NUMERICAL RADIUS INEQUALITIES FOR THE EXTENDED GENERALIZED ALUTHGE TRANSFORM OF BOUNDED OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a *contraction* $V \in \mathcal{B}(H)$, i.e. $0 \leq V^*V \leq I$, an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$ we define the operator

$$\Delta_{t,V}(T) := |T|^t V |T|^{1-t}$$

that we *call the extended generalized Aluthge transform*. In this paper we provide several numerical radius inequalities concerning the extended generalized Aluthge transform $\Delta_{t,V}(T)$. The cases of usual generalized Aluthge, Dougal and Aluthge transforms are also presented.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leq w(T)\|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda|\omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T + V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{1.3}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [10], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [11] improved the inequality (1.3) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{1.5}$$

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for any operator $T\in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [9]:

If for an operator $T\in B(H)$ we denote $|T|:=(T^*T)^{1/2}$, then

$$\omega^r(T)\le\frac{1}{2}\left\||T|^{2\alpha r}+|T^*|^{2(1-\alpha)r}\right\| \tag{1.6}$$

and

$$\omega^{2r}(T)\le\left\|\alpha|T|^{2r}+(1-\alpha)|T^*|^{2r}\right\|, \tag{1.7}$$

where $\alpha\in(0,1)$ and $r\ge 1$.

If we take $\alpha=\frac{1}{2}$ and $r=1$ we get from (1.6) that

$$\omega(T)\le\frac{1}{2}\||T|+|T^*|\| \tag{1.8}$$

and from (1.7) that

$$\omega^2(T)\le\frac{1}{2}\left\||T|^2+|T^*|^2\right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [8] and [4].

Let $T=U|T|$ be the *polar decomposition* of the bounded linear operator T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T}:=|T|^{1/2}U|T|^{1/2}$, see [1].

The following properties of $\widetilde{T}$ are as follows:

(i) $\left\|\widetilde{T}\right\|\le\|T\|$,

(ii) $\omega\left(\widetilde{T}\right)\le\omega(T)$,

(iii) $r\left(\widetilde{T}\right)=\omega(T)$,

(iv) $\omega\left(\widetilde{T}\right)\le\left\|T^2\right\|^{1/2}(\le\|T\|)$, [13].

Utilizing this transform T. Yamazaki, [13] obtained in 2007 the following refinement of Kittaneh's inequality (1.4):

$$\omega(T)\le\frac{1}{2}\left(\|T\|+\omega\left(\widetilde{T}\right)\right)\le\frac{1}{2}\left(\|T\|+\left\|T^2\right\|^{1/2}\right) \tag{1.10}$$

for any operator $T\in B(H)$.

We remark that if $\widetilde{T}=0$, then obviously $w(T)=\frac{1}{2}\|T\|$.

For a *contraction* $V\in\mathcal{B}(H)$, i.e. $0\le V^*V\le I$ and an operator $T\in\mathcal{B}(H)$ and $t\in[0,1]$ we define the operator

$$\Delta_{t,V}(T):=|T|^tV|T|^{1-t}$$

that we *call the extended generalized Aluthge transform.*

We assume in what follows that $|T|^0:=I$.

For $t=1$ we have

$$\widehat{T}_V:=\Delta_{1,V}(T)=|T|V,$$

that we call *the extended Dougal transform*, for $t=1/2$,

$$\widetilde{T}_V=\Delta_{1/2,V}(T):=|T|^{1/2}V|T|^{1/2},$$

that we *call the extended Aluthge transform* and for $t=0$,

$$T_V:=\Delta_{0,V}(T)=V|T|.$$

An operator $U \in \mathcal{B}(H)$ is called a *partial isometry* if $\|Ux\| = \|x\|$ for all $x \in \mathcal{N}^{\perp}(U)$.

Now, let $x \in H$, then there exists a unique $x_1 \in \mathcal{N}(U)$ and a unique $x_2 \in \mathcal{N}^{\perp}(U)$ such that $x = x_1 + x_2$. Then

$$0 \leq \langle U^* U x, x \rangle = \|Ux\|^2 = \|Ux_1 + Ux_2\|^2 = \|Ux_2\|^2 = \|x_2\|^2.$$

By the fact that $x_1 \perp x_2$,

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2.$$

Therefore

$$0 \leq \langle U^* U x, x \rangle \leq \|x\|^2,$$

which shows that U is a contraction on H.

Let $T \in \mathcal{B}(H)$ and $T = U|T|$ the polar decomposition of T with U a partial isometry. Then

$$T_U = U|T| = T,$$

$$\widetilde{T}_U = |T|^{1/2} U |T|^{1/2} = \widetilde{T}$$

is the usual *Aluthge transform* and

$$\widehat{T}_U = |T| U = \widehat{T}$$

is the usual *Dougal transform.*

For $t \in (0,1)$

$$\Delta_{t,U}(T) = |T|^t U |T|^{1-t} =: \Delta_t(T)$$

is the *generalized Aluthge transform* introduced in by Cho and Tanahashi in [7].

Abu-Omar and Kittaneh [2] improved on inequality (1.10) using generalized Aluthge transform to prove that

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \min_{t \in [0,1]} \omega(\Delta_t(T))\right).$$

For $t = 1$ this also gives the following result for the *Dougal transform*

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \omega\left(\widehat{T}\right)\right). \tag{1.11}$$

In [3] Bunia et al. also proved that

$$\omega(T) \leq \min_{t \in [0,1]}\left\{\frac{1}{2}\omega(\Delta_t(T)) + \frac{1}{4}\left(\|T\|^{2t} + \|T\|^{2(1-t)}\right)\right\},$$

which for $t = 1/2$ gives (1.10) as well.

