Integral inequalities involving new conformable derivatives

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Abstract

Here, we first describe the new conformable derivatives and integrals ([4]). Then, we establish Opial, Ostrowski, Poincaré, Sobolev, Polya and Hilbert-Pachpatte type integral inequalities.

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1 Background

All in this section come from [4].

The motivation though comes from control theory [5]. A proportional derivative (PD) controller for controller output u at time t with two tuning parameters is given by

$$u(t) = k_p E(t) + k_d \frac{d}{dt} E(t), \tag{1}$$

where k_p is the proportional gain, k_d is the derivative gain, and E is the error between the state variable and the process variable.

The above motivate the following:

Definition 1 *(A class of New Conformable Derivatives) ([4]) Let $\alpha \in [0,1]$, and let the functions $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous such that*

$$\begin{gathered} \lim_{\alpha \to 0+} k_1(\alpha,t) = 1, \quad \lim_{\alpha \to 0+} k_0(\alpha,t) = 0, \ \ \forall\, t \in \mathbb{R}, \\ \lim_{\alpha \to 1-} k_1(\alpha,t) = 0, \quad \lim_{\alpha \to 1-} k_0(\alpha,t) = 1, \ \ \forall\, t \in \mathbb{R}, \\ k_1(\alpha,t) \neq 0,\ \alpha \in [0,1),\ k_0(\alpha,t) \neq 0, \quad \alpha \in (0,1], \ \ \forall\, t \in \mathbb{R}. \end{gathered} \tag{2}$$

Then, the following differentiable operator D^{α}, defined via

$$D^{\alpha} f(t) = k_1(\alpha, t) f(t) + k_0(\alpha, t) f'(t) \tag{3}$$

is the New Conformable derivative given that $f'(t)$ exists for $t \in \mathbb{R}$.

Here, k_1 is a type of proportional gain k_p, k_0 is a type of derivative gain k_d, f is the error, and $u = D^{\alpha} f$ is the controller output. For example, one may choose $k_1 := (1-\alpha)\omega^{\alpha}$ and $k_0 := \alpha\omega^{1-\alpha}$ for any $\omega \in (0, \infty)$; or $k_1 := (1-\alpha)|t|^{\alpha}$ and $k_0 := \alpha |t|^{1-\alpha}$ on $\mathbb{R} - \{0\}$, so that

$$D^{\alpha} f(t) = (1-\alpha)|t|^{\alpha} f(t) + \alpha |t|^{1-\alpha} f'(t). \tag{4}$$

Another class of conformable derivatives is

$$D^{\alpha} f(t) = \cos\left(\alpha \frac{\pi}{2}\right) |t|^{\alpha} f(t) + \sin\left(\alpha \frac{\pi}{2}\right) |t|^{1-\alpha} f'(t). \tag{5}$$

In general we have that $D^{\beta} D^{\alpha} \neq D^{\alpha} D^{\beta}$ for $\alpha, \beta \in [0,1]$.

Definition 2 *([4]) (Conformable Exponential Function). Let $\alpha \in (0,1]$, the points $s, t \in \mathbb{R}$ with $s \leq t$, and let the function $p : [s,t] \to \mathbb{R}$ be continuous. Let $k_0, k_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy (2) with $\frac{p}{k_0}$ and $\frac{k_1}{k_0}$ Riemann integrable on $[s,t]$. Then the exponential function with respect to D^{α} in (3) is defined to be*

$$e_p(t,s) := e^{\int_s^t \frac{p(\tau) - k_1(\alpha,\tau)}{k_0(\alpha,\tau)} d\tau}, \qquad e_0(t,s) := e^{-\int_s^t \frac{k_1(\alpha,\tau)}{k_0(\alpha,\tau)} d\tau}. \tag{6}$$

Using (3) and (6) we have the following basic results.

Lemma 3 *([4]) (Basic Derivatives). Let the conformable differential operator D^{α} be given as in (3), where $\alpha \in [0,1]$. Let the function $p : [s,t] \to \mathbb{R}$ be continuous. Let $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (2), with $\frac{p}{k_0}$ and $\frac{k_1}{k_0}$ Riemann integrable on $[s,t]$. Assume the functions f and g are differentiable as needed. Then*

(i) $D^{\alpha}[af + bg] = aD^{\alpha}[f] + bD^{\alpha}[g]$ for all $a, b \in \mathbb{R}$;

(ii) $D^{\alpha} c = ck_1(\alpha, \cdot)$ for all constants $c \in \mathbb{R}$;

(iii) $D^{\alpha}[fg] = fD^{\alpha}[g] + gD^{\alpha}[f] - fgk_1(\alpha, \cdot)$;

(iv) $D^{\alpha}\left[\frac{f}{g}\right] = \frac{gD^{\alpha}[f] - fD^{\alpha}[g]}{g^2} + \frac{f}{g}k_1(\alpha, \cdot)$;

(v) for $\alpha \in (0,1]$ and fixed $s \in \mathbb{R}$, the exponential function satisfies

$$D_t^{\alpha}[e_p(t,s)] = p(t) e_p(t,s) \tag{7}$$

for $e_p(t,s)$ given in (6);

(vi) for $\alpha \in (0,1]$ and for the exponential function e_0 given in (6), we have

$$D^{\alpha}\left[\int_a^t \frac{f(s) e_0(t,s)}{k_0(\alpha,s)} ds\right] = f(t). \tag{8}$$

Definition 4 *([4]) (Integrals). Let* $\alpha \in (0,1]$ *and* $t_0 \in \mathbb{R}$. *In the light of (6) and Lemma 3 (v) & (vi), define the antiderivative via*

$$\int D^{\alpha} f(t)\, d_{\alpha} t = f(t) + c e_0(t, t_0), \quad c \in \mathbb{R}.$$

Similarly, define the integral of f *over* $[a,b]$ *as*

$$\int_a^t f(s)\, e_0(t,s)\, d_{\alpha} s := \int_a^t \frac{f(s)\, e_0(t,s)}{k_0(t,s)} ds, \quad d_{\alpha} s := \frac{1}{k_0(\alpha, s)} ds; \tag{9}$$

recall that

$$e_0(t,s) := e^{-\int_s^t \frac{k_1(\alpha,\tau)}{k_0(\alpha,\tau)} d\tau} = e^{-\int_s^t k_1(\alpha,\tau) d_{\alpha}\tau}$$

from (6).

