sum_{\begin{cases}k_r=\lceil na_r\rceil\\ \left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^\beta}\end{cases}}^{\lfloor nb_r\rfloor}\Psi(nx_r-k_r)\right) \tag{47}$$

$$=\left(\sum_{\begin{cases}k_r=\lceil na_r\rceil\\ \left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^\beta}\end{cases}}^{\lfloor nb_r\rfloor}\Psi(nx_r-k_r)\right)$$

$$\leq\sum_{\begin{cases}k_r=-\infty\\ \left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^\beta}\end{cases}}^{\infty}\Psi(nx_r-k_r)\overset{\text{(by Theorem 7)}}{\leq}e^4\cdot e^{-2n^{(1-\beta)}}.$$

We have proved that

(v)

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor} \Theta\left(nx-k\right) \leq e^{4}\cdot e^{-2n^{(1-\beta)}}, \tag{48}$$

$0<\beta<1$, $n\in\mathbb{N}$, $x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$.

By Theorem 8 clearly we obtain

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta\left(nx-k\right)}=\frac{1}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\Psi\left(nx_i-k_i\right)\right)} \tag{49}$$

$$<\frac{1}{\left(\Psi\left(1\right)\right)^{N}}=\left(4.1488766\right)^{N}.$$

That is,

(vi) it holds

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta\left(nx-k\right)}<\frac{1}{\left(\Psi\left(1\right)\right)^{N}}=\left(4.1488766\right)^{N}, \tag{50}$$

$\forall\ x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$, $n\in\mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty} \Theta\left(nx-k\right) \leq e^{4}\cdot e^{-2n^{(1-\beta)}}, \tag{51}$$

$0<\beta<1$, $n\in\mathbb{N}$, $x\in\mathbb{R}^{N}$.

Also we get that

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta\left(nx-k\right)\neq 1, \tag{52}$$

for at least some $x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$.

Let $f\in C\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$ and $n\in\mathbb{N}$ such that $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

We introduce and define the multivariate positive linear neural network operator ($x:=(x_1,...,x_N)\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$)

$$F_n\left(f,x_1,...,x_N\right):=F_n\left(f,x\right):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Theta\left(nx-k\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta\left(nx-k\right)} \tag{53}$$

$$:=\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}...\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor}f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\Psi\left(nx_i-k_i\right)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\Psi\left(nx_i-k_i\right)\right)}.$$

For large enough n we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We study here the pointwise and uniform convergence of $F_n(f)$ to f with rates.

For convinience we call

$$F_n^*(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Theta(nx - k) \tag{54}$$

$$:= \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Psi(nx_i - k_i)\right),$$

$\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

That is

$$F_n(f, x) := \frac{F_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx - k)}, \tag{55}$$

$\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$F_n(f, x) - f(x) = \frac{F_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx - k)}. \tag{56}$$

Consequently we derive

$$|F_n(f, x) - f(x)| \leq (4.1488766)^N \left| F_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx - k) \right|, \tag{57}$$

$\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$. When $f \in C_B(\mathbb{R}^N)$ we define,

$$\overline{F}_n(f, x) := \overline{F}_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Theta(nx - k) := \tag{58}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Psi(nx_i - k_i)\right),$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$, $N \geq 1$, the multivariate quasi-interpolation neural network operator. Here we present a series of multivariate neural network approximations to a function given with rates. We first give

Theorem 9. *Let* $f \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $n, N \in \mathbb{N}$. *Then*

i)

$$|F_n(f, x) - f(x)| \leq (4.1488766)^N \cdot$$

$$\left\{\omega_1\left(f, \frac{1}{n^\beta}\right) + 2e^4 \|f\|_\infty e^{-2n^{(1-\beta)}}\right\} =: \lambda_1, \tag{59}$$

ii)

$$\|F_n(f) - f\|_\infty \leq \lambda_1. \tag{60}$$

Next we present

Theorem 10. *Let $f \in C_B\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x \in \mathbb{R}^N$, $n, N \in \mathbb{N}$. Then*

i)

$$\left|\overline{F}_n(f,x)-f(x)\right| \leq \omega_1\left(f,\frac{1}{n^\beta}\right)+2e^4\|f\|_\infty e^{-2n^{(1-\beta)}} =: \lambda_2, \tag{61}$$

ii)

$$\left\|\overline{F}_n(f)-f\right\|_\infty \leq \lambda_2. \tag{62}$$

In the next we discuss high order of approximation by using the smoothness of f. We give

Theorem 11. *Let $f \in C^m\left(\prod_{i=1}^N [a_i,b_i]\right)$, $0<\beta<1$, $n,m,N \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N [a_i,b_i]\right)$. Then*

i)

$$\left|F_n(f,x)-f(x)-\sum_{j=1}^m\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!}\right)F_n\left(\prod_{i=1}^N(\cdot-x_i)^{\alpha_i},x\right)\right)\right| \leq \tag{63}$$

$$(4.1488766)^N \cdot \left\{\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\right.$$

$$\left.\left(\frac{2e^4\|b-a\|_\infty^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)e^{-2n^{(1-\beta)}}\right\},$$

ii)

$$|F_n(f,x)-f(x)| \leq (4.1488766)^N \cdot \tag{64}$$

$$\left\{\sum_{j=1}^m\left(\sum_{|\alpha|=j}\left(\frac{|f_\alpha(x)|}{\prod_{i=1}^N\alpha_i!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^N(b_i-a_i)^{\alpha_i}\right)\cdot e^4e^{-2n^{(1-\beta)}}\right]\right)+\right.$$

$$\left.\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{2e^4\|b-a\|_\infty^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)e^{-2n^{(1-\beta)}}\right\},$$

iii)

$$\|F_n(f)-f\|_\infty \leq (4.1488766)^N \cdot \tag{65}$$

$$\left\{\sum_{j=1}^m\left(\sum_{|\alpha|=j}\left(\frac{\|f_\alpha\|_\infty}{\prod_{i=1}^N\alpha_i!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^N(b_i-a_i)^{\alpha_i}\right)e^4e^{-2n^{(1-\beta)}}\right]\right)+\right.$$

$$\left.\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{2e^4\|b-a\|_\infty^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)e^{-2n^{(1-\beta)}}\right\},$$

iv) Assume $f_\alpha(x_0)=0$, for all $\alpha : |\alpha|=1,...,m$; $x_0 \in \left(\prod_{i=1}^N [a_i,b_i]\right)$. Then

$$|F_n(f,x_0)-f(x_0)| \leq (4.1488766)^N \cdot \tag{66}$$

$$\left\{\frac{N^m}{m!n^{m\beta}}\omega_1^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{2e^4\|b-a\|_\infty^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)e^{-2n^{(1-\beta)}}\right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