If V is a contraction, then $\|V\| \leq 1$ and since $\|V^*\| = \|V\|$, hence V^* is also a contraction. Observe that

$$\Delta_{t,V}^*(T) := \left(|T|^t V |T|^{1-t}\right)^* = |T|^{1-t} V^* |T|^t = \Delta_{1-t,V^*}(T)$$

for all $t \in [0,1]$. Therefore

$$(T_V)^* = \widehat{T}_{V^*}, \quad \left(\widehat{T}_V\right)^* = T_{V^*}$$

and

$$\left(\widetilde{T}_V\right)^* = \widetilde{T}_{V^*}.$$

Since $\|V^*V\| = \|VV^*\| = \|V\|^2$ and V is a contraction, then

$$\left\| \frac{V^*V \pm VV^*}{2} \right\| \leq \|V\|^2 \leq 1$$

showing that

$$W := \frac{V^*V \pm VV^*}{2}$$

is a contraction and we can consider the transform

$$\Delta_{t, \frac{V^*V \pm VV^*}{2}}(T) := |T|^t \left(\frac{V^*V \pm VV^*}{2} \right) |T|^{1-t}$$

for $t \in [0,1]$.

For a contraction V, we have

$$\mathrm{Im}(V) := \frac{V - V^*}{2i}, \ \ \mathrm{Re}(V) := \mathrm{Re}\left(\frac{V + V^*}{2} \right)$$

and since

$$\|\mathrm{Im}(V)\| = \left\| \frac{V - V^*}{2i} \right\| \leq \|V\| \leq 1 \text{ and } \|\mathrm{Re}(V)\| \leq \|V\| \leq 1,$$

hence $\mathrm{Im}(V)$ and $\mathrm{Re}(V)$ are contractions as well. We can then consider the transforms

$$\Delta_{t,\mathrm{Im}(V)}(T) := |T|^t \,\mathrm{Im}(V) \,|T|^{1-t} \text{ and } \Delta_{t,\mathrm{Re}(V)}(T) := |T|^t \,\mathrm{Re}(V) \,|T|^{1-t}$$

for $t \in [0,1]$.

For $T \in \mathcal{B}(H)$ we define

$$T_+ := \frac{1}{2}(|T| + T) \text{ and } T_- := \frac{1}{2}(|T| - T).$$

If U is the partial isometry in the polar representation of T, then

$$V := \frac{I \pm U}{2}$$

is a contraction and

$$\Delta_{t, \frac{I \pm U}{2}}(T) := |T|^t \frac{I \pm U}{2} |T|^{1-t} = \frac{|T| \pm \Delta_t(T)}{2}.$$

In particular, we get

$$T_{\frac{I \pm U}{2}} = \frac{|T| \pm T}{2} = T_\pm, \widehat{T}_{\frac{I \pm U}{2}} = \frac{|T| \pm \widehat{T}}{2}$$

and

$$\widetilde{T}_{\frac{I \pm U}{2}} = \frac{|T| \pm \widetilde{T}}{2}$$

for any operator $T \in \mathcal{B}(H)$.

Motivated by the above results, in this paper we provide several numerical radius inequalities concerning the extended generalized Aluthge transform $\Delta_{t,V}(T)$. The cases of usual generalized Aluthge, Dougal and Aluthge transforms are also presented.

2. Main Results

We use the following recent inequality for the product of two operators obtained in [6]:

Lemma 1. *For any $B, C \in \mathcal{B}(H)$ we have*

$$\omega^2(BC) \le \frac{1}{2}\left\| |B^*|^4 + |C|^4 \right\|. \tag{2.1}$$

We can state the following result:

Theorem 1. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$, we have*

$$\begin{aligned}\omega^2(T_V) &\le \frac{1}{2}\left\| V|T|^{4(1-t)}V^* + |T|^{4t} \right\| \le \frac{1}{2}\left\| \|T\|^{4(1-t)} I + |T|^{4t} \right\| \\ &\le \frac{1}{2}\left(\|T\|^{4(1-t)} + \|T\|^{4t} \right)\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\omega^2\left(\widehat{T}_V\right) &\le \frac{1}{2}\left\| |T|^{4(1-t)} + V^*|T|^{4t}V \right\| \le \frac{1}{2}\left\| |T|^{4(1-t)} + \|T\|^{4t} I \right\| \\ &\le \frac{1}{2}\left(\|T\|^{4(1-t)} + \|T\|^{4t} \right).\end{aligned} \tag{2.3}$$

We also have

$$\begin{aligned}\omega^2(\Delta_{t,V}(T)) &\le \frac{1}{2}\left\| \|T\|^{2t} |T|^t |V^*|^4 |T|^t + |T|^{4(1-t)} \right\| \\ &\le \frac{1}{2}\left\| \|T\|^{2t} |T|^{2t} + |T|^{4(1-t)} \right\| \le \frac{1}{2}\left(\|T\|^{4(1-t)} + \|T\|^{4t} \right)\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\omega^2(\Delta_{t,V}(T)) &\le \frac{1}{2}\left\| |T|^{4t} + \|T\|^{2(1-t)} |T|^{1-t} |V|^4 |T|^{1-t} \right\| \\ &\le \frac{1}{2}\left\| |T|^{4t} + \|T\|^{2(1-t)} |T|^{(1-t)} \right\| \le \frac{1}{2}\left(\|T\|^{4(1-t)} + \|T\|^{4t} \right).\end{aligned} \tag{2.5}$$

Proof. If we take $B = V|T|^{1-t}$ and $C = |T|^t$, then we get $BC = V|T| = T_V$ and

$$|B^*|^2 = BB^* = V|T|^{1-t}|T|^{1-t}V^* = V|T|^{2(1-t)}V^*.$$

Observe that

$$\begin{aligned}0 &\le |B^*|^4 = |B^*|^2|B^*|^2 = V|T|^{2(1-t)}V^*V|T|^{2(1-t)}V^* \\ &\le V|T|^{2(1-t)} I |T|^{2(1-t)}V^* = V|T|^{4(1-t)}V^* \\ &\le \|T\|^{4(1-t)} VV^* \le \|T\|^{4(1-t)} I.\end{aligned}$$

Since $|C|^4 = |T|^{4t}$, then

$$0 \le |B^*|^4 + |C|^4 \le V|T|^{4(1-t)}V^* + |T|^{4t} \le \|T\|^{4(1-t)} I + |T|^{4t}$$

and by (2.1) we derive (2.2).