Lemma 5 *([4]) (Basic Integrals). Let the conformable differential operator* D^{α} *be given as in (3), the integral be given as in (9) with* $\alpha \in (0,1]$. *Let the functions* k_0, k_1 *be continuous and satisfy (2), and let* f *and* g *be differentiable as needed. Then*

(i) the derivative of the definite integral of f *is given by*

$$D^{\alpha}\left[\int_a^t f(s)\, e_0(t,s)\, d_{\alpha} s\right] = f(t); \tag{10}$$

(ii) the definite integral of the derivative of f *is given by*

$$\int_a^t D^{\alpha}[f(s)]\, e_0(t,s)\, d_{\alpha} s = f(s)\, e_0(t,s)\,|_{s=a}^{t} := f(t) - f(a)\, e_0(t,a); \tag{11}$$

(iii) an integration by parts formula is given by

$$\int_a^b f(t)\, D^{\alpha}[g(t)]\, e_0(b,t)\, d_{\alpha} t = f(t)\, g(t)\, e_0(b,t)\,|_{t=a}^{b} -$$

$$\int_a^b g(t)\left(D^{\alpha}[f(t)] - k_1(\alpha,t) f(t)\right) e_0(b,t)\, d_{\alpha} t; \tag{12}$$

(iv) a version of the Leibniz rule for differentiation of an integral is given by

$$D^{\alpha}\left[\int_a^t f(t,s)\, e_0(t,s)\, d_{\alpha} s\right] = \tag{13}$$

$$\int_a^t \left(D_t^{\alpha}[f(t,s)] - k_1(\alpha,t) f(t,s)\right) e_0(t,s)\, d_{\alpha} s + f(t,t),$$

using (15); or, if e_0 *is absent,*

$$D^{\alpha}\left[\int_a^t f(t,s)\, d_{\alpha} s\right] = f(t,t) + \int_a^t D_t^{\alpha}[f(t,s)]\, d_{\alpha} s. \tag{14}$$

Definition 6 *([4]) (Partial Conformable Derivatives). Let* $\alpha \in [0,1]$*, and let the functions* $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ *be continuous and satisfy (2). Given a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *such that* $\frac{\partial}{\partial t} f(t,s)$ *exists for each fixed* $s \in \mathbb{R}$*, define the partial differential operator* D_t^α *via*

$$D_t^\alpha f(t,s) = k_1(\alpha,t) f(t,s) + k_0(\alpha,t) \frac{\partial}{\partial t} f(t,s). \tag{15}$$

Next comes the important Taylor's formula.

Theorem 7 *([4]) Let* $n \in \mathbb{N}$*,* $\alpha \in (0,1]$*, and suppose* f *is* n *times continuously differentiable on* $[t_0,\infty)$*. Let* $t, s \in [t_0,\infty)$*, and define the functions* h_k *by*

$$h_0(t,s) \equiv 1 \quad \text{and} \quad h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\, d_\alpha \tau \quad \text{for } k \in \mathbb{N}_0. \tag{16}$$

Then

$$f(t) = e_0(t,s) \sum_{k=0}^{n-1} (-1)^k h_k(s,t) (D^\alpha)^k f(s) +$$

$$(-1)^{n-1} \int_s^t h_{n-1}(\tau,t) (D^\alpha)^n f(\tau) e_0(t,\tau)\, d_\alpha \tau \tag{17}$$

for $t \in [t_0,\infty)$*.*

Example 8 *([4]) For* $\alpha \in (0,1]$*, let* $\omega_0, \omega_1 \in (0,\infty)$*, let* k_1 *satisfy (2), and take*

$$k_0(\alpha,t) \equiv \alpha \omega_0^{1-\alpha}. \tag{18}$$

By (9),

$$d_\alpha \tau = \frac{1}{k_0(\alpha,\tau)} d\tau = \frac{1}{\alpha\omega_0^{1-\alpha}} d\tau.$$

Letting $h_0(t,s) \equiv 1$*, we calculate* h_1 *via (16) to get*

$$h_1(t,s) = \int_s^t h_0(\tau,s)\, d_\alpha\tau = \frac{1}{\alpha\omega_0^{1-\alpha}} \int_s^t 1 d\tau = \frac{t-s}{\alpha\omega_0^{1-\alpha}};$$

additionally,

$$h_2(t,s) = \int_s^t h_1(\tau,s)\, d_\alpha\tau = \frac{1}{2!}\left(\frac{t-s}{\alpha\omega_0^{1-\alpha}}\right)^2.$$

In general we have that

$$h_n(t,s) = \frac{1}{n!}\left(\frac{t-s}{\alpha\omega_0^{1-\alpha}}\right)^n. \tag{19}$$

Note that at $\alpha = 1$ *we have*

$$h_n(t,s) = \frac{1}{n!}(t-s)^n$$

as expected.