2.3 Combining 2.1 and 2.2

For the next theorems we call

$$T_1(x) := \Phi(x)$$

$$T_2(x) := \Theta(x), x \in \mathbb{R}^{\mathbb{N}}, N \in \mathbb{N}.$$

We also set

$$\gamma_{1,N} := (5.250312578)^N,$$

$$\gamma_{2,N} := (4.1488766)^N.$$

Furthermore we set,

$$\alpha_{1,n}(\beta) := 3.1992e^{-n^{(1-\beta)}},$$

$$\alpha_{2,n}(\beta) := e^4 e^{-2n^{(1-\beta)}},$$

where $0 < \beta < 1, n \in \mathbb{N}$.
We define,

$${}_1L_n(f,x) := G_n(f,x),$$

$${}_2L_n(f,x) := F_n(f,x), x \in \mathbb{R}^{\mathbb{N}}, n \in \mathbb{N}$$

and

$${}_1\overline{L}_n(f,x) := \overline{G}_n(f,x),$$

$${}_2\overline{L}_n(f,x) := \overline{F}_n(f,x), x \in \mathbb{R}^{\mathbb{N}}, n \in \mathbb{N}.$$

Notice that

$$6.3984e^{-n^{(1-\beta)}} = 2a_{1,n}(\beta).$$

Theorem 12. *Let $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $n, N \in \mathbb{N}$ and $k = 1, 2$. Then*

i)

$$|{}_kL_n(f,x) - f(x)| \leq \gamma_{k,N} \left\{\omega_1\left(f, \frac{1}{n^\beta}\right) + 2\alpha_{k,n}(\beta)\|f\|_\infty\right\} =: \lambda_1, \tag{67}$$

ii)

$$\|{}_kL_n(f) - f\|_\infty \leq \lambda_1. \tag{68}$$

Proof. From Theorems 1 and 9. □
Next we present

Theorem 13. *Let $f \in C_B(\mathbb{R}^N)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $n, N \in \mathbb{N}$ and $k = 1, 2$. Then*

i)

$$\left|{}_k\overline{L}_n(f,x) - f(x)\right| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + 2\alpha_{k,n}(\beta)\|f\|_\infty =: \lambda_2, \tag{69}$$

ii)

$$\left\|{}_k\overline{L}_n(f) - f\right\|_\infty \leq \lambda_2. \tag{70}$$

Proof. From Theorems 2 and 10. $\square$

In the next we discuss high order of approximation by using the smoothness of f.

We give

Theorem 14. *Let $f \in C^m\left(\prod_{i=1}^N [a_i, b_i]\right)$, $0 < \beta < 1$, $n, m, N \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ and $k = 1, 2$. Then*

i)

$$\left| {}_kL_n(f, x) - f(x) - \sum_{j=1}^m \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) {}_kL_n\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) \right| \leq \tag{71}$$

$$\gamma_{k,N} \cdot \left\{ \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left(\frac{2\alpha_{k,n} \|b - a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \right\},$$

ii)

$$|{}_kL_n(f, x) - f(x)| \leq \gamma_{k,N} \cdot \tag{72}$$

$$\left\{ \sum_{j=1}^m \left(\sum_{|\alpha|=j} \left(\frac{|f_\alpha(x)|}{\prod_{i=1}^N \alpha_i!} \right) \left[\frac{1}{n^{\beta j}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \cdot \alpha_{k,n}(\beta) \right] \right) + \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left(\frac{2\alpha_{k,n}(\beta) \|b - a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \right\},$$

iii)

$$\|{}_kL_n(f) - f\|_\infty \leq \gamma_{k,N} \cdot \tag{73}$$

$$\left\{ \sum_{j=1}^N \left(\sum_{|\alpha|=j} \left(\frac{\|f_\alpha\|_\infty}{\prod_{i=1}^N \alpha_i!} \right) \left[\frac{1}{n^{\beta j}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \alpha_{k,n}(\beta) \right] \right) + \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left(\frac{2\alpha_{k,n}(\beta) \|b - a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \right\},$$

iv) Assume $f_\alpha(x_0) = 0$, for all $\alpha : |\alpha| = 1, ..., m$; $x_0 \in \left(\prod_{i=1}^N [a_i, b_i]\right)$. Then

$$|{}_kL_n(f, x_0) - f(x_0)| \leq \gamma_{k,N} \cdot \tag{74}$$

$$\left\{ \frac{N^m}{m! n^{m\beta}} \omega_1^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \left(\frac{2\alpha_{k,n}(\beta) \|b - a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

Proof. From Theorems 3 and 11. $\square$

Next we apply Theorem 14 for m=1.

Corollary 15. *Let $f \in C^1\left(\prod_{i=1}^N [a_i, b_i]\right)$, $0 < \beta < 1$, $n, m, N \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ and $k = 1, 2$. Then*

i)

$$\left| {}_kL_n(f, x) - f(x) - \sum_{i=1}^N \frac{\partial f}{\partial x_i} {}_kL_n((\cdot - x_i), x) \right| \leq \tag{75}$$

$$\gamma_{k,N}\cdot\left\{\frac{N}{n^\beta}\max_{i\in\{1,\dots,N\}}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{n^\beta}\right)+\right.$$
$$\left.\left(2\alpha_{k,n}N\left\|b-a\right\|_\infty\max_{i\in\{1,\dots,N\}}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty\right)\right\},$$

ii)

$$\left|{}_kL_n\left(f,x\right)-f\left(x\right)\right|\leq\gamma_{k,N}\cdot \tag{76}$$
$$\left\{\sum_{i=1}^N\left|\frac{\partial f(x)}{\partial x_i}\right|\left[\frac{1}{n^\beta}+\left(b_i-a_i\right)\cdot\alpha_{k,n}\left(\beta\right)\right]+\right.$$
$$\left.\frac{N}{n^\beta}\max_{i\in\{1,\dots,N\}}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{n^\beta}\right)+\left(2\alpha_{k,n}\left(\beta\right)N\left\|b-a\right\|_\infty\max_{i\in\{1,\dots,N\}}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty\right)\right\},$$

