Trigonometric and Hyperbolic Poincaré, Sobolev and Hilbert-Pachpatte type inequalities

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Abstract

In this article based on trigonometric and hyperbolic type Taylor formulae we establish Poincaré, Sobolev and Hilbert-Pachpatte type inequalities of different kinds specific and general.

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1 Main Results

We start with a collection of Poincaré type inequalities.

Theorem 1 *Let $f \in C^2([a,b], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f(a) = f'(a) = 0$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|f\|_{L_q([a,b])} \leq \left(\int_a^b \left(\int_a^x |\sin(x-t)|^p \, dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \|f + f''\|_{L_q([a,b])} . \tag{1}$$

Proof. Since $f(a) = f'(a) = 0$, by Corollary 3.4 of [1] we have

$$f(x) = \int_a^x (f''(t) + f(t)) \sin(x-t) \, dt, \tag{2}$$

$\forall\, x \in [a,b]$.

It follows by Hölder's inequality that

$$|f(x)| \leq \int_a^x |f''(t) + f(t)| \, |\sin(x-t)| \, dt \leq$$

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$$\left(\int_a^x |\sin(x-t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_a^x |(f''+f)(t)|^q\,dt\right)^{\frac{1}{q}} \leq \tag{3}$$

$$\left(\int_a^x |\sin(x-t)|^p\,dt\right)^{\frac{1}{p}} \|f''+f\|_{L_q([a,b])}.$$

Hence

$$|f(x)|^q \leq \left(\int_a^x |\sin(x-t)|^p\,dt\right)^{\frac{q}{p}} \|f''+f\|_{L_q([a,b])}^q, \tag{4}$$

and

$$\int_a^b |f(x)|^q\,dx \leq \left(\int_a^b \left(\int_a^x |\sin(x-t)|^p\,dt\right)^{\frac{q}{p}} dx\right) \|f''+f\|_{L_q([a,b])}^q. \tag{5}$$

We have proved that

$$\left(\int_a^b |f(x)|^q\,dx\right)^{\frac{1}{q}} \leq \left(\int_a^b \left(\int_a^x |\sin(x-t)|^p\,dt\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \|f''+f\|_{L_q([a,b])}. \tag{6}$$

■

It follows

Theorem 2 *All as in Theorem 1. Then*

$$\|f\|_{L_q([a,b])} \leq \left(\int_a^b \left(\int_a^x |\sinh(x-t)|^p\,dt\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \|f-f''\|_{L_q([a,b])}. \tag{7}$$

Proof. As similar to Theorem 1 is omitted. It is based on Corollary 3.5 of [1]. ■

We continue with

Theorem 3 *Let $f \in C^4([a,b],K)$, where $K=\mathbb{R}$ or $\mathbb{C}$, such that $f(a)=f'(a)=f''(a)=f'''(a)=0$, and $p,q>1: \frac{1}{p}+\frac{1}{q}=1$. Then*

$$\|f\|_{L_q([a,b])} \leq$$

$$\frac{1}{2}\left(\int_a^b \left(\int_a^x |\sinh(x-t)-\sin(x-t)|^p\,dt\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \|f-f''''\|_{L_q([a,b])}. \tag{8}$$

Proof. As similar to Theorem 1 is omitted. It is based on Corollary 3.6 of [1]. ■

It follows

Theorem 4 *Let $f \in C^4([a,b], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f(a) = f'(a) = f''(a) = f'''(a) = 0$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Also let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. Then*

$$\|f\|_{L_q([a,b])} \leq$$

$$\frac{1}{|\alpha| |\beta| |\beta^2 - \alpha^2|} \left(\int_a^b \left(\int_a^x |\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t))|^p \, dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}$$

$$\left\| f'''' + (\alpha^2 + \beta^2) f'' + \alpha^2\beta^2 f \right\|_{L_q([a,b])} . \tag{9}$$

Proof. As similar to Theorem 1 is omitted. It is based on Corollary 3.7 of [1]. ■

Next comes

Theorem 5 *Let $f \in C^4([a,b], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f^{(i)}(a) = 0$, $i = 0, 1, 2, 3$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then*

$$\|f\|_{L_q([a,b])} \leq$$

$$\frac{1}{2|\alpha|^3} \left(\int_a^b \left(\int_a^x |\sin(\alpha(x-t)) - \alpha(x-t)\cos(\alpha(x-t))|^p \, dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}$$

$$\left\| f^{(4)} + 2\alpha^2 f'' + \alpha^4 f \right\|_{L_q([a,b])} . \tag{10}$$