Example 9 *([4]) For $\alpha \in (0,1]$, let $\omega_0, \omega_1 \in (0,\infty)$, let k_1 satisfy (2), and this time take*

$$k_0(\alpha, t) = \alpha (\omega_0 t)^{1-\alpha}, \quad t \in [0,\infty). \tag{20}$$

By (9),

$$d_\alpha \tau = \frac{\tau^{\alpha-1}}{\alpha \omega_0^{1-\alpha}} d\tau.$$

Again starting with $h_0(t,s) \equiv 1$, we see that

$$h_1(t,s) = \int_s^t h_0(\tau, s) d_\alpha \tau = \frac{1}{\alpha \omega_0^{1-\alpha}} \int_s^t \tau^{\alpha-1} d\tau = \frac{t^\alpha - s^\alpha}{\alpha^2 \omega_0^{1-\alpha}},$$

and

$$h_2(t,s) = \int_s^t h_1(\tau, s) d_\alpha \tau = \frac{1}{2!} \left(\frac{t^\alpha - s^\alpha}{\alpha^2 \omega_0^{1-\alpha}} \right)^2.$$

Continuing, we find that

$$h_n(t,s) = \frac{1}{n!} \left(\frac{t^\alpha - s^\alpha}{\alpha^2 \omega_0^{1-\alpha}} \right)^n, \tag{21}$$

which is just $\frac{1}{n!}(t-s)^n$ at $\alpha = 1$.

2 Main Results

Motivation comes from [1]-[3]. We need

Remark 10 *(to Lemma 5 (iii))*
When $f = g$ we get

$$\int_a^b f(t) D^\alpha [f(t)] e_0(b,t) d_\alpha t = f^2(b) - f^2(a) e_0(b,a) -$$

$$\int_a^b f(t) D^\alpha [f(t)] e_0(b,t) d_\alpha t + \int_a^b k_1(\alpha, t) f^2(t) e_0(b,t) d_\alpha t. \tag{22}$$

Therefore it holds

$$\int_a^b f(t) D^\alpha [f(t)] e_0(b,t) d_\alpha t = \left(\frac{f^2(b) - f^2(a) e_0(b,a)}{2} \right) + \tag{23}$$

$$\frac{1}{2} \int_a^b f^2(t) k_1(\alpha, t) e_0(b,t) d_\alpha t.$$

We present the following Opial's type inequality:

Theorem 11 *All as in Theorem 7. Further assume that* $t \geq s$, $(D^{\alpha})^{k} f(s) = 0$, *for* $k = 0, 1, ..., n-1$; $\alpha \in (0,1]$, $n \in \mathbb{N}$; *and* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_{s}^{t} |f(w)| \, |(D^{\alpha})^{n} f(w)| \, e_0(t,w) \, d_{\alpha}w \leq$$

$$\left[\int_{s}^{t} \left(\int_{s}^{w} |h_{n-1}(\tau,w)|^{p} e_0(w,\tau) \, d_{\alpha}\tau\right) e_0(t,w) \, d_{\alpha}w\right]^{\frac{1}{p}}$$

$$\left\{ \frac{\left(\int_{s}^{t} |(D^{\alpha})^{n} f(w)|^{q} e_0(t,w) \, d_{\alpha}w\right)^{2}}{2} + \right. \tag{24}$$

$$\left. \frac{1}{2}\left[\int_{s}^{t} \left(\int_{s}^{w} |(D^{\alpha})^{n} f(\tau)|^{q} e_0(w,\tau) \, d_{\alpha}\tau\right)^{2} k_1(\alpha,w) \, e_0(t,w) \, d_{\alpha}w\right]\right\}^{\frac{1}{q}}.$$

Proof. Since $(D^{\alpha})^{k} f(s) = 0$, for $k = 0, 1, ..., n-1$, $t \geq s$, we have

$$f(t) = (-1)^{n-1} \int_{s}^{t} h_{n-1}(\tau,t) \, (D^{\alpha})^{n} f(\tau) \, e_0(t,\tau) \, d_{\alpha}\tau, \tag{25}$$

and

$$f(w) = (-1)^{n-1} \int_{s}^{w} h_{n-1}(\tau,w) \, (D^{\alpha})^{n} f(\tau) \, e_0(w,\tau) \, d_{\alpha}\tau, \tag{26}$$

for all $s \leq w \leq t$.

By Hölder's inequality we obtain

$$|f(w)| \leq \int_{s}^{w} |h_{n-1}(\tau,w)| \, |(D^{\alpha})^{n} f(\tau)| \, e_0(w,\tau) \, d_{\alpha}\tau =$$

$$\int_{s}^{w} |h_{n-1}(\tau,w)| \, (e_0(w,\tau))^{\frac{1}{p}} \, |(D^{\alpha})^{n} f(\tau)| \, (e_0(w,\tau))^{\frac{1}{q}} \, d_{\alpha}\tau \leq$$

$$\left(\int_{s}^{w} |h_{n-1}(\tau,w)|^{p} (e_0(w,\tau)) \, d_{\alpha}\tau\right)^{\frac{1}{p}} \left(\int_{s}^{w} |(D^{\alpha})^{n} f(\tau)|^{q} (e_0(w,\tau)) \, d_{\alpha}\tau\right)^{\frac{1}{q}}. \tag{27}$$

Call

$$z(w) := \int_{s}^{w} |(D^{\alpha})^{n} f(\tau)|^{q} (e_0(w,\tau)) \, d_{\alpha}\tau, \quad z(s) = 0, \tag{28}$$

$s \leq w \leq t$.

Then (by (10))

$$D^{\alpha} z(w) = |(D^{\alpha})^{n} f(w)|^{q}, \tag{29}$$

and

$$|(D^{\alpha})^{n} f(w)| = (D^{\alpha} z(w))^{\frac{1}{q}}, \text{ all } s \leq w \leq t. \tag{30}$$

Therefore we have (all $s \leq w \leq t$)

$$|f(w)| \left|(D^{\alpha})^n f(w)\right| e_0(t,w) \leq$$

$$\left[\left(\int_s^w |h_{n-1}(\tau,w)|^p e_0(w,\tau)\, d_\alpha\tau\right) e_0(t,w)\right]^{\frac{1}{p}} \left[z(w) D^\alpha z(w) e_0(t,w)\right]^{\frac{1}{q}}. \quad (31)$$