iii)

$$\left\|{}_kL_n\left(f\right)-f\right\|_\infty\leq\gamma_{k,N}\cdot \tag{77}$$
$$\left\{\sum_{i=1}^N\left|\frac{\partial f}{\partial x_i}\right|_\infty\left[\frac{1}{n^\beta}+\left(b_i-a_i\right)\alpha_{k,n}\left(\beta\right)\right]+\right.$$
$$\left.\frac{N}{n^\beta}\max_{i\in\{1,\dots,N\}}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{n^\beta}\right)+\left(2\alpha_{k,n}\left(\beta\right)N\left\|b-a\right\|_\infty\max_{i\in\{1,\dots,N\}}\|\frac{\partial f}{\partial x_i}\|_\infty\right)\right\}.$$

iv) Assume $\frac{\partial f}{\partial x_i}\left(x_0\right)=0, i=1,\dots,N,$ *where* $x_0\in\left(\prod_{i=1}^N\left[a_i,b_i\right]\right)$. *Then*

$$\left|{}_kL_n\left(f,x_0\right)-f\left(x_0\right)\right|\leq\gamma_{k,N}\cdot \tag{78}$$
$$\left\{\frac{N}{n^\beta}\max_{i\in\{1,\dots,N\}}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{n^\beta}\right)+\left(2\alpha_{k,n}\left(\beta\right)N\left\|b-a\right\|_\infty\max_{i\in\{1,\dots,N\}}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty\right)\right\},$$

notice in the last the extremely high rate of convergence at $n^{-2\beta}$.

3 Multiple Time Separating Random Functions

Let $(\Omega,\mathcal{F},P)$ be a probability space, $\omega\in\Omega; Y_1,Y_2,\dots,Y_m, m\in\mathbb{N}$, be real-valued random variables on Ω with finite expectations, and $h_1(t),h_2(t),\dots h_m(t):\prod_{j=1}^N I_j\to\mathbb{R}$, where I_j are infinite subsets of $\mathbb{R}$ for every $j=1,\dots,N$. Typically here I_j is an infinite length interval of $\mathbb{R}$, usualy $I_j=\mathbb{R}$ or $I_j=\mathbb{R}_+$, for $j=1,\dots,N$.
Clearly, then

$$Y\left(t,\omega\right):=\sum_{i=1}^m h_i(t)Y_i(\omega),\ t\in\prod_{j=1}^N I_j, \tag{79}$$

is a quite common time separating random function.
We can assume that $h_i\in C^r(\prod_{j=1}^N I_j), i=1,2,...,m; r\in\mathbb{N}$. Consequently, we have that the expectation

$$\left(EY\right)\left(t\right)=\sum_{i=1}^m h_i(t)EY_i\in C\left(\prod_{j=1}^N I_j\right)\text{ or }C^r\left(\prod_{j=1}^N I_j\right). \tag{80}$$

A classical example of multiple time separating process is

$$\left(\sin\left(\prod_{j=1}^N t_j\right)\right)Y_1(\omega)+\left(\cos\left(\prod_{j=1}^N t_j\right)\right)Y_2(\omega),\ t_j\in I_j,$$

for $j = 1, \ldots, N$.

Notice that $\left| \sin \left(\prod_{j=1}^{N} t_j \right) \right| \leq 1$ and $\left| \cos \left(\prod_{j=1}^{N} t_j \right) \right| \leq 1$.

Another typical example is

$$\left(\sinh \left(\prod_{j=1}^{N} t_j \right) \right) Y_1(\omega) + \left(\cosh \left(\prod_{j=1}^{N} t_j \right) \right) Y_2(\omega), \ t_j \in I_j, \text{ for } j = 1, \ldots, N. \tag{81}$$

In this article we will apply the main results of Section 2.3, to $f(t) = (EY)(t)$. We will finish with several applications.

4 Main Results

We present the following stochastic approximation result.

Theorem 16. *Let $(EY)(t)$ as in (80), $t \in \prod_{j=1}^{N} I_j$. Here $I_j = [t_{1,j}, t_{2,j}]$, where $t_{1,j}, t_{2,j} \in \mathbb{R}$ with $t_{1,j} < t_{2,j}$, for every $j = 1, \ldots, N$. Let also $0 < \beta < 1$, $n, N \in \mathbb{N}$ and $k = 1, 2$. Then*

i)

$$|{}_kL_n((EY), t) - (EY)(t)| \leq \gamma_{k,N}$$

$$\left\{ \omega_1 \left((EY), \frac{1}{n^\beta} \right) + 2\alpha_{k,n}(\beta) \|(EY)\|_\infty \right\} =: \lambda_1, \tag{82}$$

ii)

$$\|{}_kL_n(EY) - (EY)\|_\infty \leq \lambda_1. \tag{83}$$

Proof. From Theorem 12. □

It follows the counter part of the previous Theorem.

Theorem 17. *Let $(EY)(t)$ as in (80), $h_i \in C_B(\mathbb{R}^\mathbb{N})$ for every $i = 1, \ldots, m$. Let also $0 < \beta < 1$, $n, N \in \mathbb{N}$ and $k = 1, 2$. Then*

i)

$$\left| {}_k\overline{L}_n((EY), t) - (EY)(t) \right| \leq \omega_1 \left((EY), \frac{1}{n^\beta} \right) + 2\alpha_{k,n}(\beta) \|(EY)\|_\infty =: \lambda_2, \tag{84}$$

ii)

$$\left\| {}_k\overline{L}_n(EY) - (EY) \right\|_\infty \leq \lambda_2. \tag{85}$$