If we take $B = |T|^{1-t}$ and $C = |T|^t V$, then we get $BC = |T|V = \widehat{T}_V$. Also

$$0 \le |C|^2 = C^*C = V^*|T|^t|T|^tV = V^*|T|^{2t}V$$

and

$$0 \leq |C|^4 = |C|^2 |C|^2 = V^* |T|^{2t} VV^* |T|^{2t} V \leq V^* |T|^{2t} I |T|^{2t} V$$
$$= V^* |T|^{4t} V \leq \|T\|^{4t} V^*V \leq \|T\|^{4t} I$$

and since $|B^*|^4 = |T|^{4(1-t)}$, then by (2.1) we derive (2.3).

Further, if we take $B = |T|^t V$ and $C = |T|^{1-t}$, then we get $BC = |T|^t V |T|^{1-t} = \Delta_{t,V}(T)$ and

$$|B^*|^2 = BB^* = |T|^t VV^* |T|^t .$$

Observe that

$$0 \leq |B^*|^4 = |T|^t VV^* |T|^t |T|^t VV^* |T|^t = |T|^t VV^* |T|^{2t} VV^* |T|^t$$
$$\leq \|T\|^{2t} |T|^t VV^*VV^* |T|^t = \|T\|^{2t} |T|^t (VV^*)^2 |T|^t$$
$$= \|T\|^{2t} |T|^t |V^*|^4 |T|^t \leq \|T\|^{2t} |T|^{2t}$$

and since $|C|^4 = |T|^{4(1-t)}$, then by (2.1) we derive (2.4).

Finally, if we take $B = |T|^t$ and $C = V |T|^{1-t}$, then we get $BC = |T|^t V |T|^{1-t} = \Delta_{t,V}(T)$ and

$$|C|^2 = C^*C = |T|^{1-t} V^*V |T|^{1-t} .$$

Observe that

$$|C|^4 = |T|^{1-t} V^*V |T|^{1-t} |T|^{1-t} V^*V |T|^{1-t} = |T|^{1-t} V^*V |T|^{2(1-t)} V^*V |T|^{1-t}$$
$$\leq \|T\|^{2(1-t)} |T|^{1-t} V^*VV^*V |T|^{1-t} = \|T\|^{2(1-t)} |T|^{1-t} (V^*V)^2 |T|^{1-t}$$
$$= \|T\|^{2(1-t)} |T|^{1-t} |V|^4 |T|^{1-t} \leq \|T\|^{2(1-t)} |T|^{(1-t)}$$

and since $|B^*|^4 = |T|^{4t}$, then by (2.1) we derive (2.5). □

Remark 1. *If we take $t = 1/2$ in (2.2) and (2.3), then we get*

$$\omega^2(T_V) \leq \frac{1}{2} \left\| V |T|^2 V^* + |T|^2 \right\| \leq \frac{1}{2} \left\| \|T\|^2 I + |T|^2 \right\| \leq \|T\|^2$$

and

$$\omega^2\left(\widehat{T}_V\right) \leq \frac{1}{2} \left\| V^* |T|^2 V + |T|^2 \right\| \leq \frac{1}{2} \left\| \|T\|^2 I + |T|^2 \right\| \leq \|T\|^2 .$$

Also, if we take $t = 1/2$ in (2.4) and (2.5), then we get

$$\omega^2\left(\widetilde{T}_V\right) \leq \frac{1}{2} \left\| \|T\|^{1/2} \widetilde{T}_{|V^*|^4} + |T|^2 \right\| \leq \frac{1}{2} \left\| \|T\| |T| + |T|^2 \right\| \leq \|T\|^2$$

and

$$\omega^2\left(\widetilde{T}_V\right) \leq \frac{1}{2} \left\| \|T\|^{1/2} \widetilde{T}_{|V|^4} + |T|^2 \right\| \leq \frac{1}{2} \left\| \|T\| |T| + |T|^2 \right\| \leq \|T\|^2 .$$

If we take $t = 0$ in Theorem 1, then we get

$$\omega^2(T_V) \leq \frac{1}{2} \left\| V |T|^4 V^* + I \right\|$$

and

$$\omega^2\left(\widehat{T}_V\right) \leq \frac{1}{2} \left\| |T|^4 + |V|^2 \right\| .$$

We also have

$$\omega^2(T_V) \leq \frac{1}{2} \left\| |V^*|^4 + |T|^4 \right\|$$

and

$$\omega^2(T_V) \le \frac{1}{2}\left\| I + \|T\|^2 |T| |V|^4 |T| \right\|.$$

If we take $t = 1$ in Theorem 1, then we get

$$\omega^2(T_V) \le \frac{1}{2}\left\| |V^*|^2 + |T|^4 \right\|$$

and

$$\omega^2\left(\widehat{T}_V\right) \le \frac{1}{2}\left\| I + V^* |T|^4 V \right\|.$$

We also have

$$\omega^2\left(\widehat{T}_V\right) \le \frac{1}{2}\left\| \|T\|^2 |T| |V^*|^4 |T| + I \right\|$$

and

$$\omega^2\left(\widehat{T}_V\right) \le \frac{1}{2}\left\| |T|^4 + |V|^4 \right\|.$$

The case of one operator is as follows:

Proposition 1. *For $T \in \mathcal{B}(H)$ and $t \in [0,1]$, we have*

$$\omega^2(T) \le \frac{1}{2}\left\| |T^*|^{4(1-t)} + |T|^{4t} \right\| \tag{2.6}$$

and

$$\omega^2(\Delta_t(T)) \le \frac{1}{2}\left\| |T|^{4(1-t)} + |T|^{4t} \right\|. \tag{2.7}$$

Proof. If $T = U|T|$ is the polar decomposition of T with U a partial isometry. As in the proof of Theorem 1 if we take $B = U|T|^{1-t}$ and $C = |T|^t$, then we get $BC = U|T| = T$ and

$$|B^*|^2 = BB^* = U|T|^{1-t}|T|^{1-t}U^* = U|T|^{2(1-t)}U^* = |T^*|^{2(1-t)},$$

where the last equality is well know. Then $|B^*|^4 = |T^*|^{4(1-t)}$, $|C|^4 = |T|^{4t}$ and by (2.1) we derive (2.6).