Next, we apply again Hölder's inequality and finally we use (23) to get that

$$\int_s^t |f(w)| \left|(D^{\alpha})^n f(w)\right| e_0(t,w)\, d_\alpha w \leq$$

$$\int_s^t \left\{\left[\left(\int_s^w |h_{n-1}(\tau,w)|^p e_0(w,\tau)\, d_\alpha\tau\right) e_0(t,w)\right]^{\frac{1}{p}} \left[z(w) D^\alpha z(w) e_0(t,w)\right]^{\frac{1}{q}}\right\} d_\alpha w$$

$$\leq \left(\int_s^t \left(\int_s^w |h_{n-1}(\tau,w)|^p e_0(w,\tau)\, d_\alpha\tau\right) e_0(t,w)\, d_\alpha w\right)^{\frac{1}{p}} \quad (32)$$

$$\left(\int_s^t z(w) D^\alpha z(w) e_0(t,w)\, d_\alpha w\right)^{\frac{1}{q}} =: (\xi).$$

By (23), we derive that

$$\int_s^t z(w) D^\alpha z(w) e_0(t,w)\, d_\alpha w =$$

$$\frac{z^2(t)}{2} + \frac{1}{2}\int_s^t z^2(w) k_1(\alpha,w) e_0(t,w)\, d_\alpha w =$$

$$\frac{\left(\int_s^t \left|(D^{\alpha})^n f(w)\right|^q e_0(t,w)\, d_\alpha w\right)^2}{2} + \quad (33)$$

$$\frac{1}{2}\left[\int_s^t \left(\int_s^w \left|(D^{\alpha})^n f(\tau)\right|^q e_0(w,\tau)\, d_\alpha\tau\right)^2 k_1(\alpha,w) e_0(t,w)\, d_\alpha w\right].$$

Consequently, we get that

$$(\xi) = \left[\int_s^t \left(\int_s^w |h_{n-1}(\tau,w)|^p e_0(w,\tau)\, d_\alpha\tau\right) e_0(t,w)\, d_\alpha w\right]^{\frac{1}{p}}$$

$$\left\{\frac{\left(\int_s^t \left|(D^{\alpha})^n f(w)\right|^q e_0(t,w)\, d_\alpha w\right)^2}{2} + \right. \quad (34)$$

$$\left.\frac{1}{2}\left[\int_s^t \left(\int_s^w \left|(D^{\alpha})^n f(\tau)\right|^q e_0(w,\tau)\, d_\alpha\tau\right)^2 k_1(\alpha,w) e_0(t,w)\, d_\alpha w\right]\right\}^{\frac{1}{q}}.$$

The claim is proved. ■

Corollary 12 *(to Theorem 11, for $n = 1$) Here f is continuously differentiable on $[t_0, \infty)$, $t, s \in [t_0, \infty)$, $t \geq s$, $f(s) = 0$, $\alpha \in (0, 1]$; and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_s^t |f(w)| |D^\alpha f(w)| e_0(t, w) d_\alpha w \leq$$

$$\left[\int_s^t \left(\int_s^w e_0(w, \tau) d_\alpha \tau\right) e_0(t, w) d_\alpha w\right]^{\frac{1}{p}}$$

$$\left\{\frac{\left(\int_s^t |D^\alpha f(w)|^q e_0(t, w) d_\alpha w\right)^2}{2} + \right. \tag{35}$$

$$\left.\frac{1}{2}\left[\int_s^t \left(\int_s^w |D^\alpha f(\tau)|^q e_0(w, \tau) d_\alpha \tau\right)^2 k_1(\alpha, w) e_0(t, w) d_\alpha w\right]\right\}^{\frac{1}{q}}.$$

Corollary 13 *(to Theorem 11, for $n = 2$) Let f be twice continuously differentiable on $[t_0, \infty)$, $t, s \in [t_0, \infty)$, $t \geq s$, $f(s) = D^\alpha f(s) = 0$, $\alpha \in (0, 1]$; and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here it is $h_1(t, s) = \int_s^t \frac{d\tau}{k_0(\alpha, \tau)}$. Then*

$$\int_s^t |f(w)| \left|(D^\alpha)^2 f(w)\right| e_0(t, w) d_\alpha w \leq$$

$$\left[\int_s^t \left(\int_s^w |h_1(\tau, w)|^p e_0(w, \tau) d_\alpha \tau\right) e_0(t, w) d_\alpha w\right]^{\frac{1}{p}}$$

$$\left\{\frac{\left(\int_s^t \left|(D^\alpha)^2 f(w)\right|^q e_0(t, w) d_\alpha w\right)^2}{2} + \right. \tag{36}$$

$$\left.\frac{1}{2}\left[\int_s^t \left(\int_s^w \left|(D^\alpha)^2 f(\tau)\right|^q e_0(w, \tau) d_\alpha \tau\right)^2 k_1(\alpha, w) e_0(t, w) d_\alpha w\right]\right\}^{\frac{1}{q}}.$$

It follows an Ostrowski like inequality.

Theorem 14 *Let $n \in \mathbb{N}$, and f is n times continuously differentiable over $[a.b] \subset \mathbb{R}$; h_k as in (16). For fixed $s \in [a, b]$, assume that $(D^\alpha)^k f(s) = 0$, $k = 1, ..., n - 1$, $\alpha \in (0, 1]$, and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left|\int_a^b (f(t) - e_0(t, s) f(s)) dt\right| \leq \int_a^b |f(t) - e_0(t, s) f(s)| dt \leq$$

$$\left[\|(D^\alpha)^n f\|_{L_q([a,s], e_0(\alpha, \cdot) d_\alpha \tau)} \left(\int_a^s \left(\int_t^s (h_{n-1}(\tau, t))^p e_0(t, \tau) d_\alpha \tau\right)^{\frac{1}{p}} dt\right) + \right. \tag{37}$$

$$\left\|(D^{\alpha})^{n} f\right\|_{L_q([s,b],d_\alpha\tau)} \left(\int_s^b \left(\int_s^t \left(\left|h_{n-1}(\tau,t)\right|\right)^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} dt\right)\Bigg].$$