Proof. From Theorem 13. □

We give

Theorem 18. *Let $(EY)(t)$ as in (80), $(EY)(t) \in C^m \left(\prod_{j=1}^{N} I_j \right)$, $t \in \prod_{j=1}^{N} I_j$. Here $I_j = [t_{1,j}, t_{2,j}]$, where $t_{1,j}, t_{2,j} \in \mathbb{R}$ with $t_{1,j} < t_{2,j}$, for $j = 1, \ldots, N, t_1 = (t_{1,1}, \ldots, t_{1,N}), t_2 = (t_{2,1}, \ldots, t_{2,N})$. Let also $0 < \beta < 1$, $n, N \in \mathbb{N}$ and $k = 1, 2$. Then* *i)*

$$\left| {}_kL_n\left(\left(EY\right),t\right)-\left(EY\right)(t)-\sum_{j^*=1}^{m}\left(\sum_{|\alpha|=j^*}\left(\frac{\left(EY\right)_\alpha(t)}{\prod_{i=1}^{N}\alpha_i!}\right){}_kL_n\left(\prod_{i=1}^{N}\left(\cdot-x_i\right)^{\alpha_i},x\right)\right)\right|\leq \tag{86}$$

$$\gamma_{k,N}\cdot\left\{\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(\left(EY\right)_\alpha,\frac{1}{n^\beta}\right)+\right.$$

$$\left.\left(\frac{2\alpha_{k,n}\left\|t_2-t_1\right\|_\infty^m\left\|\left(EY\right)_\alpha\right\|_{\infty,m}^{\max}N^m}{m!}\right)\right\},$$

ii)

$$\left|{}_kL_n\left(\left(EY\right),t\right)-\left(EY\right)(t)\right|\leq\gamma_{k,N}\cdot \tag{87}$$

$$\left\{\sum_{j^*=1}^{m}\left(\sum_{|\alpha|=j^*}\left(\frac{\left|\left(EY\right)_\alpha(t)\right|}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta j^*}}+\left(\prod_{i=1}^{N}\left(t_{2,i}-t_{1,i}\right)^{\alpha_i}\right)\cdot\alpha_{k,n}(\beta)\right]\right)+\right.$$

$$\left.\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(\left(EY\right)_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{2\alpha_{k,n}(\beta)\left\|t_2-t_1\right\|_\infty^m\left\|\left(EY\right)_\alpha\right\|_{\infty,m}^{\max}N^m}{m!}\right)\right\},$$

iii)

$$\left\|{}_kL_n\left(\left(EY\right)\right)-\left(EY\right)\right\|_\infty\leq\gamma_{k,N}\cdot \tag{88}$$

$$\left\{\sum_{j^*=1}^{N}\left(\sum_{|\alpha|=j^*}\left(\frac{\left\|\left(EY\right)_\alpha\right\|_\infty}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta j^*}}+\left(\prod_{i=1}^{N}\left(t_{2,i}-t_{1,i}\right)^{\alpha_i}\right)\alpha_{k,n}(\beta)\right]\right)+\right.$$

$$\left.\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{2\alpha_{k,n}(\beta)\left\|t_2-t_1\right\|_\infty^m\left\|f_\alpha\right\|_{\infty,m}^{\max}N^m}{m!}\right)\right\},$$

iv) Assume $\left(EY\right)_\alpha(t_0)=0$*, for all* $\alpha:|\alpha|=1,...,m;$ $x_0\in\left(\prod_{i=1}^{N}\left[t_{1,i},t_{2,i}\right]\right)$*. Then*

$$\left|{}_kL_n\left(\left(EY\right),t_0\right)-\left(EY\right)(t_0)\right|\leq\gamma_{k,N}\cdot \tag{89}$$

$$\left\{\frac{N^m}{m!n^{m\beta}}\omega_1^{\max}\left(\left(EY\right)_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{2\alpha_{k,n}(\beta)\left\|t_2-t_1\right\|_\infty^m\left\|f_\alpha\right\|_{\infty,m}^{\max}N^m}{m!}\right)\right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$*.*

Proof. From Theorem 14. □
Next we apply Theorem 18 for m=1.

Corollary 19. *Let* $\left(EY\right)(t)$ *as in (80),* $\left(EY\right)(t)\in C^1\left(\prod_{j=1}^{N}I_j\right)$, $t\in\prod_{j=1}^{N}I_j$*. Here* $I_j=\left[t_{1,j},t_{2,j}\right]$*, where* $t_{1,j},t_{2,j}\in\mathbb{R}$ *with* $t_{1,j}<t_{2,j}$ *for* $j=1,\ldots,N$, $t_1=\left(t_{1,1},\ldots,t_{1,N}\right)$, $t_2=\left(t_{2,1},\ldots,t_{2,N}\right)$*. Let also* $0<\beta<1$*,* $n,m,N\in\mathbb{N}$*, and* $k=1,2$*. Then*

i)

$$\left|{}_kL_n\left(\left(EY\right),t\right)-\left(EY\right)(t)-\sum_{i=1}^{N}\frac{\partial\left(EY\right)}{\partial t_i}\,{}_kL_n\left(\left(\cdot-t_i\right),t\right)\right|\leq \tag{90}$$

$$\gamma_{k,N}\cdot\left\{\frac{N}{n^\beta}\max_{i\in\{1,\ldots,N\}}\omega_1\left(\frac{\partial\left(EY\right)}{\partial t_i},\frac{1}{n^\beta}\right)+\right.$$

$$\left(2\alpha_{k,n} N \left\|t_2 - t_1\right\|_\infty \max_{i\in\{1,\dots,N\}} \left\|\frac{\partial (EY)}{\partial t_i}\right\|_\infty\right)\Bigg\},$$

ii)

$$\left|{}_kL_n\left((EY),t\right) - (EY)(t)\right| \leq \gamma_{k,N} \cdot \tag{91}$$

$$\Bigg\{\sum_{i=1}^{N}\left|\frac{\partial (EY)(t)}{\partial t_i}\right|\left[\frac{1}{n^\beta} + (t_{2,i} - t_{1,i})\cdot \alpha_{k,n}(\beta)\right] +$$

$$\frac{N}{n^\beta}\max_{i\in\{1,\dots,N\}}\omega_1\left(\frac{\partial (EY)}{\partial t_i}, \frac{1}{n^\beta}\right) + \left(2\alpha_{k,n}(\beta) N \left\|t_2 - t_1\right\|_\infty \max_{i\in\{1,\dots,N\}} \left\|\frac{\partial (EY)}{\partial t_i}\right\|_\infty\right)\Bigg\},$$

iii)

$$\left\|{}_kL_n\left((EY)\right) - (EY)\right\|_\infty \leq \gamma_{k,N} \cdot \tag{92}$$

$$\Bigg\{\sum_{i=1}^{N}\left|\frac{\partial (EY)}{\partial t_i}\right|_\infty \left[\frac{1}{n^\beta} + (t_{2,i} - t_{1,i})\,\alpha_{k,n}(\beta)\right] +$$