Proof. As similar to Theorem 1 is omitted. It is based on Corollary 3.8 of [1]. ■

We continue with

Theorem 6 *All as in Theorem 4. Then*

$$\|f\|_{L_q([a,b])} \leq$$

$$\frac{1}{|\alpha| |\beta| |\beta^2 - \alpha^2|} \left(\int_a^b \left(\int_a^x |\alpha \sinh(\beta(x-t)) - \beta \sinh(\alpha(x-t))|^p \, dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}$$

$$\left\| f^{(4)} - (\alpha^2 + \beta^2) f'' + \alpha^2\beta^2 f \right\|_{L_q([a,b])} . \tag{11}$$

Proof. As similar to Theorem 1 is omitted. It is based on Corollary 3.9 of [1]. ■

We also give

Theorem 7 *All as in Theorem 5. Then*

$$\|f\|_{L_q([a,b])} \leq$$

$$\frac{1}{2|\alpha|^3}\left(\int_a^b \left(\int_a^x |\alpha(x-t)\cosh(\alpha(x-t)) - \sinh(\alpha(x-t))|^p \, dt\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}}$$

$$\left\| f^{(4)} - 2\alpha^2 f'' + \alpha^4 f \right\|_{L_q([a,b])}. \tag{12}$$

Proof. As similar to Theorem 1 is omitted. It is based on Corollary 3.10 of [1]. ■

We make

Remark 8 *The following come from [1]. Let K denote $\mathbb{R}$ or $\mathbb{C}$, $[a,b] \subset \mathbb{R}$. For $c = (c_0, ..., c_n) \in K^{n+1}$ with $c_n = 1$, let the n-th order linear differential operator*

$$D_c : C^n([a,b], K) \to C([a,b], K)$$

be defined by the formula

$$D_c(f) := c_n f^{(n)} + ... + c_1 f' + c_0 f \quad (f \in C^n([a,b], K)).$$

Let $\omega_c \in C^n(\mathbb{R}, \mathbb{C})$ denote the unique solution of the initial value problem

$$D_c(\omega_c) = 0, \;\; \omega_c^{(l)}(0) = \delta_{l,n-1} \quad (l \in \{0, ..., n-1\}). \tag{13}$$

The function ω_c will be called the characteristic solution of $D_c(\omega) = 0$.

Define

$$(T_{a,c} f)(x) := \sum_{j=0}^{n-1} \left(f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(x-a) \right), \tag{14}$$

if $f^{(j)}(a) = 0$, $j = 0, 1, ..., n-1$, then

$$(T_{a,c} f)(x) = 0, \;\; \forall \; x \in [a,b].$$

Theorem 9 *([1]) Let $n \in \mathbb{N}$, $c = (c_0, ..., c_n) \in K^{n+1}$ with $c_n = 1$. Then for all $f \in C^n([a,b], K)$, $x \in [a,b]$, we have*

$$f(x) = (T_{a,c} f)(x) + \int_a^x D_c(f)(t) \, \omega_c(x-t) \, dt. \tag{15}$$

If $f^{(j)}(a) = 0$, $j = 0, 1, ..., n-1$, then

$$f(x) = \int_a^x D_c(f)(t) \, \omega_c(x-t) \, dt, \tag{16}$$

$\forall \; x \in [a,b]$.

We give the following general Poincaré type inequality.

Theorem 10 *All as in Remark 8 and* $f^{(j)}(a) = 0$, $j = 0, 1, ..., n-1$, $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\|f\|_{L_q([a,b])} \leq \left(\int_a^b \left(\int_a^x |\omega_c (x-t)|^p \, dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \|D_c (f)\|_{L_q([a,b])} . \tag{17}$$

Proof. As similar to Theorem 1 is omitted. It is based on Remark 8 and Theorem 9. ■

Next follow Sobolev type inequalities.

Theorem 11 *All as in Theorem 1,* $r > 0$. *Then*

$$\|f\|_{L_r([a,b])} \leq \left(\int_a^b \left(\int_a^x |\sin (x-t)|^p \, dt \right)^{\frac{r}{p}} dx \right)^{\frac{1}{r}} \|f'' + f\|_{L_q([a,b])} . \tag{18}$$

Proof. As in (3) we have

$$|f(x)| \leq \left(\int_a^x |\sin (x-t)|^p \, dt \right)^{\frac{1}{p}} \|f'' + f\|_{L_q([a,b])} . \tag{19}$$

and (by $r > 0$)

$$|f(x)|^r \leq \left(\int_a^x |\sin (x-t)|^p \, dt \right)^{\frac{r}{p}} \|f'' + f\|_{L_q([a,b])}^r , \tag{20}$$

$\forall \, x \in [a,b]$.