Proof. Here $(D^{\alpha})^{k} f(s) = 0$, $k = 1, ..., n-1$; $t, s \in [a,b]$, s is fixed.
We have by (17) that

$$f(t) - e_0(t,s) f(s) = (-1)^{n-1} \int_s^t h_{n-1}(\tau,t) (D^{\alpha})^{n} f(\tau) e_0(t,\tau)\, d_\alpha\tau.$$

Let $t \geq s$, then by Hölder's inequality, we get

$$\left|f(t) - e_0(t,s) f(s)\right| \leq \int_s^t \left|h_{n-1}(\tau,t)\right| \left|(D^{\alpha})^{n} f(\tau)\right| e_0(t,\tau)\, d_\alpha\tau =$$

$$\int_s^t \left(\left|h_{n-1}(\tau,t)\right| (e_0(t,\tau))^{\frac{1}{p}}\right) \left(\left|(D^{\alpha})^{n} f(\tau)\right| e_0(t,\tau)^{\frac{1}{q}}\right) d_\alpha\tau \leq$$

$$\left(\int_s^t \left|h_{n-1}(\tau,t)\right|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} \left(\int_s^t \left|(D^{\alpha})^{n} f(\tau)\right|^q e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{q}} \leq \tag{38}$$

$$\left(\int_s^t \left|h_{n-1}(\tau,t)\right|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} \left(\int_s^t \left|(D^{\alpha})^{n} f(\tau)\right|^q d_\alpha\tau\right)^{\frac{1}{q}} =$$

$$\left\|(D^{\alpha})^{n} f\right\|_{L_q([s,b],d_\alpha\tau)} \left(\int_s^t \left|h_{n-1}(\tau,t)\right|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}}.$$

So when, $s \leq t \leq b$, we got that

$$\left|f(t) - e_0(t,s) f(s)\right| \leq \tag{39}$$

$$\left\|(D^{\alpha})^{n} f\right\|_{L_q([s,b],d_\alpha\tau)} \left(\int_s^t \left|h_{n-1}(\tau,t)\right|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}}.$$

Let now $t \leq s$, then

$$-\left(f(t) - e_0(t,s) f(s)\right) = (-1)^{n-1} \int_t^s h_{n-1}(\tau,t) (D^{\alpha})^{n} f(\tau) e_0(t,\tau)\, d_\alpha\tau. \tag{40}$$

Hence, it holds (again by Hölder's inequality)

$$\left|f(t) - e_0(t,s) f(s)\right| \leq \int_t^s h_{n-1}(\tau,t) \left|(D^{\alpha})^{n} f(\tau)\right| e_0(t,\tau)\, d_\alpha\tau = \tag{41}$$

$$\int_t^s h_{n-1}(\tau,t) (e_0(t,\tau))^{\frac{1}{p}} \left|(D^{\alpha})^{n} f(\tau)\right| e_0(t,\tau)^{\frac{1}{q}}\, d_\alpha\tau \leq$$

$$\left(\int_t^s (h_{n-1}(\tau,t))^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} \left(\int_t^s \left|(D^{\alpha})^{n} f(\tau)\right|^q e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{q}} \leq$$

$$\left(\int_t^s \left(h_{n-1}(\tau,t)\right)^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} \left(\int_a^s \left|(D^\alpha)^n f(\tau)\right|^q e_0(\alpha,\tau)\, d_\alpha\tau\right)^{\frac{1}{q}} =$$

$$\left(\int_t^s \left(h_{n-1}(\tau,t)\right)^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} \left\|(D^\alpha)^n f\right\|_{L_q([a,s],e_0(\alpha,\cdot)d_\alpha\tau)}. \tag{42}$$

Thus, when, $a \leq t \leq s$, we found that

$$|f(t) - e_0(t,s) f(s)| \leq$$

$$\left\|(D^\alpha)^n f\right\|_{L_q([a,s],e_0(\alpha,\cdot)d_\alpha\tau)} \left(\int_t^s \left(h_{n-1}(\tau,t)\right)^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}}. \tag{43}$$

Consequently, it holds

$$\left|\int_a^b \left(f(t) - e_0(t,s) f(s)\right) dt\right| \leq \int_a^b |f(t) - e_0(t,s) f(s)|\, dt =$$

$$\int_a^s |f(t) - e_0(t,s) f(s)|\, dt + \int_s^b |f(t) - e_0(t,s) f(s)|\, dt \leq \tag{44}$$

$$\left[\left(\int_a^s \left(\int_t^s \left(h_{n-1}(\tau,t)\right)^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} dt\right) \left\|(D^\alpha)^n f\right\|_{L_q([a,s],e_0(\alpha,\cdot)d_\alpha\tau)} +\right.$$

$$\left.\left(\int_s^b \left(\int_s^t \left(|h_{n-1}(\tau,t)|\right)^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{1}{p}} dt\right) \left\|(D^\alpha)^n f\right\|_{L_q([s,b],d_\alpha\tau)}\right].$$

The claim is proved. ■

Next comes a Poincaré type inequality.

Theorem 15 *Let all as in Theorem 14, including $f(s) = 0$, and $a \leq s \leq t \leq b$. Then*

$$\|f\|_{L_q([s,b])} \leq \left(\int_s^b \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau)\, d_\alpha\tau\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} \left\|(D^\alpha)^n f\right\|_{L_q([s,b],d_\alpha\tau)}. \tag{45}$$

Proof. Since $(D^\alpha)^k f(s) = 0$, for $k = 0, 1, ..., n-1$; $t \geq s$, we have

$$f(t) = (-1)^{n-1} \int_s^t h_{n-1}(\tau,t) (D^\alpha)^n f(\tau) e_0(t,\tau)\, d_\alpha\tau. \tag{46}$$

By Hölder's inequality we obtain

$$|f(t)| \leq \int_s^t |h_{n-1}(\tau,t)| \left|(D^\alpha)^n f(\tau)\right| e_0(t,\tau)\, d_\alpha\tau =$$

$$\int_s^t |h_{n-1}(\tau,t)| (e_0(t,\tau))^{\frac{1}{p}} |(D^\alpha)^n f(\tau)| e_0(t,\tau)^{\frac{1}{q}} d_\alpha \tau \leq$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau) d_\alpha \tau\right)^{\frac{1}{p}} \left(\int_s^t |(D^\alpha)^n f(\tau)|^q e_0(t,\tau) d_\alpha \tau\right)^{\frac{1}{q}} \leq \tag{47}$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau) d_\alpha \tau\right)^{\frac{1}{p}} \left(\int_s^b |(D^\alpha)^n f(\tau)|^q e_0(t,\tau) d_\alpha \tau\right)^{\frac{1}{q}},$$

$\forall\ t \in [s,b]$.