$$\frac{N}{n^\beta}\max_{i\in\{1,\dots,N\}}\omega_1\left(\frac{\partial (EY)}{\partial t_i}, \frac{1}{n^\beta}\right) + \left(2\alpha_{k,n}(\beta) N \left\|t_2 - t_1\right\|_\infty \max_{i\in\{1,\dots,N\}} \|\frac{\partial (EY)}{\partial t_i}\|_\infty\right)\Bigg\}.$$

iv) Assume $\frac{\partial(EY)}{\partial t_i}(t_0) = 0, i = 1,\dots,N,$ *where* $t_0 \in \left(\prod_{i=1}^N [t_{1,i}, t_{2,i}]\right)$*. Then*

$$\left|{}_kL_n\left((EY),t_0\right) - (EY)(t_0)\right| \leq \gamma_{k,N} \cdot \tag{93}$$

$$\Bigg\{\frac{N}{n^\beta}\max_{i\in\{1,\dots,N\}}\omega_1\left(\frac{\partial (EY)}{\partial t_i}, \frac{1}{n^\beta}\right) + \left(2\alpha_{k,n}(\beta) N \left\|t_2 - t_1\right\|_\infty \max_{i\in\{1,\dots,N\}} \left\|\frac{\partial (EY)}{\partial t_i}\right\|_\infty\right)\Bigg\},$$

notice in the last the extremely high rate of convergence at $n^{-2\beta}$.

5 Applications

For the next applications we consider (Ω, F, P) be a probability space and Y_0, Y_1, Y_2 be real valued random variables on Ω with finite expectations. We consider the stochastic processes $Z_i(t,\omega)$ for $i = 1,2,\dots,7$, where $t = (t_1,\dots,t_N) \in \mathbb{R}^{\mathbb{N}}$ and $\omega \in \Omega$ as follows:

$$Z_1(t,\omega) = \left[\left(\sum_{j=1}^{N}(t_j - t_{j,0})\right)^{\mu+1} + 1\right] Y_0(\omega), \tag{94}$$

where $t_0 = (t_{1,0},\dots,t_{N,0}) \in \mathbb{R}^{\mathbb{N}}$ and $\mu \in \mathbb{N}$ are fixed;

$$Z_2(t,\omega) = \sin\left(\xi\sum_{j=1}^{N} t_j\right) Y_1(\omega) + \cos\left(\xi\sum_{j=1}^{N} t_j\right) Y_2(\omega), \tag{95}$$

where $\xi > 0$ is fixed;

$$Z_3(t,\omega) = \sinh\left(\mu\sum_{j=1}^{N} t_j\right) Y_1(\omega) + \cosh\left(\mu\sum_{j=1}^{N} t_j\right) Y_2(\omega), \tag{96}$$

where $\mu > 0$ is fixed;

$$Z_4(t,\omega) = \mathit{sech}\left(\mu\sum_{j=1}^{N} t_j\right) Y_1(\omega) + \tanh\left(\mu\sum_{j=1}^{N} t_j\right) Y_2(\omega), \tag{97}$$

where $\mu > 0$ is fixed.
Here $sechx := \frac{1}{\cosh\left(\sum_{j=1}^{N} x_j\right)} = \frac{2}{\exp\left(\sum_{j=1}^{N} x_j\right)+\exp\left(-\sum_{j=1}^{N} x_j\right)}, x = (x_1, \ldots, x_n) \in \mathbb{R}^{\mathbb{N}}$.

$$Z_5(t,\omega) = \exp\left(-\ell_1 \prod_{j=1}^{N} t_j\right) Y_1(\omega) + \exp\left(-\ell_2 \prod_{j=1}^{N} t_j\right) Y_2(\omega), \tag{98}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_6(t,\omega) = \frac{1}{1+\exp\left(-\ell_1 \prod_{j=1}^{N} t_j\right)} Y_1(\omega) + \frac{1}{1+\exp\left(-\ell_2 \prod_{j=1}^{N} t_j\right)} Y_2(\omega), \tag{99}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_7(t,\omega) = e^{-e^{-\mu_1 \prod_{j=1}^{N} t_j}} Y_1(\omega) + e^{-e^{-\mu_2 \prod_{j=1}^{N} t_j}} Y_2(\omega), \tag{100}$$

where $\mu_1, \mu_2 > 0$ are fixed;
The expectations of $Z_i, i = 1, 2, ..., 7$ are

$$(EZ_1)(t) = \left[\left(\sum_{j=1}^{N} (t_j - t_{j,0})\right)^{\mu+1} + 1\right] E(Y_0), \tag{101}$$

$$(EZ_2)(t) = \sin\left(\xi \sum_{j=1}^{N} t_j\right) E(Y_1) + \cos\left(\xi \sum_{j=1}^{N} t_j\right) E(Y_2), \tag{102}$$

$$(EZ_3)(t) = \sinh\left(\mu \sum_{j=1}^{N} t_j\right) E(Y_1) + \cosh\left(\mu \sum_{j=1}^{N} t_j\right) E(Y_2), \tag{103}$$

$$(EZ_4)(t) = sech\left(\mu \sum_{j=1}^{N} t_j\right) E(Y_1) + \tanh\left(\mu \sum_{j=1}^{N} t_j\right) E(Y_2), \tag{104}$$

$$(EZ_5)(t) = \exp\left(-\ell_1 \prod_{j=1}^{N} t_j\right) E(Y_1) + \exp\left(-\ell_2 \prod_{j=1}^{N} t_j\right) E(Y_2), \tag{105}$$

$$(EZ_6)(t) = \frac{1}{1+\exp\left(-\ell_1 \prod_{j=1}^{N} t_j\right)} E(Y_1) + \frac{1}{1+\exp\left(-\ell_2 \prod_{j=1}^{N} t_j\right)} E(Y_2), \tag{106}$$

$$(EZ_7)(t) = e^{-e^{-\mu_1 \prod_{j=1}^{N} t_j}} E(Y_1) + e^{-e^{-\mu_2 \prod_{j=1}^{N} t_j}} E(Y_2), \tag{107}$$

For the next $(EZ_i)(t), i = 1, 2, \ldots, 7$ are as defined in relations between (101) and (107) respectively.
We present the following result.

