A GENERALIZATION OF BUZANO'S INEQUALITY IN TERMS OF TWO OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. If A, $B\in\mathcal{B}(H)$ such that

$$\nabla(B,A):=2\operatorname{Re}(B^{*}A)-|B|^{2}\geq 0,$$

then for x, $y\in H$,

$$\frac{1}{2}\left[\left\langle |A|^{2}x,x\right\rangle^{1/2}\left\langle |A|^{2}y,y\right\rangle^{1/2}+\left|\left\langle |A|^{2}x,y\right\rangle\right|\right]\geq|\langle\nabla(B,A)x,y\rangle|.$$

For $A=I$ and $B=P$, we get the recent generalization of Buzano inequlity for projections,

$$\frac{1}{2}\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right]\geq|\langle Px,y\rangle|.$$

Applications for operator norm and numerical radius inequalities are also provided.

1. Introduction

Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\|\,\|y\|\geq|\langle x,y\rangle|\text{ for any }x,y\in H.\tag{1.1}$$

The equality case holds in (1.1) if and only if there exists a constant $\lambda\in\mathbb{K}$ such that $x=\lambda y$.

In 1985 the author [3] (see also [5, p. 38]) established the following refinement of (1.1):

$$\|x\|\,\|y\|\geq|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|+|\langle x,e\rangle\langle e,y\rangle|\geq|\langle x,y\rangle|\tag{1.2}$$

for any x, y, $e\in H$ with $\|e\|=1$.

Using the triangle inequality for modulus we have by (1.2) that

$$\|x\|\,\|y\|\geq|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|+|\langle x,e\rangle\langle e,y\rangle|\geq 2|\langle x,e\rangle\langle e,y\rangle|-|\langle x,y\rangle|,$$

which implies the *Buzano inequality* [2]

$$\frac{1}{2}\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right]\geq|\langle x,e\rangle\langle e,y\rangle|\tag{1.3}$$

that holds for any x, y, $e\in H$ with $\|e\|=1$.

In [8] we obtained the following result that extends Buzano's inequality for projections:

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

Key words and phrases. Bounded linear operators, Numerical radius, Schwarz inequality, Buzano inequality.

Theorem 1. *Let $P : H \to H$ be an orthogonal projection on H. Then for any $x, y \in H$ we have the inequality*

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x, y\rangle|\right] \geq |\langle Px, y\rangle| \,. \tag{1.4}$$

For many inequalities related to Schwarz's inequality in inner product spaces, see [4] and [5].

Let $(H; \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [9, p. 1]:

$$W(T) = \{\langle Tx, x\rangle\,,\ x \in H,\ \|x\| = 1\}\,.$$

The *numerical radius* $\omega(T)$ of an operator T on H is defined by [9, p. 8]:

$$\omega(T) = \sup\{|\lambda|\,, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|\,, \|x\| = 1\}\,.$$

It is well known that $\omega(\cdot)$ is a norm on the Banach algebra $B(H)$ and the following inequality holds true

$$\omega(T) \leq \|T\| \leq 2\omega(T)\,, \text{ for any } T \in B(H)\,.$$

Utilizing Buzano's inequality (1.3) we obtained the following inequality for the numerical radius [6]:

Theorem 2. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a bounded linear operator on H. Then*

$$\omega^2(T) \leq \frac{1}{2}\left[\omega\left(T^2\right) + \|T\|^2\right]. \tag{1.5}$$

The constant $\frac{1}{2}$ is best possible in (1.5).

By utilizing Theorem 1 we also obtained in [8] the following result as well:

Theorem 3. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$\omega(BPA) \leq \frac{1}{2}\left[\|A\|\,\|B\| + \omega(BA)\right] \tag{1.6}$$

and

$$\|BPA\| \leq \frac{1}{2}\left[\|A\|\,\|B\| + \|BA\|\right]. \tag{1.7}$$

Also, we have:

Corollary 1. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$w(BPA) \leq \frac{1}{2}w(BA) + \frac{1}{4}\left\||A|^2 + |B^*|^2\right\|. \tag{1.8}$$

In particular, we have

$$w(APA) \leq \frac{1}{2}w\left(A^2\right) + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|. \tag{1.9}$$

For several inequalities related to the numerical radius, see [7] and the recent monograph [1].