Finally, if we take $B = |T|^t$ and $C = U|T|^{1-t}$, then we get $BC = |T|^t U|T|^{1-t} = \Delta_t(T)$ and, since U is an isometry on ran$(|T|)$,

$$|C|^2 = C^*C = |T|^{1-t}U^*U|T|^{1-t} = |T|^{2(1-t)}.$$

Then $|C|^4 = |T|^{4(1-t)}$, $|B|^4 = |T|^{4t}$ and by (2.1) we derive (2.7). □

Remark 2. *The inequality (2.6) also follows by (1.6) for $r = 2$.*

We use the following inequality as well [6]:

Lemma 2. *For any B, $C \in \mathcal{B}(H)$ we have*

$$\omega^2(BC) \le \frac{1}{2}\left(\|B\|^2\|C\|^2 + \omega\left(|B^*|^2|C|^2\right)\right). \tag{2.8}$$

We have:

Theorem 2. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ and $t \in [0,1]$, we have*

$$\begin{aligned}\omega^2(T_V) &\le \frac{1}{2}\left(\|V\|^2\|T\|^2 + \omega\left(V|T|^{2(1-t)}V^*|T|^{2t}\right)\right) \\ &\le \frac{1}{2}\left(\|T\|^2 + \omega\left(V|T|^{2(1-t)}V^*|T|^{2t}\right)\right)\end{aligned} \tag{2.9}$$

and

$$\omega^2\left(\widehat{T}_V\right) \le \frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(V^*|T|^{2t}V|T|^{2(1-t)}\right)\right) \le \frac{1}{2}\left(\|T\|^2+\omega\left(V^*|T|^{2t}V|T|^{2(1-t)}\right)\right). \tag{2.10}$$

Also, we have

$$\omega^2\left(\Delta_{t,V}(T)\right) \le \frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(|T|^tVV^*|T|^{2-t}\right)\right) \le \frac{1}{2}\left(\|T\|^2+\omega\left(|T|^tVV^*|T|^{2-t}\right)\right) \tag{2.11}$$

and

$$\omega^2\left(\Delta_{t,V}(T)\right) \le \frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(|T|^{1+t}V^*V|T|^{1-t}\right)\right) \le \frac{1}{2}\left(\|T\|^2+\omega\left(|T|^{1+t}V^*V|T|^{1-t}\right)\right). \tag{2.12}$$

Proof. If we take $B=V|T|^{1-t}$ and $C=|T|^t$, then we get $BC=V|T|=T_V$ and

$$|B^*|^2=V|T|^{2(1-t)}V^*,\ \ |C|^2=|T|^{2t}.$$

By (2.8) we get

$$\omega^2(T_V) \le \frac{1}{2}\left\|V|T|^{1-t}\right\|^2\left\||T|^t\right\|^2+\frac{1}{2}\omega\left(V|T|^{2(1-t)}V^*|T|^{2t}\right) \le \frac{1}{2}\|V\|^2\|T\|^2+\frac{1}{2}\omega\left(V|T|^{2(1-t)}V^*|T|^{2t}\right),$$

which proves (2.9).

If we take $B=|T|^{1-t}$ and $C=|T|^tV$, then we get $BC=|T|V=\widehat{T}_V$. Also

$$|B^*|^2=|T|^{2(1-t)},\ \ |C|^2=V^*|T|^t|T|^tV=V^*|T|^{2t}V.$$

By (2.8) we get

$$\omega^2\left(\widehat{T}_V\right) \le \frac{1}{2}\left\||T|^{2(1-t)}\right\|^2\left\|V^*|T|^{2t}V\right\|^2+\frac{1}{2}\omega\left(|T|^{2(1-t)}V^*|T|^{2t}V\right) \le \frac{1}{2}\|V\|^2\|T\|^2+\frac{1}{2}\omega\left(|T|^{2(1-t)}V^*|T|^{2t}V\right),$$

which proves (2.10).

Further, if we take $B=|T|^tV$ and $C=|T|^{1-t}$, then we get $BC=|T|^tV|T|^{1-t}=\Delta_{t,V}(T)$ and

$$|B^*|^2=BB^*=|T|^tVV^*|T|^t,\ \ |C|^2=|T|^{2(1-t)}.$$

By (2.8) we get

$$\omega^2\left(\Delta_{t,V}(T)\right) \le \frac{1}{2}\left\||T|^tV\right\|^2\|T\|^{2(1-t)}+\frac{1}{2}\omega\left(|T|^tVV^*|T|^t|T|^{2(1-t)}\right) \le \frac{1}{2}\|V\|^2\|T\|^2+\frac{1}{2}\omega\left(|T|^tVV^*|T|^{2-t}\right),$$

which proves (2.11).