Proposition 20. *Let* $t \in \prod_{j=1}^{N} I_j$. *Here* $I_j = [t_{1,j}, t_{2,j}]$, *where* $t_{1,j}, t_{2,j} \in \mathbb{R}$ *with* $t_{1,j} < t_{2,j}$, *for every* $j = 1, \ldots, N$. *Let also* $0 < \beta < 1$, $n, N \in \mathbb{N}$ *and* $k = 1, 2$. *Then, for* $i = 1, 2, \ldots, 7$

i)

$$|{}_kL_n\left(\left(EZ_i\right), t\right) - \left(EZ_i\right)(t)| \leq \gamma_{k,N}$$

$$\left\{\omega_1\left(\left(EZ_i\right), \frac{1}{n^\beta}\right) + 2\alpha_{k,n}(\beta)\left\|\left(EZ_i\right)\right\|_\infty\right\} =: \lambda_1, \tag{108}$$

ii)

$$\left\|{}_kL_n\left(EZ_i\right) - \left(EZ_i\right)\right\|_\infty \leq \lambda_1. \tag{109}$$

Proof. From Theorem 16. □

Next we present.

Proposition 21. *Let* $0 < \beta < 1$, $n, N \in \mathbb{N}$ *and* $k = 1, 2$. *Then, for* $i \in \{2, 4, 6, 7\}$,

i)

$$\left|{}_k\overline{L}_n\left(\left(EZ_i\right), t\right) - \left(EZ_i\right)(t)\right| \leq \omega_1\left(\left(EZ_i\right), \frac{1}{n^\beta}\right) + 2\alpha_{k,n}(\beta)\left\|\left(EZ_i\right)\right\|_\infty =: \lambda_2, \tag{110}$$

ii)

$$\left\|{}_k\overline{L}_n\left(EZ_i\right) - \left(EZ_i\right)\right\|_\infty \leq \lambda_2. \tag{111}$$

Proof. Notice that for every $t \in \mathbb{R}$ we have that:

- for $Z_2(t, \omega)$, $\left|\sin\left(\xi \sum_{i=1}^{N} t_i\right)\right| \leq 1$ and $\left|\cos\left(\xi \sum_{i=1}^{N} t_i\right)\right| \leq 1$,
- for $Z_4(t, \omega)$, $\left|sech\left(\mu \sum_{i=1}^{N} t_i\right)\right| \leq 1$ and $\left|tanh\left(\mu \sum_{i=1}^{N} t_i\right)\right| \leq 1$,
- for $Z_6(t, \omega)$, $0 < \frac{1}{1+\exp\left(-\ell_1 \prod_{i=1}^{N} t_i\right)} < 1$ and $0 < \frac{1}{1+\exp\left(-\ell_2 \prod_{i=1}^{N} t_i\right)} < 1$,
- for $Z_7(t, \omega)$, $0 < e^{-e^{-\mu_1 \prod_{i=1}^{N} t_i}} < 1$ and $0 < e^{-e^{-\mu_2 \prod_{i=1}^{N} t_i}} < 1$.

Thus, the results come from Theorem 18. □

Proposition 22. *Let* $t \in \prod_{i=j}^{N} I_j$. *Here* $I_j = [t_{1,j}, t_{2,j}]$, *where* $t_{1,j}, t_{2,j} \in \mathbb{R}$ *with* $t_{1,j} < t_{2,j}$ *for* $j = 1, \ldots, N, t_1 = (t_{1,1}, \ldots, t_{1,N}), t_2 = (t_{2,1}, \ldots, t_{2,N})$. *Let also* $0 < \beta < 1$, $n, m, N \in \mathbb{N}$, *and* $k = 1, 2$. *Then, for* $i = 1, \ldots, 7$

i)

$$\left|{}_kL_n\left(\left(EZ_i\right), t\right) - \left(EZ_i\right)(t) - \sum_{j=1}^{N} \frac{\partial\left(EZ_i\right)}{\partial t_j}\, {}_kL_n\left(\left(\cdot - t_j\right), t\right)\right| \leq \tag{112}$$

$$\gamma_{k,N} \cdot \left\{\frac{N}{n^\beta} \max_{j \in \{1, \ldots, N\}} \omega_1\left(\frac{\partial\left(EZ_i\right)}{\partial t_j}, \frac{1}{n^\beta}\right) + \right.$$

$$\left.\left(2\alpha_{k,n} N \left\|t_2 - t_1\right\|_\infty \max_{j \in \{1, \ldots, N\}} \left\|\frac{\partial\left(EZ_i\right)}{\partial t_j}\right\|_\infty\right)\right\}.$$

ii) Assume $\frac{\partial (EZ_i)}{\partial t_j}(t_0)=0, j=1,\ldots,N,$ *where* $t_0 \in \left(\prod_{j=1}^{N}[t_{1,j},t_{2,j}]\right)$. *Then*

$$\left|{}_kL_n\left(\left(EZ_i\right),t_0\right)-\left(EZ_i\right)(t_0)\right| \leq \gamma_{k,N}\cdot \tag{113}$$

$$\left\{\frac{N}{n^{\beta}}\max_{j\in\{1,\ldots,N\}}\omega_1\left(\frac{\partial\left(EZ_i\right)}{\partial t_j},\frac{1}{n^{\beta}}\right)+\left(2\alpha_{k,n}\left(\beta\right)N\left\|t_2-t_1\right\|_{\infty}\max_{j\in\{1,\ldots,N\}}\left\|\frac{\partial\left(EZ_i\right)}{\partial t_j}\right\|_{\infty}\right)\right\},$$

notice in the last the extremely high rate of convergence at $n^{-2\beta}$.

Proof. From Corolary 19 □

6 Specific Applications

Let $(\Omega,\mathcal{F},P)$, where Ω is the set of non-negative integers, be a probability space, $Y_{1,1},Y_{2,1}$ be real-valued random variables on Ω following Poisson distributions with parameters $\lambda_1,\lambda_2\in(0,\infty)$ respectively.

We consider the stochastic process $W_1(t,\omega)$, where $t=(t_1,\ldots,t_N)\in\mathbb{R}^{\mathbb{N}}$ and $\omega\in\Omega$ as follows:

$$W_1(t,\omega)=\sin\left(\xi\sum_{j=1}^{N}t_j\right)Y_{1,1}(\omega)+\cos\left(\xi\sum_{j=1}^{N}t_j\right)Y_{2,1}(\omega), \tag{114}$$

where $\xi>0$ is fixed.