Motivated by the above results, we provide in this paper a generalization of Buzano's inequality for two operators satisfying a certain condition that incorporates the case when one is a projection and then also extends the previous result

(1.4). Applications for operator norm and numerical radius inequalities are also provided.

2. Main Results

We start to the following result:

Theorem 4. *Let A, $B\in\mathcal{B}(H)$ such that $|B-A|^2\le|A|^2$ or, equivalently*

$$\nabla(B,A):=2\operatorname{Re}(B^*A)-|B|^2\ge 0.$$

Then for x, $y\in H$,

$$\left\langle|A|^2x,x\right\rangle^{1/2}\left\langle|A|^2y,y\right\rangle^{1/2}\ge|\langle\nabla(B,A)x,y\rangle|+\left|\left\langle|A|^2x,y\right\rangle-\langle\nabla(B,A)x,y\rangle\right|\ge\left|\left\langle|A|^2x,y\right\rangle\right|. \tag{2.1}$$

Also

$$\frac{1}{2}\left[\left\langle|A|^2x,x\right\rangle^{1/2}\left\langle|A|^2y,y\right\rangle^{1/2}+\left|\left\langle|A|^2x,y\right\rangle\right|\right]\ge|\langle\nabla(B,A)x,y\rangle| \tag{2.2}$$

for x, $y\in H$.

Proof. Observe that for $x\in H$,

$$\begin{aligned}\|Ax-Bx\|^2&=\langle Ax,Ax\rangle-\langle Ax,Bx\rangle-\langle Bx,Ax\rangle+\langle Bx,Bx\rangle\\&=\langle A^*Ax,x\rangle-\langle B^*Ax,x\rangle-\overline{\langle B^*Ax,x\rangle}+\langle B^*Bx,x\rangle\\&=\left\langle|A|^2x,x\right\rangle-2\operatorname{Re}\langle B^*Ax,x\rangle+\left\langle|B|^2x,x\right\rangle.\end{aligned}$$

Also

$$\begin{aligned}\langle 2\operatorname{Re}(B^*A)x,x\rangle&=\langle(B^*A+A^*B)x,x\rangle=\langle B^*Ax,x\rangle+\langle A^*Bx,x\rangle\\&=\langle B^*Ax,x\rangle+\langle x,B^*Ax\rangle=\langle B^*Ax,x\rangle+\overline{\langle B^*Ax,x\rangle}\\&=2\operatorname{Re}\langle B^*Ax,x\rangle,\end{aligned}$$

which gives that

$$\begin{aligned}\|Ax-Bx\|^2&=\left\langle|A|^2x,x\right\rangle-\langle 2\operatorname{Re}(B^*A)x,x\rangle+\left\langle|B|^2x,x\right\rangle\\&=\left\langle|A|^2x,x\right\rangle-\left\langle\left[2\operatorname{Re}(B^*A)-|B|^2\right]x,x\right\rangle\\&=\left\langle|A|^2x,x\right\rangle-\langle\nabla(B,A)x,x\rangle\ge 0\end{aligned}$$

for $x\in H$.

Similarly, for all $y\in H$,

$$\|Ay-By\|^2=\left\langle|A|^2y,y\right\rangle-\langle\nabla(B,A)y,y\rangle\ge 0.$$

Also, we have

$$\begin{aligned}\langle Ax-Bx,Ay-By\rangle &= \langle Ax,Ay\rangle-\langle Bx,Ay\rangle-\langle Ax,By\rangle+\langle Bx,By\rangle\\ &= \left\langle |A|^2 x,y\right\rangle-\langle A^*Bx,y\rangle-\langle B^*Ax,y\rangle+\left\langle |B|^2 x,y\right\rangle\\ &= \left\langle |A|^2 x,y\right\rangle-\left\langle \left[A^*B+B^*A-|B|^2\right]x,y\right\rangle\\ &= \left\langle |A|^2 x,y\right\rangle-\left\langle \left[2\operatorname{Re}(B^*A)-|B|^2\right]x,y\right\rangle\\ &= \left\langle |A|^2 x,y\right\rangle-\langle \nabla(B,A)x,y\rangle\end{aligned}$$

for all $x,y\in H$.