Hence it holds

$$|f(t)|^q \leq$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau) d_\alpha \tau\right)^{\frac{q}{p}} \left(\int_s^b |(D^\alpha)^n f(\tau)|^q e_0(t,\tau) d_\alpha \tau\right) \leq$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau) d_\alpha \tau\right)^{\frac{q}{p}} \left(\int_s^b |(D^\alpha)^n f(\tau)|^q d_\alpha \tau\right) = \tag{48}$$

$$\left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau) d_\alpha \tau\right)^{\frac{q}{p}} \|(D^\alpha)^n f\|^q_{L_q([s,b],d_\alpha \tau)}.$$

Consequently, we get that

$$\int_s^b |f(t)|^q dt \leq$$

$$\left(\int_s^b \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau) d_\alpha \tau\right)^{\frac{q}{p}} dt\right) \|(D^\alpha)^n f\|^q_{L_q([s,b],d_\alpha \tau)}. \tag{49}$$

Thus, we derive

$$\left(\int_s^b |f(t)|^q dt\right)^{\frac{1}{q}} \leq$$

$$\left(\int_s^b \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau) d_\alpha \tau\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} \|(D^\alpha)^n f\|_{L_q([s,b],d_\alpha \tau)}. \tag{50}$$

The claim is proved. ■

It follows a Sobolev type inequality.

Theorem 16 *All as in Theorem 15, $r \geq 1$. Then*

$$\|f\|_{L_r([s,b])} \leq \left(\int_s^b \left(\int_s^t |h_{n-1}(\tau,t)|^p e_0(t,\tau) d_\alpha \tau\right)^{\frac{r}{p}} dt\right)^{\frac{1}{r}} \|(D^\alpha)^n f\|_{L_q([s,b],d_\alpha \tau)}. \tag{51}$$

Proof. As in (47) we obtain

$$|f(t)| \leq \left(\int_s^t |h_{n-1}(\tau, t)|^p e_0(t, \tau) \, d_\alpha \tau \right)^{\frac{1}{p}} \left(\int_s^b |(D^\alpha)^n f(\tau)|^q e_0(t, \tau) \, d_\alpha \tau \right)^{\frac{1}{q}} \tag{52}$$

$$\leq \left(\int_s^t |h_{n-1}(\tau, t)|^p e_0(t, \tau) \, d_\alpha \tau \right)^{\frac{1}{p}} \|(D^\alpha)^n f\|_{L_q([s,b], d_\alpha \tau)}, \tag{53}$$

$\forall \ t \in [s, b]$.

Hence, by $r \geq 1$, we obtain

$$|f(t)|^r \leq \left(\int_s^t |h_{n-1}(\tau, t)|^p e_0(t, \tau) \, d_\alpha \tau \right)^{\frac{r}{p}} \|(D^\alpha)^n f\|_{L_q([s,b], d_\alpha \tau)}^r, \tag{54}$$

$\forall \ t \in [s, b]$.

Consequently, it holds

$$\int_s^b |f(t)|^r \, dt \leq$$

$$\left(\int_s^b \left(\int_s^t |h_{n-1}(\tau, t)|^p e_0(t, \tau) \, d_\alpha \tau \right)^{\frac{r}{p}} dt \right) \|(D^\alpha)^n f\|_{L_q([s,b], d_\alpha \tau)}^r. \tag{55}$$

Finally we get

$$\left(\int_s^b |f(t)|^r \, dt \right)^{\frac{1}{r}} \leq$$

$$\left(\int_s^b \left(\int_s^t |h_{n-1}(\tau, t)|^p e_0(t, \tau) \, d_\alpha \tau \right)^{\frac{r}{q}} dt \right)^{\frac{1}{r}} \|(D^\alpha)^n f\|_{L_q([s,b], d_\alpha \tau)}. \tag{56}$$

The claim is proved. ■

We continue with Polya type inequalities.

Corollary 17 *(to Theorem 14) Let $n \in \mathbb{N}$, and $f \in C^n([a, b])$; h_k as in (16). For fixed $s \in [a, b]$, assume that $(D^\alpha)^k f(s) = 0$, $k = 0, 1, ..., n-1$, $\alpha \in (0, 1]$, and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \int_a^b f(t) \, dt \right| \leq \int_a^b |f(t)| \, dt \leq$$

$$\left[\|(D^\alpha)^n f\|_{L_q([a,s], e_0(\alpha, \cdot) d_\alpha \tau)} \left(\int_a^s \left(\int_t^s (h_{n-1}(\tau, t))^p e_0(t, \tau) \, d_\alpha \tau \right)^{\frac{1}{p}} dt \right) + \right.$$

$$\left. \|(D^\alpha)^n f\|_{L_q([s,b], d_\alpha \tau)} \left(\int_s^b \left(\int_s^t (|h_{n-1}(\tau, t)|)^p e_0(t, \tau) \, d_\alpha \tau \right)^{\frac{1}{p}} dt \right) \right]. \tag{57}$$

Proof. By (37), just set $f(s) = 0$. ■

We give the following result.