Finally, if we take $B=|T|^t$ and $C=V|T|^{1-t}$, then we get $BC=|T|^tV|T|^{1-t}=\Delta_{t,V}(T)$ and

$$|B^*|^2=|T|^{2t},\ \ |C|^2=C^*C=|T|^{1-t}V^*V|T|^{1-t}.$$

By (2.8) we get

$$\begin{aligned}\omega^2\left(\Delta_{t,V}(T)\right) &\leq \frac{1}{2}\|T\|^{2t}\left\|V|T|^{1-t}\right\|^2+\frac{1}{2}\omega\left(|T|^{1+t}V^*V|T|^{1-t}\right)\\ &\leq \frac{1}{2}\|V\|^2\|T\|^2+\frac{1}{2}\omega\left(|T|^{1+t}V^*V|T|^{1-t}\right),\end{aligned}$$

which proves (2.11). □

Remark 3. *If we take $t=1/2$ in Theorem 2, then we get*

$$\omega^2(T_V)\leq\frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(V|T|V^*|T|\right)\right)\leq\frac{1}{2}\left(\|T\|^2+\omega\left(V|T|V^*|T|\right)\right)$$

and

$$\omega^2\left(\widehat{T}_V\right)\leq\frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(V^*|T|V|T|\right)\right)\leq\frac{1}{2}\left(\|T\|^2+\omega\left(V^*|T|V|T|\right)\right).$$

Also, we have

$$\begin{aligned}\omega^2\left(\widetilde{T}_V\right)&\leq\frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(|T|^{1/2}VV^*|T|^{3/2}\right)\right)\\&\leq\frac{1}{2}\left(\|T\|^2+\omega\left(|T|^{1/2}VV^*|T|^{3/2}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\omega^2\left(\widetilde{T}_V\right)&\leq\frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(|T|^{3/2}V^*V|T|^{1/2}\right)\right)\\&\leq\frac{1}{2}\left(\|T\|^2+\omega\left(|T|^{3/2}V^*V|T|^{1/2}\right)\right).\end{aligned}$$

If we take $t=0$ in Theorem 2, then we get

$$\omega^2(T_V)\leq\frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(V|T|^2V^*\right)\right)\leq\frac{1}{2}\left(\|T\|^2+\omega\left(V|T|^2V^*\right)\right)$$

and

$$\omega^2\left(\widehat{T}_V\right)\leq\frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(V^*V|T|^2\right)\right)\leq\frac{1}{2}\left(\|T\|^2+\omega\left(V^*V|T|^2\right)\right).$$

Also, we have

$$\omega^2(T_V)\leq\frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(VV^*|T|^2\right)\right)\leq\frac{1}{2}\left(\|T\|^2+\omega\left(VV^*|T|^2\right)\right)$$

and

$$\omega^2(T_V)\leq\frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(|T|V^*V|T|\right)\right)\leq\frac{1}{2}\left(\|T\|^2+\omega\left(|T|V^*V|T|\right)\right).$$

If we take $t=1$ in Theorem 2, then we get

$$\omega^2(T_V)\leq\frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(VV^*|T|^2\right)\right)\leq\frac{1}{2}\left(\|T\|^2+\omega\left(VV^*|T|^2\right)\right)$$

and

$$\omega^2\left(\widehat{T}_V\right)\leq\frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(V^*|T|^2V\right)\right)\leq\frac{1}{2}\left(\|T\|^2+\omega\left(V^*|T|^2V\right)\right).$$

Also, we have

$$\omega^2\left(\widehat{T}_V\right)\leq\frac{1}{2}\left(\|V\|^2\|T\|^2+\omega\left(|T|VV^*|T|\right)\right)\leq\frac{1}{2}\left(\|T\|^2+\omega\left(|T|VV^*|T|\right)\right)$$

and

$$\omega^2\left(\widehat{T}_V\right) \le \frac{1}{2}\left(\|V\|^2\|T\|^2 + \omega\left(|T|^2 V^*V\right)\right) \le \frac{1}{2}\left(\|T\|^2 + \omega\left(|T|^2 V^*V\right)\right).$$

We also have:

Proposition 2. *For $T \in \mathcal{B}(H)$ and $t \in [0,1]$, we have*

$$\omega^2(T) \le \frac{1}{2}\left(\|T\|^2 + \omega\left(|T^*|^{2(1-t)}|T|^{2t}\right)\right), \tag{2.13}$$

see also [6],

$$\begin{aligned} \omega^2\left(\widehat{T}\right) &\le \frac{1}{2}\left(\|T\|^{2(1-t)}\left\|U^*|T|^{2t}U\right\| + \omega\left(|T|^{2(1-t)}U^*|T|^{2t}U\right)\right) \\ &\le \frac{1}{2}\left(\|T\|^2 + \omega\left(|T|^{2(1-t)}U^*|T|^{2t}U\right)\right), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \omega^2(\Delta_{t,V}(T)) &\le \frac{1}{2}\left(\left\||T|^t U\right\|^2\|T\|^{2(1-t)} + \omega\left(|T|^t UU^*|T|^{2-t}\right)\right) \\ &\le \frac{1}{2}\left(\|T\|^2 + \omega\left(|T|^t UU^*|T|^{2-t}\right)\right). \end{aligned} \tag{2.15}$$

Proof. Let $B = U|T|^{1-t}$ and $C = |T|^t$, then we get $BC = U|T| = T$,

$$|B|^2 = \left(U|T|^{1-t}\right)^* U|T|^{1-t} = |T|^{1-t}U^*U|T|^{1-t} = |T|^{2(1-t)}$$

and then

$$\|B\|^2 = \left\||B|^2\right\| = \|T\|^{2(1-t)}$$

and $\|C\|^2 = \|T\|^{2t}$.

Also,

$$|B^*|^2|C|^2 = U|T|^{1-t}|T|^{1-t}U^*|T|^{2t} = U|T|^{2(1-t)}U^*|T|^{2t} = |T^*|^{2(1-t)}|T|^{2t}$$

and by Lemma 2 we obtain (2.13).