Since $E\left(Y_{1,1}\right)=\lambda_1$ and $E\left(Y_{2,1}\right)=\lambda_2$, the expectations of $Z_{2,1}$ are

$$\left(EW_1\right)(t)=\sin\left(\xi\sum_{j=1}^{N}t_j\right)\lambda_1+\cos\left(\xi\sum_{j=1}^{N}t_j\right)\lambda_2, \tag{115}$$

furthermore, for $j^*=1,\ldots,N$ we have that

$$\frac{\partial\left(EW_1\right)(t)}{\partial t_{j^*}}=\xi\left[\cos\left(\xi\sum_{j=1}^{N}t_j\right)\lambda_1-\sin\left(\xi\sum_{j=1}^{N}t_j\right)\lambda_2\right]=:D_1(t). \tag{116}$$

Notice that

$$\frac{\partial\left(EW_1\right)(t)}{\partial t_{j^*}}=\frac{\partial\left(EW_1\right)(t)}{\partial t_{i^*}}=D_1(t)$$

for every $j^*,i^*=1,\ldots,N$.

For the next we consider $(\Omega,\mathcal{F},P)$, where $\Omega=\mathbb{R}$, be a probability space, $Y_{1,2},Y_{2,2}$ on Ω following Gaussian distributions with expectations $\hat{\mu}_1,\hat{\mu}_2\in\mathbb{R}$ respectively. We consider the stochastic process $W_2(t,\omega)$, where $t=(t_1,\ldots,t_N)\in\mathbb{R}^{\mathbb{N}}$ and $\omega\in\Omega$ as follows:

$$W_2(t,\omega)=sech\left(\mu\sum_{j=1}^{N}t_j\right)Y_{1,2}(\omega)+\tanh\left(\mu\sum_{j=1}^{N}t_j\right)Y_{2,2}(\omega), \tag{117}$$

where $\mu>0$ is fixed.

Since $E\left(Y_{1,1}\right)=\hat{\mu}_1$ and $E\left(Y_{2,1}\right)=\hat{\mu}_2$, the expectations of $Z_{2,1}$ are

$$\left(EW_2\right)(t)=\hat{\mu}_1 sech\left(\mu\sum_{j=1}^{N}t_j\right)+\hat{\mu}_2\tanh\left(\mu\sum_{j=1}^{N}t_j\right), \tag{118}$$

furthermore, for $j^* = 1, \dots, N$ we have that

$$\frac{\partial (EW_2)(t)}{\partial t_{j^*}} = -\mu \left[\hat{\mu}_1 sech \left(\mu \sum_{j=1}^{N} t_j \right) \tanh \left(\mu \sum_{j=1}^{N} t_j \right) + \hat{\mu}_2 \left(\tanh^2 \left(\mu \sum_{j=1}^{N} t_j \right) - 1 \right) \right] =: D_2(t) \tag{119}$$

Notice that

$$\frac{\partial (EW_2)(t)}{\partial t_{j^*}} = \frac{\partial (EW_2)(t)}{\partial t_{i^*}} = D_2(t)$$

for every $j^*, i^* = 1, \dots, N$.

Next, we consider $(\Omega, \mathcal{F}, P)$, where $\Omega = [0, \infty)$, be a probability space, $Y_{1,3}, Y_{2,3}$ be real-valued random variables on Ω following Weibull distributions with scale parameters 1 and shape parameters $\gamma_1, \gamma_2 \in (0, \infty)$ respectively.

We consider the stochastic process $W_3(t, \omega)$, where $t = (t_1, \dots, t_N) \in \mathbb{R}^{\mathbb{N}}$ and $\omega \in \Omega$ as follows:

$$W_3(t, \omega) = \frac{1}{1 + \exp\left(-\ell_1 \prod_{j=1}^{N} t_j \right)} Y_{1,3}(\omega) + \frac{1}{1 + \exp\left(-\ell_2 \prod_{j=1}^{N} t_j \right)} Y_{2,3}(\omega), \tag{120}$$

where $\ell_1, \ell_2 > 0$ are fixed;

Since $E(Y_{1,3}) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)$ and $E(Y_{2,3}) = \Gamma\left(1 + \frac{1}{\gamma_2}\right)$, where $\Gamma(\cdot)$ is the Gamma function, The expectations of $Z_{i,3}, i = 1, 2, 3, 5$, are

$$(EW_3)(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right) \frac{1}{1 + \exp\left(-\ell_1 \prod_{j=1}^{N} t_j \right)} + \Gamma\left(1 + \frac{1}{\gamma_2}\right) \frac{1}{1 + \exp\left(-\ell_2 \prod_{j=1}^{N} t_j \right)}, \tag{121}$$

furthermore, for $j = 1, \dots, N$ we have that

$$r\frac{\partial (EW_3)(t)}{\partial t_j} = \left(\prod_{r=1, r\neq j}^{N} t_r \right) \left[\frac{\ell_1 \Gamma\left(1 + \frac{1}{\gamma_1}\right) \exp\left(-\ell_1 \prod_{r=1}^{N} t_r \right)}{\left(1 + \exp\left(-\ell_1 \prod_{r=1}^{N} t_r \right)\right)^2} + \frac{\ell_2 \Gamma\left(1 + \frac{1}{\gamma_2}\right) \exp\left(-\ell_2 \prod_{r=1}^{N} t_r \right)}{\left(1 + \exp\left(-\ell_2 \prod_{r=1}^{N} t_r \right)\right)^2} \right] \tag{122}$$

Notice that

$$\frac{\partial (EW_3)(t)}{\partial t_j} = \left(\prod_{i=1, i\neq j}^{N} t_i \right) D_3(t)$$

for every $j = 1, \dots, N$, where

$$D_3(t) =: \frac{\ell_1 \Gamma\left(1 + \frac{1}{\gamma_1}\right) \exp\left(-\ell_1 \prod_{j=1}^{N} t_j \right)}{\left(1 + \exp\left(-\ell_1 \prod_{j=1}^{N} t_j \right)\right)^2} + \frac{\ell_2 \Gamma\left(1 + \frac{1}{\gamma_2}\right) \exp\left(-\ell_2 \prod_{j=1}^{N} t_j \right)}{\left(1 + \exp\left(-\ell_2 \prod_{j=1}^{N} t_j \right)\right)^2} \tag{123}$$

We present the following result.