Theorem 18 *Let $n \in \mathbb{N}$, $f \in C^n([a,b])$; h_k as in (16). Assume that $(D^{\alpha})^k f(a) = (D^{\alpha})^k f(b) = 0$, $k = 0, 1, ..., n-1$, $\alpha \in (0,1]$, and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt \leq$$

$$\|(D^{\alpha})^n f\|_{L_q\left(\left[a, \frac{a+b}{2}\right], d_{\alpha}\tau\right)} \left(\int_a^{\frac{a+b}{2}} \left(\int_a^t (|h_{n-1}(\tau,t)|)^p e_0(t,\tau)\, d_{\alpha}\tau \right)^{\frac{1}{p}} dt \right) +$$

$$\|(D^{\alpha})^n f\|_{L_q\left(\left[\frac{a+b}{2}, b\right], e_0(\alpha,\cdot) d_{\alpha}\tau\right)} \left(\int_{\frac{a+b}{2}}^b \left(\int_t^b (h_{n-1}(\tau,t))^p e_0(t,\tau)\, d_{\alpha}\tau \right)^{\frac{1}{p}} dt \right).$$

Proof. Let $s = a$ and instead of b we choose $\frac{a+b}{2}$. Then, by (57), we derive

$$\left| \int_a^{\frac{a+b}{2}} f(t)\, dt \right| \leq \int_a^{\frac{a+b}{2}} |f(t)|\, dt \leq$$

$$\|(D^{\alpha})^n f\|_{L_q\left(\left[a, \frac{a+b}{2}\right], d_{\alpha}\tau\right)} \left(\int_a^{\frac{a+b}{2}} \left(\int_a^t (|h_{n-1}(\tau,t)|)^p e_0(t,\tau)\, d_{\alpha}\tau \right)^{\frac{1}{p}} dt \right). \quad (59)$$

Next, assume that $s = b$ and instead of a we choose $\frac{a+b}{2}$.

Then, again by (57), we obtain

$$\left| \int_{\frac{a+b}{2}}^b f(t)\, dt \right| \leq \int_{\frac{a+b}{2}}^b |f(t)|\, dt \leq$$

$$\|(D^{\alpha})^n f\|_{L_q\left(\left[\frac{a+b}{2}, b\right], e_0(\alpha,\cdot) d_{\alpha}\tau\right)} \left(\int_{\frac{a+b}{2}}^b \left(\int_t^b (h_{n-1}(\tau,t))^p e_0(t,\tau)\, d_{\alpha}\tau \right)^{\frac{1}{p}} dt \right). \quad (60)$$

Finally, we have that (by (59), (60))

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt = \int_a^{\frac{a+b}{2}} |f(t)|\, dt + \int_{\frac{a+b}{2}}^b |f(t)|\, dt \leq$$

$$\|(D^{\alpha})^n f\|_{L_q\left(\left[a, \frac{a+b}{2}\right], d_{\alpha}\tau\right)} \left(\int_a^{\frac{a+b}{2}} \left(\int_a^t (|h_{n-1}(\tau,t)|)^p e_0(t,\tau)\, d_{\alpha}\tau \right)^{\frac{1}{p}} dt \right) +$$

$$\|(D^{\alpha})^n f\|_{L_q\left(\left[\frac{a+b}{2}, b\right], e_0(\alpha,\cdot) d_{\alpha}\tau\right)} \left(\int_{\frac{a+b}{2}}^b \left(\int_t^b (h_{n-1}(\tau,t))^p e_0(t,\tau)\, d_{\alpha}\tau \right)^{\frac{1}{p}} dt \right). \quad (61)$$

The claim is proved. ■

We finish with a Hilbert-Pachpatte type inequality.

Theorem 19 *Here $j=1,2$. Let $n_j \in \mathbb{N}$, and $f_j \in C^{n_j}([a_j,b_j])$; h_{kj} as in (16). For fixed $s_j \in [a_j,b_j]$, assume that $(D^{\alpha})^{k_j} f_j(s_j)=0$, $k_j=0,1,...,n_j-1$, $\alpha \in (0,1]$, and let $p,q>1: \frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{s_1}^{b_1}\int_{s_2}^{b_2} \frac{|f_1(t_1)|\,|f_2(t_2)|\,dt_1dt_2}{\left[\frac{\left(\int_{s_1}^{t_1}(h_{n_1-1}(\tau_1,t_1))^p e_0(t_1,\tau_1)d_\alpha\tau_1\right)}{p}+\frac{\left(\int_{s_2}^{t_2}(h_{n_2-1}(\tau_2,t_2))^q e_0(t_2,\tau_2)d_\alpha\tau_2\right)}{q}\right]} \leq$$

$$(b_1-s_1)(b_2-s_2)\left(\int_{s_1}^{b_1}|(D^{\alpha})^{n_1} f_1(\tau_1)|^q d_\alpha\tau_1\right)^{\frac{1}{q}}\left(\int_{s_2}^{b_2}|(D^{\alpha})^{n_2} f_2(\tau_2)|^p d_\alpha\tau_2\right)^{\frac{1}{p}}. \tag{62}$$

Proof. Here $j=1,2$. Since $(D^{\alpha})^{k_j} f_j(s_j)=0$, $k_j=0,1,...,n_j-1$; $t_j \geq s_j$, we have

$$f_j(t_j)=(-1)^{n_j-1}\int_{s_j}^{t_j} h_{n_j-1}(\tau_j,t_j)(D^{\alpha})^{n_j} f_j(\tau_j)\,e_0(t_j,\tau_j)\,d_\alpha\tau_j. \tag{63}$$