Thus

$$\int_a^b |f(x)|^r \, dx \leq \left(\int_a^b \left(\int_a^x |\sin (x-t)|^p \, dt \right)^{\frac{r}{p}} dx \right) \|f'' + f\|_{L_q([a,b])}^r , \tag{21}$$

and

$$\left(\int_a^b |f(x)|^r \, dx \right)^{\frac{1}{r}} \leq \left(\int_a^b \left(\int_a^x |\sin (x-t)|^p \, dt \right)^{\frac{r}{p}} dx \right)^{\frac{1}{r}} \|f'' + f\|_{L_q([a,b])} , \tag{22}$$

proving the claim. ■

Theorem 12 *All as in Theorem 2,* $r > 0$. *Then*

$$\|f\|_{L_r([a,b])} \leq \left(\int_a^b \left(\int_a^x |\sinh (x-t)|^p \, dt \right)^{\frac{r}{p}} dx \right)^{\frac{1}{r}} \|f - f''\|_{L_q([a,b])} . \tag{23}$$

Proof. As similar to Theorem 11 is omitted. ■

Theorem 13 *All as in Theorem 3, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq$$

$$\frac{1}{2}\left(\int_a^b \left(\int_a^x |\sinh(x-t) - \sin(x-t)|^p \, dt\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}} \|f - f''''\|_{L_q([a,b])}. \tag{24}$$

Proof. As similar to Theorem 11 is omitted. ■

Theorem 14 *All as in Theorem 4, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq$$

$$\frac{1}{|\alpha||\beta|\left|\beta^2-\alpha^2\right|}\left(\int_a^b \left(\int_a^x |\beta\sin(\alpha(x-t)) - \alpha\sin(\beta(x-t))|^p \, dt\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}}$$
$$\left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f\right\|_{L_q([a,b])}. \tag{25}$$

Proof. As similar to Theorem 11 is omitted. ■

Theorem 15 *All as in Theorem 5, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq$$

$$\frac{1}{2|\alpha|^3}\left(\int_a^b \left(\int_a^x |\sin(\alpha(x-t)) - \alpha(x-t)\cos(\alpha(x-t))|^p \, dt\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}}$$
$$\left\|f^{(4)} + 2\alpha^2 f'' + \alpha^4 f\right\|_{L_q([a,b])}. \tag{26}$$

Proof. As similar to Theorem 11 is omitted. ■

Theorem 16 *All as in Theorem 6, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq$$

$$\frac{1}{|\alpha||\beta|\left|\beta^2-\alpha^2\right|}\left(\int_a^b \left(\int_a^x |\alpha\sinh(\beta(x-t)) - \beta\sinh(\alpha(x-t))|^p \, dt\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}}$$
$$\left\|f^{(4)} - \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f\right\|_{L_q([a,b])}. \tag{27}$$

Proof. As similar to Theorem 11 is omitted. ■

Theorem 17 *All as in Theorem 7, $r>0$. Then*

$$\|f\|_{L_r([a,b])} \leq$$

$$\frac{1}{2|\alpha|^3}\left(\int_a^b\left(\int_a^x |\alpha(x-t)\cosh(\alpha(x-t))-\sinh(\alpha(x-t))|^p\,dt\right)^{\frac{r}{p}}dx\right)^{\frac{1}{r}}$$

$$\left\|f^{(4)}-2\alpha^2 f''+\alpha^4 f\right\|_{L_q([a,b])}. \quad (28)$$

Proof. It is omitted. ■

Theorem 18 *All as in Theorem 10, $r>0$. Then*

$$\|f\|_{L_r([a,b])} \leq \left(\int_a^b\left(\int_a^x |\omega_c(x-t)|^p\,dt\right)^{\frac{r}{p}}dx\right)^{\frac{1}{r}}\|D_c(f)\|_{L_q([a,b])}. \quad (29)$$

Proof. It is omitted. ■

We continue with a collection of Hilbert-Pachaptted inequalities.