If we take $B = |T|^{1-t}$ and $C = |T|^t U$, then we get $BC = |T|U = \widehat{T}$. Also

$$|B^*|^2 = |T|^{2(1-t)},\ \ |C|^2 = U^*|T|^t|T|^t U = U^*|T|^{2t}U$$

and

$$|B^*|^2|C|^2 = |T|^{2(1-t)}U^*|T|^{2t}U$$

and by Lemma 2 we obtain (2.14).

Further, if we take $B = |T|^t U$and $C = |T|^{1-t}$, then we get $BC = |T|^t U|T|^{1-t} = \Delta_{t,V}(T)$,

$$|B^*|^2 = BB^* = |T|^t UU^*|T|^t,\ \ |C|^2 = |T|^{2(1-t)}$$

and

$$|B^*|^2|C|^2 = |T|^t UU^*|T|^t|T|^{2(1-t)} = |T|^t UU^*|T|^{2-t}$$

and by Lemma 2 we obtain (2.15). □

Remark 4. *The case $t = 1/2$ in (2.13) was obtained in* [6] *as,*

$$\omega^2(T) \le \frac{1}{2}\left(\|T\|^2 + \omega(|T^*|\,|T|)\right) \le \frac{1}{2}\left(\|T\|^2 + \left\|T^2\right\|\right). \tag{2.16}$$

If we take $t = 1/2$ in (2.14), then we get

$$\omega^2\left(\widehat{T}\right) \le \frac{1}{2}\left(\|T\|\left\||T|^{1/2}U\right\|^2 + \omega(|T|\,U^*\,|T|\,U)\right). \tag{2.17}$$

From (2.15) we obtain

$$\omega^2\left(\widetilde{T}\right) \le \frac{1}{2}\left(\|T\|\left\||T|^{1/2}U\right\|^2 + \omega\left(\left|U^*\,|T|^{1/2}\right|^2 |T|\right)\right). \tag{2.18}$$

If we take $t = 0$ in (2.15) then we get

$$\omega^2(T) \le \frac{1}{2}\left(\|T\|^2 + \omega\left(UU^*\,|T|^2\right)\right). \tag{2.19}$$

3. Related Results

In [6] the author also obtained:

Lemma 3. *For any B, $C \in \mathcal{B}(H)$ we have*

$$\omega^2(BC) \le \left\|\alpha\,|B^*|^2 + (1-\alpha)\,|C|^2\right\| \|B\|^{2(1-\alpha)}\,\|C\|^{2\alpha}$$

for all $\alpha \in [0,1]$. In particular,

$$\omega^2(BC) \le \frac{1}{2}\left\||B^*|^2 + |C|^2\right\| \|B\|\,\|C\|.$$

We can state the following result:

Theorem 3. *For a contraction $V \in \mathcal{B}(H)$, an operator $T \in \mathcal{B}(H)$ and t, $\alpha \in [0,1]$, we have*

$$\begin{aligned}\omega^2(T_V) &\le \left\|\alpha V\,|T|^{2(1-t)}\,V^* + (1-\alpha)\,|T|^{2t}\right\| \left\|V\,|T|^{2(1-t)}\,V^*\right\|^{1-\alpha} \|T\|^{2t\alpha} \\ &\le \left\|\alpha V\,|T|^{2(1-t)}\,V^* + (1-\alpha)\,|T|^{2t}\right\| \|T\|^{2[(1-t)(1-\alpha)+t\alpha]}\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\omega^2\left(\widehat{T}_V\right) &\le \left\|\alpha\,|T|^{2(1-t)} + (1-\alpha)\,V^*\,|T|^{2t}\,V\right\| \left\|V^*\,|T|^{2t}\,V\right\|^{1-\alpha} \|T\|^{2t\alpha} \\ &\le \left\|\alpha\,|T|^{2(1-t)} + (1-\alpha)\,V^*\,|T|^{2t}\,V\right\| \|T\|^{2[(1-t)(1-\alpha)+t\alpha]}.\end{aligned} \tag{3.2}$$

Also, we have

$$\begin{aligned}&\omega^2(\Delta_{t,V}(T)) \\ &\le \left\|\alpha\,|T|^t\,VV^*\,|T|^t + (1-\alpha)\,|T|^{2(1-t)}\right\| \left\||T|^t\,VV^*\,|T|^t\right\|^{1-\alpha} \|T\|^{2(1-t)\alpha} \\ &\le \left\|\alpha\,|T|^t\,VV^*\,|T|^t + (1-\alpha)\,|T|^{2(1-t)}\right\| \|T\|^{2[(1-t)(1-\alpha)+t\alpha]}\end{aligned} \tag{3.3}$$

and

$$\omega^2(\Delta_{t,V}(T)) \tag{3.4}$$
$$\leq \left\| \alpha |T|^{2t} + (1-\alpha)|T|^{1-t} V^* V |T|^{1-t} \right\| \|T\|^{2t(1-\alpha)} \left\| |T|^{1-t} V^* V |T|^{1-t} \right\|^{2\alpha}$$
$$\leq \left\| \alpha |T|^{2t} + (1-\alpha)|T|^{1-t} V^* V |T|^{1-t} \right\| \|T\|^{2[t(1-\alpha)+\alpha(1-t)]}.$$

Proof. If we take $B = V|T|^{1-t}$ and $C = |T|^t$, then we get $BC = V|T| = T_V$ and

$$|B^*|^2 = BB^* = V|T|^{1-t}|T|^{1-t}V^* = V|T|^{2(1-t)}V^*.$$

By Lemma 3 for $\alpha \in [0,1]$ we get

$$\omega^2(T_V) \leq \left\| \alpha V|T|^{2(1-t)}V^* + (1-\alpha)|T|^{2t} \right\| \left\| V|T|^{2(1-t)}V^* \right\|^{1-\alpha} \|T\|^{2t\alpha}$$
$$\leq \left\| \alpha V|T|^{2(1-t)}V^* + (1-\alpha)|T|^{2t} \right\| \|T\|^{2(1-t)(1-\alpha)} \|T\|^{2t\alpha}$$
$$= \left\| \alpha V|T|^{2(1-t)}V^* + (1-\alpha)|T|^{2t} \right\| \|T\|^{2[(1-t)(1-\alpha)+t\alpha]},$$

which proves (3.1).