As in (27) we get

$$|f_1(t_1)| \leq \left(\int_{s_1}^{t_1}|h_{n_1-1}(\tau_1,t_1)|^p e_0(t_1,\tau_1)\,d_\alpha\tau_1\right)^{\frac{1}{p}}$$

$$\left(\int_{s_1}^{t_1}|(D^{\alpha})^{n_1} f_1(\tau_1)|^q e_0(t_1,\tau_1)\,d_\alpha\tau_1\right)^{\frac{1}{q}}, \tag{64}$$

and

$$|f_2(t_2)| \leq \left(\int_{s_2}^{t_2}|h_{n_2-1}(\tau_2,t_2)|^q e_0(t_2,\tau_2)\,d_\alpha\tau_2\right)^{\frac{1}{q}}$$

$$\left(\int_{s_2}^{t_2}|(D^{\alpha})^{n_2} f_2(\tau_2)|^p e_0(t_2,\tau_2)\,d_\alpha\tau_2\right)^{\frac{1}{p}}. \tag{65}$$

Hence

$$|f_1(t_1)|\,|f_2(t_2)| \leq$$

$$\left(\int_{s_1}^{t_1}|h_{n_1-1}(\tau_1,t_1)|^p e_0(t_1,\tau_1)\,d_\alpha\tau_1\right)^{\frac{1}{p}}\left(\int_{s_2}^{t_2}|h_{n_2-1}(\tau_2,t_2)|^q e_0(t_2,\tau_2)\,d_\alpha\tau_2\right)^{\frac{1}{q}}$$

$$\left(\int_{s_1}^{t_1}|(D^{\alpha})^{n_1} f_1(\tau_1)|^q e_0(t_1,\tau_1)\,d_\alpha\tau_1\right)^{\frac{1}{q}}\left(\int_{s_2}^{t_2}|(D^{\alpha})^{n_2} f_2(\tau_2)|^p e_0(t_2,\tau_2)\,d_\alpha\tau_2\right)^{\frac{1}{p}} \tag{66}$$

(using Young's inequality for $a^*, b^* \geq 0$, $a^{*\frac{1}{p}} b^{*\frac{1}{q}} \leq \frac{a^*}{p} + \frac{b^*}{q}$)

$$\leq \left[\frac{\left(\int_{s_1}^{t_1} \left| h_{n_1-1}\left(\tau_1, t_1\right)\right|^p e_0\left(t_1, \tau_1\right) d_\alpha \tau_1 \right)}{p} + \frac{\left(\int_{s_2}^{t_2} \left| h_{n_2-1}\left(\tau_2, t_2\right)\right|^q e_0\left(t_2, \tau_2\right) d_\alpha \tau_2 \right)}{q} \right]$$

$$\left(\int_{s_1}^{t_1} \left| \left(D^\alpha\right)^{n_1} f_1\left(\tau_1\right)\right|^q e_0\left(t_1, \tau_1\right) d_\alpha \tau_1 \right)^{\frac{1}{q}} \left(\int_{s_2}^{t_2} \left| \left(D^\alpha\right)^{n_2} f_2\left(\tau_2\right)\right|^p e_0\left(t_2, \tau_2\right) d_\alpha \tau_2 \right)^{\frac{1}{p}}. \tag{67}$$

Thus, it holds

$$\int_{s_1}^{b_1} \int_{s_2}^{b_2} \frac{\left|f_1\left(t_1\right)\right| \left|f_2\left(t_2\right)\right| dt_1 dt_2}{\left[\frac{\left(\int_{s_1}^{t_1} \left(\left| h_{n_1-1}(\tau_1, t_1) \right| \right)^p e_0(t_1, \tau_1) d_\alpha \tau_1 \right)}{p} + \frac{\left(\int_{s_2}^{t_2} \left(\left| h_{n_2-1}(\tau_2, t_2) \right| \right)^q e_0(t_2, \tau_2) d_\alpha \tau_2 \right)}{q} \right]}$$

(denominator can be zero only when both $t_1 = s_1$ and $t_2 = s_2$)

$$\leq \left(\int_{s_1}^{b_1} \left(\int_{s_1}^{t_1} \left| \left(D^\alpha\right)^{n_1} f_1\left(\tau_1\right)\right|^q e_0\left(t_1, \tau_1\right) d_\alpha \tau_1 \right)^{\frac{1}{q}} dt_1 \right)$$

$$\left(\int_{s_2}^{b_2} \left(\int_{s_2}^{t_2} \left| \left(D^\alpha\right)^{n_2} f_2\left(\tau_2\right)\right|^p e_0\left(t_2, \tau_2\right) d_\alpha \tau_2 \right)^{\frac{1}{p}} dt_2 \right) \leq$$

$$\left(\int_{s_1}^{b_1} \left(\int_{s_1}^{t_1} \left| \left(D^\alpha\right)^{n_1} f_1\left(\tau_1\right)\right|^q d_\alpha \tau_1 \right)^{\frac{1}{q}} dt_1 \right) \tag{68}$$

$$\left(\int_{s_2}^{b_2} \left(\int_{s_2}^{t_2} \left| \left(D^\alpha\right)^{n_2} f_2\left(\tau_2\right)\right|^p d_\alpha \tau_2 \right)^{\frac{1}{p}} dt_2 \right) \leq$$

$$\left(\int_{s_1}^{b_1} \left(\int_{s_1}^{b_1} \left| \left(D^\alpha\right)^{n_1} f_1\left(\tau_1\right)\right|^q d_\alpha \tau_1 \right)^{\frac{1}{q}} dt_1 \right)$$

$$\left(\int_{s_2}^{b_2} \left(\int_{s_2}^{b_2} \left| \left(D^\alpha\right)^{n_2} f_2\left(\tau_2\right)\right|^p d_\alpha \tau_2 \right)^{\frac{1}{p}} dt_2 \right) =$$

$$\left(b_1 - s_1\right)\left(b_2 - s_2\right) \left(\int_{s_1}^{b_1} \left| \left(D^\alpha\right)^{n_1} f_1\left(\tau_1\right)\right|^q d_\alpha \tau_1 \right)^{\frac{1}{q}} \left(\int_{s_2}^{b_2} \left| \left(D^\alpha\right)^{n_2} f_2\left(\tau_2\right)\right|^p d_\alpha \tau_2 \right)^{\frac{1}{p}},$$

proving the claim. ■

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