Theorem 19 *Here $j=1,2$. Let $f_j\in C^2([a_j,b_j],K)$, where $K=\mathbb{R}$ or $\mathbb{C}$, such that $f_j(a_j)=f_j'(a_j)=0$, and $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)||f_2(x_2)|\,dx_1dx_2}{\left[\frac{\int_{a_1}^{x_1}|\sin(x_1-t_1)|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}|\sin(x_2-t_2)|^q dt_2}{q}\right]} \leq$$

$$(b_1-a_1)(b_2-a_2)\|f_1''+f_1\|_{L_q([a_1,b_1])}\|f_2''+f_2\|_{L_p([a_2,b_2])}. \quad (30)$$

Proof. As in (3) we have

$$|f_1(x_1)| \leq \left(\int_{a_1}^{x_1}|\sin(x_1-t_1)|^p\,dt_1\right)^{\frac{1}{p}}\|f_1''+f_1\|_{L_q([a_1,b_1])}, \quad (31)$$

$\forall\ x_1\in[a_1,b_1]$,

and

$$|f_2(x_2)| \leq \left(\int_{a_2}^{x_2}|\sin(x_2-t_2)|^q\,dt_2\right)^{\frac{1}{q}}\|f_2''+f_2\|_{L_p([a_2,b_2])}. \quad (32)$$

$\forall\ x_2\in[a_2,b_2]$.

Hence we have

$$|f_1(x_1)||f_2(x_2)| \leq \left(\int_{a_1}^{x_1}|\sin(x_1-t_1)|^p\,dt_1\right)^{\frac{1}{p}}\left(\int_{a_2}^{x_2}|\sin(x_2-t_2)|^q\,dt_2\right)^{\frac{1}{q}}$$

$$\|f_1''+f_1\|_{L_q([a_1,b_1])}\|f_2''+f_2\|_{L_p([a_2,b_2])} \leq \quad (33)$$

(using Young's inequality for $a,b \geq 0$, $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$)

$$\left[\frac{\int_{a_1}^{x_1} |\sin(x_1 - t_1)|^p \, dt_1}{p} + \frac{\int_{a_2}^{x_2} |\sin(x_2 - t_2)|^q \, dt_2}{q} \right]$$

$$\|f_1'' + f_1\|_{L_q([a_1,b_1])} \|f_2'' + f_2\|_{L_p([a_2,b_2])} . \tag{34}$$

So far we have

$$\frac{|f_1(x_1)| |f_2(x_2)|}{\left[\frac{\int_{a_1}^{x_1} |\sin(x_1-t_1)|^p dt_1}{p} + \frac{\int_{a_2}^{x_2} |\sin(x_2-t_2)|^q dt_2}{q} \right]} \leq$$

$$\|f_1'' + f_1\|_{L_q([a_1,b_1])} \|f_2'' + f_2\|_{L_p([a_2,b_2])} , \tag{35}$$

$\forall \, (x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]$.

The denominator in (35) can be zero only when both $x_1 = a_1$ and $x_2 = a_2$.

Therefore we obtain (30), by integrating (35) over $[a_1, b_1] \times [a_2, b_2]$. ■

Theorem 20 *All as in Theorem 19. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(x_1)| |f_2(x_2)| \, dx_1 dx_2}{\left[\frac{\int_{a_1}^{x_1} |\sinh(x_1-t_1)|^p dt_1}{p} + \frac{\int_{a_2}^{x_2} |\sinh(x_2-t_2)|^q dt_2}{q} \right]} \leq$$

$$(b_1 - a_1)(b_2 - a_2) \|f_1 - f_1''\|_{L_q([a_1,b_1])} \|f_2 - f_2''\|_{L_p([a_2,b_2])} . \tag{36}$$

Proof. As similar to Theorem 19 is omitted. ■

Theorem 21 *Here $j = 1, 2$. Let $f_j \in C^4([a_j, b_j], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f_j^{(i)}(a_j) = 0$, $i = 0, 1, 2, 3$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(x_1)| |f_2(x_2)| \, dx_1 dx_2}{\left[\frac{\int_{a_1}^{x_1} |\sinh(x_1-t_1) - \sin(x_1-t_1)|^p dt_1}{p} + \frac{\int_{a_2}^{x_2} |\sinh(x_2-t_2) - \sin(x_2-t_2)|^q dt_2}{q} \right]} \leq$$

$$\frac{(b_1 - a_1)(b_2 - a_2)}{4} \left\| f_1 - f_1^{(4)} \right\|_{L_q([a_1,b_1])} \left\| f_2 - f_2^{(4)} \right\|_{L_p([a_2,b_2])} . \tag{37}$$