If we take $B = |T|^{1-t}$ and $C = |T|^t V$, then we get $BC = |T|V = \widehat{T}_V$. Also

$$|B^*|^2 = |T|^{2(1-t)}, \; |C|^2 = V^*|T|^t|T|^t V = V^*|T|^{2t}V.$$

From Lemma 3 we get

$$\omega^2\left(\widehat{T}_V\right) \leq \left\| \alpha|T|^{2(1-t)} + (1-\alpha)V^*|T|^{2t}V \right\| \|T\|^{2(1-t)(1-\alpha)} \left\| V^*|T|^{2t}V \right\|^{\alpha}$$
$$\leq \left\| \alpha|T|^{2(1-t)} + (1-\alpha)V^*|T|^{2t}V \right\| \|T\|^{2(1-t)(1-\alpha)} \|T\|^{2t\alpha}$$
$$= \left\| \alpha|T|^{2(1-t)} + (1-\alpha)V^*|T|^{2t}V \right\| \|T\|^{2[(1-t)(1-\alpha)+t\alpha]},$$

which proves (3.2).

Further, if we take $B = |T|^t V$ and $C = |T|^{1-t}$, then we get $BC = |T|^t V |T|^{1-t} = \Delta_{t,V}(T)$ and

$$|B^*|^2 = BB^* = |T|^t VV^*|T|^t, \; |C|^2 = |T|^{2(1-t)}.$$

By Lemma 3 we get

$$\omega^2(\Delta_{t,V}(T))$$
$$\leq \left\| \alpha|T|^t VV^*|T|^t + (1-\alpha)|T|^{2(1-t)} \right\| \left\| |T|^t VV^*|T|^t \right\|^{1-\alpha} \|T\|^{2(1-t)\alpha}$$
$$\leq \left\| \alpha|T|^t VV^*|T|^t + (1-\alpha)|T|^{2(1-t)} \right\| \|T\|^{2t(1-\alpha)} \|T\|^{2(1-t)\alpha}$$
$$= \left\| \alpha|T|^t VV^*|T|^t + (1-\alpha)|T|^{2(1-t)} \right\| \|T\|^{2[(1-t)(1-\alpha)+t\alpha]},$$

which proves (3.3).

Finally, if we take $B = |T|^t$ and $C = V|T|^{1-t}$, then we get $BC = |T|^t V |T|^{1-t} = \Delta_{t,V}(T)$ and

$$|B^*|^2 = |T|^{2t}, \; |C|^2 = C^*C = |T|^{1-t}V^*V|T|^{1-t}.$$

From Lemma 3 we get

$$\begin{aligned}
&\omega^2\left(\Delta_{t,V}(T)\right)\\
&\leq \left\|\alpha\left|T\right|^{2t}+(1-\alpha)\left|T\right|^{1-t}V^*V\left|T\right|^{1-t}\right\|\left\|T\right\|^{2t(1-\alpha)}\left\|\left|T\right|^{1-t}V^*V\left|T\right|^{1-t}\right\|^{2\alpha}\\
&\leq \left\|\alpha\left|T\right|^{2t}+(1-\alpha)\left|T\right|^{1-t}V^*V\left|T\right|^{1-t}\right\|\left\|T\right\|^{2t(1-\alpha)}\left\|T\right\|^{2\alpha(1-t)}\\
&= \left\|\alpha\left|T\right|^{2t}+(1-\alpha)\left|T\right|^{1-t}V^*V\left|T\right|^{1-t}\right\|\left\|T\right\|^{2[t(1-\alpha)+\alpha(1-t)]},
\end{aligned}$$

which proves (3.4). □

Remark 5. *If we take $t=1/2$ in Theorem 3, then for $\alpha\in[0,1]$,*

$$\begin{aligned}
\omega^2\left(T_V\right) &\leq \left\|\alpha V\left|T\right|V^*+(1-\alpha)\left|T\right|\right\|\left\|V\left|T\right|V^*\right\|^{1-\alpha}\left\|T\right\|^{\alpha}\\
&\leq \left\|\alpha V\left|T\right|V^*+(1-\alpha)\left|T\right|\right\|\left\|T\right\|
\end{aligned}$$

and

$$\begin{aligned}
\omega^2\left(\widehat{T}_V\right) &\leq \left\|\alpha\left|T\right|+(1-\alpha)V^*\left|T\right|V\right\|\left\|V^*\left|T\right|V\right\|^{1-\alpha}\left\|T\right\|^{\alpha}\\
&\leq \left\|\alpha\left|T\right|+(1-\alpha)V^*\left|T\right|V\right\|\left\|T\right\|.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\omega^2\left(\widetilde{T}_V\right)\\
&\leq \left\|\alpha\left|T\right|^{1/2}VV^*\left|T\right|^{1/2}+(1-\alpha)\left|T\right|\right\|\left\|\left|T\right|^{1/2}VV^*\left|T\right|^{1/2}\right\|^{1-\alpha}\left\|T\right\|^{\alpha}\\
&\leq \left\|\alpha\left|T\right|^{1/2}VV^*\left|T\right|^{1/2}+(1-\alpha)\left|T\right|\right\|\left\|T\right\|
\end{aligned}$$

and

$$\begin{aligned}
&\omega^2\left(\widetilde{T}_V\right)\\
&\leq \left\|\alpha\left|T\right|+(1-\alpha)\left|T\right|^{1/2}V^*V\left|T\right|^{1/2}\right\|\left\|T\right\|^{1-\alpha}\left\|\left|T\right|^{1/2}V^*V\left|T\right|^{1/2}\right\|^{2\alpha}\\
&\leq \left\|\alpha\left|T\right|+(1-\alpha)\left|T\right|^{1/2}V^*V\left|T\right|^{1/2}\right\|\left\|T\right\|.
\end{aligned}$$