Proposition 23. *Let* $t \in \prod_{j=1}^{N} I_j$. *Here* $I_j = [t_{1,j}, t_{2,j}]$, *where* $t_{1,j}, t_{2,j} \in \mathbb{R}$ *with* $t_{1,j} < t_{2,j}$, *for every* $j = 1, \ldots, N$. *Let also* $0 < \beta < 1$, $n, N \in \mathbb{N}$ *and* $k = 1, 2$. *Then for* $i = 1, 2, 3$,

i)

$$\left|{}_kL_n\left(\left(EW_i\right), t\right) - \left(EW_i\right)(t)\right| \leq \gamma_{k,N}$$

$$\left\{\omega_1\left(\left(EW_i\right), \frac{1}{n^\beta}\right) + 2\alpha_{k,n}(\beta)\left\|\left(EW_i\right)\right\|_\infty\right\} =: \lambda_1, \tag{124}$$

ii)

$$\left\|{}_kL_n\left(EW_i\right) - \left(EW_i\right)\right\|_\infty \leq \lambda_1. \tag{125}$$

Proof. From Proposition 20. □

Next we present.

Proposition 24. *Let* $0 < \beta < 1$, $n, N \in \mathbb{N}$ *and* $k = 1, 2$. *Then for* $i = 1, 2, 3$,

i)

$$\left|{}_k\overline{L}_n\left(\left(EW_i\right), t\right) - \left(EW_i\right)(t)\right| \leq \omega_1\left(\left(EW_i\right), \frac{1}{n^\beta}\right) + 2\alpha_{k,n}(\beta)\left\|\left(EW_i\right)\right\|_\infty =: \lambda_2, \tag{126}$$

ii)

$$\left\|{}_k\overline{L}_n\left(EW_i\right) - \left(EW_i\right)\right\|_\infty \leq \lambda_2. \tag{127}$$

Proof. From Proposition 21. □

Proposition 25. *Let* $t \in \prod_{i=j}^{N} I_j$. *Here* $I_j = [t_{1,j}, t_{2,j}]$, *where* $t_{1,j}, t_{2,j} \in \mathbb{R}$ *with* $t_{1,j} < t_{2,j}$ *for* $j = 1, \ldots, N, t_1 = (t_{1,1}, \ldots, t_{1,N}), t_2 = (t_{2,1}, \ldots, t_{2,N})$. *Let also* $0 < \beta < 1$, $n, m, N \in \mathbb{N}$, *and* $k = 1, 2$. *Then for* $i = 1, 2$,

i)

$$\left|{}_kL_n\left(\left(EW_i\right), t\right) - \left(EW_i\right)(t) - D_i(t)\sum_{j=1}^{N} {}_kL_n\left(\left(\cdot - t_j\right), t\right)\right| \leq \tag{128}$$

$$\gamma_{k,N} \cdot \left\{\frac{N}{n^\beta}\omega_1\left(D_i(t), \frac{1}{n^\beta}\right) + \left(2\alpha_{k,n}N\left\|t_2 - t_1\right\|_\infty\left\|D_i(t)\right\|_\infty\right)\right\},$$

ii) Assume $D_i(t_0) = 0$, *where* $t_0 \in \left(\prod_{j=1}^{N}[t_{1,j}, t_{2,j}]\right)$. *Then*

$$\left|{}_kL_n\left(\left(EW_i\right), t_0\right) - \left(EW_i\right)(t_0)\right| \leq \gamma_{k,N} \cdot \tag{129}$$

$$\left\{\frac{N}{n^\beta}\omega_1\left(D_i(t), \frac{1}{n^\beta}\right) + \left(2\alpha_{k,n}(\beta)N\left\|t_2 - t_1\right\|_\infty\left\|D_i(t)\right\|_\infty\right)\right\},$$

notice in the last the extremely high rate of convergence at $n^{-2\beta}$.

Proof. From Proposition 22 □

Proposition 26. *Let* $t \in \prod_{i=j}^{N} I_j$. *Here* $I_j = [t_{1,j}, t_{2,j}]$, *where* $t_{1,j}, t_{2,j} \in \mathbb{R}$ *with* $t_{1,j} < t_{2,j}$ *for* $j = 1, \ldots, N, t_1 = (t_{1,1}, \ldots, t_{1,N}), t_2 = (t_{2,1}, \ldots, t_{2,N})$. *Let also* $0 < \beta < 1$, $n, m, N \in \mathbb{N}$, *and* $k = 1, 2$. *Then*

i)

$$\left| {}_kL_n\left((EW_3), t\right) - (EW_3)(t) - D_3(t) \sum_{j=1}^{N} \left(\prod_{i=1, i\neq j}^{N} t_i \right) {}_kL_n\left((\cdot - t_j), t\right) \right| \leq \tag{130}$$

$$\gamma_{k,N} \cdot \left\{ \frac{N}{n^\beta} \max_{j \in \{1,\ldots,N\}} \omega_1 \left(D_3(t) \prod_{i=1, i\neq j}^{N} t_i, \frac{1}{n^\beta} \right) + \right.$$

$$\left. \left(2\alpha_{k,n} N \left\| t_2 - t_1 \right\|_\infty \max_{j \in \{1,\ldots,N\}} \left\| D_3(t) \prod_{i=1, i\neq j}^{N} t_i \right\|_\infty \right) \right\},$$

ii) Assume that there exist $t_0 = (t_{0,1}, t_{0,2}, \ldots, t_{0,N})$, *where* $t_0 \in \left(\prod_{j=1}^{N} [t_{1,j}, t_{2,j}] \right)$, *such that* $t_{0,k} = t_{0,l} = 0$, *for* $k, l \in \{1, \ldots, N\}$, *with* $k \neq l$. *Then*

$$\left| {}_kL_n\left((EW_3), t_0\right) - (EW_3)(t_0) \right| \leq \gamma_{k,N} \cdot \tag{131}$$

$$\left\{ \frac{N}{n^\beta} \max_{j \in \{1,\ldots,N\}} \omega_1 \left(D_3(t) \prod_{i=1, i\neq j}^{N} t_j, \frac{1}{n^\beta} \right) + \left(2\alpha_{k,n}(\beta) N \left\| t_2 - t_1 \right\|_\infty \max_{j \in \{1,\ldots,N\}} \left\| \frac{\partial (EW_3)}{\partial t_j} \right\|_\infty \right) \right\},$$

notice in the last the extremely high rate of convergence at $n^{-2\beta}$.

Proof. From Proposition 22 □

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