By Schwarz's inequality, we get

$$\|Ax-Bx\|^2\|Ay-By\|^2\geq|\langle Ax-Bx,Ay-By\rangle|^2,$$

which gives that

$$\begin{aligned}&\left(\left\langle |A|^2 x,x\right\rangle-\langle \nabla(B,A)x,x\rangle\right)\left(\left\langle |A|^2 y,y\right\rangle-\langle \nabla(B,A)y,y\rangle\right)\\ &\geq\left|\left\langle |A|^2 x,y\right\rangle-\langle \nabla(B,A)x,y\rangle\right|^2\end{aligned}\tag{2.3}$$

for all $x,y\in H$.

By the elementary inequality $(ac-bd)^2\geq\left(a^2-b^2\right)\left(c^2-d^2\right)$, which holds for any real numbers a, b, c, d,

$$\begin{aligned}&\left(\left\langle |A|^2 x,x\right\rangle^{1/2}\left\langle |A|^2 y,y\right\rangle^{1/2}-\langle \nabla(B,A)x,x\rangle^{1/2}\langle \nabla(B,A)y,y\rangle^{1/2}\right)^2\\ &\geq\left(\left\langle |A|^2 x,x\right\rangle-\langle \nabla(B,A)x,x\rangle\right)\left(\left\langle |A|^2 y,y\right\rangle-\langle \nabla(B,A)y,y\rangle\right)\end{aligned}\tag{2.4}$$

for all $x,y\in H$.

Since $\left\langle |A|^2 x,x\right\rangle^{1/2}\geq\langle \nabla(B,A)x,x\rangle^{1/2}$ and $\left\langle |A|^2 y,y\right\rangle^{1/2}\geq\langle \nabla(B,A)y,y\rangle^{1/2}$, then

$$\left\langle |A|^2 x,x\right\rangle^{1/2}\left\langle |A|^2 y,y\right\rangle^{1/2}-\langle \nabla(B,A)x,x\rangle^{1/2}\langle \nabla(B,A)y,y\rangle^{1/2}\geq 0$$

for all $x,y\in H$.

By (2.3) and (2.4) we get

$$\begin{aligned}&\left(\left\langle |A|^2 x,x\right\rangle^{1/2}\left\langle |A|^2 y,y\right\rangle^{1/2}-\langle \nabla(B,A)x,x\rangle^{1/2}\langle \nabla(B,A)y,y\rangle^{1/2}\right)^2\\ &\geq\left|\left\langle |A|^2 x,y\right\rangle-\langle \nabla(B,A)x,y\rangle\right|^2\end{aligned}$$

for all $x,y\in H$, and by taking the square root, we get

$$\begin{aligned}&\left\langle |A|^2 x,x\right\rangle^{1/2}\left\langle |A|^2 y,y\right\rangle^{1/2}-\langle \nabla(B,A)x,x\rangle^{1/2}\langle \nabla(B,A)y,y\rangle^{1/2}\\ &\geq\left|\left\langle |A|^2 x,y\right\rangle-\langle \nabla(B,A)x,y\rangle\right|,\end{aligned}$$

which gives that

$$\left\langle |A|^2 x, x\right\rangle^{1/2} \left\langle |A|^2 y, y\right\rangle^{1/2} \geq \langle \nabla (B, A) x, x\rangle^{1/2} \langle \nabla (B, A) y, y\rangle^{1/2} + \left| \left\langle |A|^2 x, y\right\rangle - \langle \nabla (B, A) x, y\rangle \right|, \tag{2.5}$$

for all $x, y \in H$.

By the Schwarz inequality for nonnegative operators we have

$$\langle \nabla (B, A) x, x\rangle^{1/2} \langle \nabla (B, A) y, y\rangle^{1/2} \geq |\langle \nabla (B, A) x, y\rangle| \tag{2.6}$$

for all $x, y \in H$.