Proof. As similar to Theorem 19 is omitted. ■

Theorem 22 *Here $j = 1, 2$. Let $f_j \in C^4([a_j, b_j], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f_j^{(i)}(a_j) = 0$, $i = 0, 1, 2, 3$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Alsi let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(x_1)| |f_2(x_2)| \, dx_1 dx_2}{\left[\frac{\int_{a_1}^{x_1} |\beta \sin(\alpha(x_1-t_1)) - \alpha \sin(\beta(x_1-t_1))|^p dt_1}{p} + \frac{\int_{a_2}^{x_2} |\beta \sin(\alpha(x_2-t_2)) - \alpha \sin(\beta(x_2-t_2))|^q dt_2}{q} \right]} \leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\alpha^2\beta^2\left(\beta^2-\alpha^2\right)^2}\left\|f_1^{(4)}+\left(\alpha^2+\beta^2\right)f_1''+\alpha^2\beta^2 f_1\right\|_{L_q([a_1,b_1])}$$
$$\left\|f_2^{(4)}+\left(\alpha^2+\beta^2\right)f_2''+\alpha^2\beta^2 f_2\right\|_{L_p([a_2,b_2])}. \quad (38)$$

Proof. It is omitted. ■

Theorem 23 *Here $j=1,2$. Let $f_j\in C^4([a_j,b_j],K)$, where $K=\mathbb{R}$ or $\mathbb{C}$, such that $f_j^{(i)}(a_j)=0$, $i=0,1,2,3$, and $p,q>1:\frac{1}{p}+\frac{1}{q}=1$; $\alpha\in\mathbb{R}$, $\alpha\neq 0$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)||f_2(x_2)|\,dx_1dx_2}{\left[\frac{\int_{a_1}^{x_1}|\sin(\alpha(x_1-t_1))-\alpha(x_1-t_1)\cos(\alpha(x_1-t_1))|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}|\sin(\alpha(x_2-t_2))-\alpha(x_2-t_2)\cos(\alpha(x_2-t_2))|^q dt_2}{q}\right]}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{4\alpha^6}\left\|f_1^{(4)}+2\alpha^2 f_1''+\alpha^4 f_1\right\|_{L_q([a_1,b_1])}$$
$$\left\|f_2^{(4)}+2\alpha^2 f_2''+\alpha^4 f_2\right\|_{L_p([a_2,b_2])}. \quad (39)$$

Proof. It is omitted. ■

Theorem 24 *All as in Theorem 22. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)||f_2(x_2)|\,dx_1dx_2}{\left[\frac{\int_{a_1}^{x_1}|\alpha\sinh(\beta(x_1-t_1))-\beta\sinh(\alpha(x_1-t_1))|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}|\alpha\sinh(\beta(x_2-t_2))-\beta\sinh(\alpha(x_2-t_2))|^q dt_2}{q}\right]}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\alpha^2\beta^2\left(\beta^2-\alpha^2\right)^2}\left\|f_1^{(4)}-\left(\alpha^2+\beta^2\right)f_1''+\alpha^2\beta^2 f_1\right\|_{L_q([a_1,b_1])}$$
$$\left\|f_2^{(4)}-\left(\alpha^2+\beta^2\right)f_2''+\alpha^2\beta^2 f_2\right\|_{L_p([a_2,b_2])}. \quad (40)$$

Proof. It is omitted. ■

Theorem 25 *All as in Theorem 23. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)||f_2(x_2)|\,dx_1dx_2}{\left[\frac{\int_{a_1}^{x_1}|\alpha(x_1-t_1)\cosh(\alpha(x_1-t_1))-\sinh(\alpha(x_1-t_1))|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}|\alpha(x_2-t_2)\cosh(\alpha(x_2-t_2))-\sinh(\alpha(x_2-t_2))|^q dt_2}{q}\right]}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{4\alpha^6}\left\|f_1^{(4)}-2\alpha^2 f_1''+\alpha^4 f_1\right\|_{L_q([a_1,b_1])}$$
$$\left\|f_2^{(4)}-2\alpha^2 f_2''+\alpha^4 f_2\right\|_{L_p([a_2,b_2])}. \quad (41)$$

Proof. It is omitted. ■

We finish with

Theorem 26 *Let $j=1,2$. Here $f_j \in C^4([a_j,b_j],K)$, where $K=\mathbb{R}$ or $\mathbb{C}$, and $f_j^{(i)}(a_j)=0$, $i=0,1,...,n-1$. All the rest are as in Remark 8 and $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)|\,|f_2(x_2)|\,dx_1dx_2}{\left[\frac{\int_{a_1}^{x_1}|\omega_c(x_1-t_1)|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}|\omega_c(x_2-t_2)|^q dt_2}{q}\right]}\leq$$

$$(b_1-a_1)(b_2-a_2)\,\|D_c(f_1)\|_{L_q([a_1,b_1])}\,\|D_c(f_2)\|_{L_p([a_2,b_2])}. \qquad (42)$$

Proof. It is omitted. ■

References

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