If in these inequalities we take $\alpha=1/2$, then we get

$$\omega^2\left(T_V\right)\leq\frac{1}{2}\left\|V\left|T\right|V^*+\left|T\right|\right\|\left\|V\left|T\right|V^*\right\|^{1/2}\left\|T\right\|^{1/2}\leq\frac{1}{2}\left\|V\left|T\right|V^*+\left|T\right|\right\|\left\|T\right\|$$

and

$$\begin{aligned}
\omega^2\left(\widehat{T}_V\right) &\leq \frac{1}{2}\left\|\left|T\right|+V^*\left|T\right|V\right\|\left\|V\left|T\right|V^*\right\|^{1/2}\left\|T\right\|^{1/2}\\
&\leq \frac{1}{2}\alpha\left|T\right|+V^*\left|T\right|V\left\|T\right\|.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\omega^2\left(\widetilde{T}_V\right) &\leq \frac{1}{2}\left\|\left|T\right|^{1/2}VV^*\left|T\right|^{1/2}+\left|T\right|\right\|\left\|\left|T\right|^{1/2}VV^*\left|T\right|^{1/2}\right\|^{1/2}\left\|T\right\|^{1/2}\\
&\leq \frac{1}{2}\left\|\left|T\right|^{1/2}VV^*\left|T\right|^{1/2}+\left|T\right|\right\|\left\|T\right\|
\end{aligned}$$

and

$$\omega^2\left(\widetilde{T}_V\right) \le \frac{1}{2}\left\||T| + |T|^{1/2} V^*V |T|^{1/2}\right\| \|T\|^{1/2} \left\||T|^{1/2} V^*V |T|^{1/2}\right\| \le \frac{1}{2}\left\||T| + |T|^{1/2} V^*V |T|^{1/2}\right\| \|T\|.$$

Proposition 3. *For $T \in \mathcal{B}(H)$ and $t \in [0,1]$, we have for*

$$\omega^2(T) \le \left\|\alpha |T^*|^{2(1-t)} + (1-\alpha)|T|^{2t}\right\| \|T\|^{2[(1-t)(1-\alpha)+t\alpha]} \tag{3.5}$$

and

$$\omega^2\left(\widehat{T}\right) \le \left\|\alpha |T|^{2(1-t)} + (1-\alpha) U^* |T|^{2t} U\right\| \left\|U^* |T|^{2t} U\right\|^{1-\alpha} \|T\|^{2t\alpha} \le \left\|\alpha |T|^{2(1-t)} + (1-\alpha) U^* |T|^{2t} U\right\| \|T\|^{2[(1-t)(1-\alpha)+t\alpha]}. \tag{3.6}$$

Also, we have

$$\omega^2\left(\widetilde{T}\right) \le \left\|\alpha |T|^t UU^* |T|^t + (1-\alpha)|T|^{2(1-t)}\right\| \left\||T|^t UU^* |T|^t\right\|^{1-\alpha} \|T\|^{2(1-t)\alpha} \le \left\|\alpha |T|^t UU^* |T|^t + (1-\alpha)|T|^{2(1-t)}\right\| \|T\|^{2[(1-t)(1-\alpha)+t\alpha]} \tag{3.7}$$

and

$$\omega^2\left(\widetilde{T}\right) \le \left\|\alpha |T|^{2t} + (1-\alpha)|T|^{2(1-t)}\right\| \|T\|^{2[t(1-\alpha)+\alpha(1-t)]}. \tag{3.8}$$

The proof follows by Theorem 3 by observing that, if $T = U|T|$ is the polar decomposition of T with U a partial isometry, then

$$U|T|^{2(1-t)} U^* = |T^*|^{2(1-t)} \text{ and } |T|^{1-t} V^*V |T|^{1-t} = |T|^{2(1-t)}.$$

If we take $t = 1/2$ in Proposition 3, then we get

$$\omega^2(T) \le \|\alpha |T^*| + (1-\alpha)|T|\| \|T\|$$

and

$$\omega^2\left(\widehat{T}\right) \le \|\alpha |T| + (1-\alpha) U^* |T| U\| \|U^* |T| U\|^{1-\alpha} \|T\|^{\alpha} \le \|\alpha |T| + (1-\alpha) U^* |T| U\| \|T\|.$$

Also, we have

$$\omega^2\left(\widetilde{T}\right) \le \left\|\alpha |T|^{1/2} UU^* |T|^{1/2} + (1-\alpha)|T|\right\| \left\||T|^{1/2} UU^* |T|^{1/2}\right\|^{1-\alpha} \|T\|^{\alpha} \le \left\|\alpha |T|^{1/2} UU^* |T|^{1/2} + (1-\alpha)|T|\right\| \|T\|.$$

For $\alpha = 1/2$ we further have

$$\omega^2(T) \le \frac{1}{2}\||T^*| + |T|\| \|T\|,$$

$$\omega^2\left(\widehat{T}\right) \le \frac{1}{2}\||T| + U^* |T| U\| \|U^* |T| U\|^{1/2} \|T\|^{1/2} \le \frac{1}{2}\||T| + U^* |T| U\| \|T\|.$$

and

$$\omega^2\left(\widetilde{T}\right) \leq \frac{1}{2}\left\| |T|^{1/2} UU^* |T|^{1/2} + |T| \right\| \left\| |T|^{1/2} UU^* |T|^{1/2} \right\|^{1/2} \|T\|^{1/2}$$
$$\leq \frac{1}{2}\left\| |T|^{1/2} UU^* |T|^{1/2} + |T| \right\| \|T\|.$$

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