By utilizing (2.5) and (2.6) we deduce the first inequality in (2.1). By the triangle inequality for the modulus we deduce the last part of (2.1).

Since, by the triangle inequality

$$\left| \left\langle |A|^2 x, y\right\rangle - \langle \nabla (B, A) x, y\rangle \right| \geq |\langle \nabla (B, A) x, y\rangle| - \left| \left\langle |A|^2 x, y\right\rangle \right|$$

then

$$|\langle \nabla (B, A) x, y\rangle| + \left| \left\langle |A|^2 x, y\right\rangle - \langle \nabla (B, A) x, y\rangle \right| \geq 2 |\langle \nabla (B, A) x, y\rangle| - \left| \left\langle |A|^2 x, y\right\rangle \right|,$$

which produces (2.2). □

Corollary 2. *Let $B \in \mathcal{B}(H)$ such that $|B - I|^2 \leq I$ or, equivalently*

$$\nabla (B) := 2 \operatorname{Re}(B^*) - |B|^2 \geq 0.$$

Then for x, $y \in H$,

$$\|x\| \|y\| \geq |\langle \nabla (B) x, y\rangle| + |\langle x, y\rangle - \langle \nabla (B) x, y\rangle| \geq |\langle x, y\rangle|. \tag{2.7}$$

Also

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y\rangle|] \geq |\langle \nabla (B) x, y\rangle| \tag{2.8}$$

for x, $y \in H$.

Remark 1. *If $B = P$, a projector, then*

$$\nabla (P) = 2P - P^2 = 2P - P = P \geq 0$$

and by (2.7) and (2.8) we derive the results obtained in [8]

$$\|x\| \|y\| \geq |\langle Px, y\rangle| + |\langle x, y\rangle - \langle Px, y\rangle| \geq |\langle x, y\rangle| \tag{2.9}$$

and

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y\rangle|] \geq |\langle Px, y\rangle| \tag{2.10}$$

for x, $y \in H$.

Corollary 3. *Let $A \in \mathcal{B}(H)$ such that $|A - I|^2 \leq |A|^2$ or, equivalently,*

$$\nabla (A) := 2 \operatorname{Re}(A) - I \geq 0.$$

Then for $x,\, y\in H$,

$$\left\langle |A|^2 x,x\right\rangle^{1/2}\left\langle |A|^2 y,y\right\rangle^{1/2} \geq |\langle \nabla(A)x,y\rangle| + \left|\left\langle |A|^2 x,y\right\rangle - \langle \nabla(A)x,y\rangle\right| \geq \left|\left\langle |A|^2 x,y\right\rangle\right|. \tag{2.11}$$

Also

$$\frac{1}{2}\left[\left\langle |A|^2 x,x\right\rangle^{1/2}\left\langle |A|^2 y,y\right\rangle^{1/2} + \left|\left\langle |A|^2 x,y\right\rangle\right|\right] \geq |\langle \nabla(A)x,y\rangle| \tag{2.12}$$

for $x,\, y\in H$.

Remark 2. *Let $C\in\mathcal{B}(H)$ such that $\operatorname{Re} C\geq 0$. By taking $A=\frac{1}{2}(I+C)$, we get that*

$$\nabla\left(\frac{1}{2}(I+C)\right) := 2\operatorname{Re}\left(\frac{1}{2}(I+C)\right) - I = \operatorname{Re} C \geq 0$$

and by (2.12) we get

$$\frac{1}{8}\left[\left\langle |I+C|^2 x,x\right\rangle^{1/2}\left\langle |I+C|^2 y,y\right\rangle^{1/2} + \left|\left\langle |I+C|^2 x,y\right\rangle\right|\right] \geq |\langle \operatorname{Re} Cx,y\rangle|$$

for $x,\, y\in H$.

This is equivalent to

$$\frac{1}{8}\left[\|x+Cx\|\,\|y+Cy\| + \left|\left\langle |I+C|^2 x,y\right\rangle\right|\right] \geq |\langle \operatorname{Re} Cx,y\rangle| \tag{2.13}$$

for $x,\, y\in H$.

If $Q\geq 0$, then by (2.13) we get

$$\frac{1}{8}\left[\|x+Qx\|\,\|y+Qy\| + \left|\left\langle (I+Q)^2 x,y\right\rangle\right|\right] \geq |\langle Qx,y\rangle| \tag{2.14}$$

for $x,\, y\in H$.

3. Applications for Numerical Radius

We have the following inequalities for the operator norm and numerical radius:

Theorem 5. *Let $A,\, B\in\mathcal{B}(H)$ such that $\nabla(B,A) := 2\operatorname{Re}(B^*A) - |B|^2 \geq 0$. Then for all $T, V\in\mathcal{B}(H)$*

$$\|V\nabla(B,A)T\| \leq \frac{1}{2}\left(\||A|T\|\,\|V|A|\| + \left\|V|A|^2 T\right\|\right) \tag{3.1}$$

and

$$\omega(V\nabla(B,A)T) \leq \frac{1}{2}\left(\||A|T\|\,\|V|A|\| + \omega\left(V|A|^2 T\right)\right). \tag{3.2}$$

Proof. From (2.2) we get, by replacing x with Tx and y with V^*y, that

$$\frac{1}{2}\left[\left\langle |A|^2 Tx,Tx\right\rangle^{1/2}\left\langle |A|^2 V^*y,V^*y\right\rangle^{1/2} + \left|\left\langle |A|^2 Tx,V^*y\right\rangle\right|\right] \geq |\langle \nabla(B,A)Tx,V^*y\rangle|,$$

which is equivalent to

$$\frac{1}{2}\left[\left\langle T^*|A|^2 Tx,x\right\rangle^{1/2}\left\langle V|A|^2 V^*y,y\right\rangle^{1/2} + \left|\left\langle V|A|^2 Tx,y\right\rangle\right|\right] \geq |\langle V\nabla(B,A)Tx,y\rangle|, \tag{3.3}$$

for x, $y \in H$.

Observe that $T^* |A|^2 T = ||A| T|^2$ and $V |A|^2 V^* = ||A| V^*|^2$ and by (3.3) we obtain

$$\begin{aligned} &|\langle V\nabla (B,A) Tx, y\rangle| \\ &\le \frac{1}{2}\left[\left\langle ||A| T|^2 x, x\right\rangle^{1/2} \left\langle ||A| V^*|^2 y, y\right\rangle^{1/2} + \left|\left\langle V |A|^2 Tx, y\right\rangle\right|\right] \end{aligned} \tag{3.4}$$

for x, $y \in H$.

Now, if we take the supremum over $\|x\| = \|y\| = 1$, then we get

$$\begin{aligned} &\|V\nabla (B,A) T\| \\ &= \sup_{\|x\|=\|y\|=1} |\langle V\nabla (B,A) Tx, y\rangle| \\ &\le \frac{1}{2} \sup_{\|x\|=\|y\|=1} \left[\left\langle ||A| T|^2 x, x\right\rangle^{1/2} \left\langle ||A| V^*|^2 y, y\right\rangle^{1/2} + \left|\left\langle V |A|^2 Tx, y\right\rangle\right|\right] \\ &\le \frac{1}{2} \sup_{\|x\|=\|y\|=1} \left[\left\langle ||A| T|^2 x, x\right\rangle^{1/2} \left\langle ||A| V^*|^2 y, y\right\rangle^{1/2}\right] \\ &+ \frac{1}{2} \sup_{\|x\|=\|y\|=1} \left|\left\langle V |A|^2 Tx, y\right\rangle\right| \\ &= \frac{1}{2} \left\| ||A| T|^2 \right\|^{1/2} \left\| ||A| V^*|^2 \right\|^{1/2} + \frac{1}{2} \left\| V |A|^2 T \right\| \\ &= \frac{1}{2} \left(\||A| T\| \|V |A|\| + \left\| V |A|^2 T \right\| \right), \end{aligned}$$

which proves (3.1).

From (3.4) we also have

$$\begin{aligned} &|\langle V\nabla (B,A) Tx, x\rangle| \\ &\le \frac{1}{2}\left[\left\langle ||A| T|^2 x, x\right\rangle^{1/2} \left\langle ||A| V^*|^2 x, x\right\rangle^{1/2} + \left|\left\langle V |A|^2 Tx, x\right\rangle\right|\right], \end{aligned} \tag{3.5}$$

for $x \in H$.

If we take the supremum over $\|x\| = 1$, then we get

$$\begin{aligned} &\omega (V\nabla (B,A) T) \\ &= \sup_{\|x\|=1} |\langle V\nabla (B,A) Tx, x\rangle| \\ &\le \frac{1}{2} \sup_{\|x\|=1} \left[\left\langle ||A| T|^2 x, x\right\rangle^{1/2} \left\langle ||A| V^*|^2 x, x\right\rangle^{1/2} + \left|\left\langle V |A|^2 Tx, x\right\rangle\right|\right] \\ &\le \frac{1}{2} \sup_{\|x\|=1} \left\langle ||A| T|^2 x, x\right\rangle^{1/2} \sup_{\|x\|=1} \left\langle ||A| V^*|^2 x, x\right\rangle^{1/2} \\ &+ \frac{1}{2} \sup_{\|x\|=1} \left|\left\langle V |A|^2 Tx, x\right\rangle\right| \\ &= \frac{1}{2} \left[\left\| ||A| T|^2 \right\|^{1/2} \left\| ||A| V^*|^2 \right\|^{1/2} + \omega\left(V |A|^2 T\right) \right] \\ &= \frac{1}{2} \left(\||A| T\| \|V |A|\| + \omega\left(V |A|^2 T\right) \right), \end{aligned}$$

which proves (3.2). □

Corollary 4. *Let $B \in \mathcal{B}(H)$ such that $\nabla(B) := 2\operatorname{Re}(B^*) - |B|^2 \geq 0$. Then for all $T, V \in \mathcal{B}(H)$*

$$\|V\nabla(B)T\| \leq \frac{1}{2}(\|T\|\|V\| + \|VT\|)$$

and

$$\omega(V\nabla(B)T) \leq \frac{1}{2}(\|T\|\|V\| + \omega(VT)).$$

Remark 3. *If we take $B = P$, a projection in H, in Corollary 4, then we recapture the results from Theorem 3.*

Corollary 5. *Let $A \in \mathcal{B}(H)$ such that $\nabla(A) := 2\operatorname{Re}(A) - I \geq 0$. Then for all $T, V \in \mathcal{B}(H)$*

$$\|V\nabla(A)T\| \leq \frac{1}{2}\left(\||A|T\|\|V|A|\| + \left\|V|A|^2 T\right\|\right)$$

and

$$\omega(V\nabla(A)T) \leq \frac{1}{2}\left(\||A|T\|\|V|A|\| + \omega\left(V|A|^2 T\right)\right).$$

Remark 4. *Let $C \in \mathcal{B}(H)$ such that $\operatorname{Re} C \geq 0$. Then for all $T, V \in \mathcal{B}(H)$*

$$\|V(\operatorname{Re} C)T\| \leq \frac{1}{8}\left(\||I+C|T\|\|V|I+C|\| + \left\|V|I+C|^2 T\right\|\right)$$

and

$$\omega(V(\operatorname{Re} C)T) \leq \frac{1}{8}\left(\||I+C|T\|\|V|I+C|\| + \omega\left(V|I+C|^2 T\right)\right).$$

If $Q \geq 0$, then by taking $C = Q$, we get

$$\|VQT\| \leq \frac{1}{8}\left(\|(I+Q)T\|\|V(I+Q)\| + \left\|V(I+Q)^2 T\right\|\right)$$

and

$$\omega(VQT) \leq \frac{1}{8}\left(\|(I+Q)T\|\|V(I+Q)\| + \omega\left(V(I+Q)^2 T\right)\right).$$

Also we have:

Theorem 6. *Let $A, B \in \mathcal{B}(H)$ such that $\nabla(B, A) := 2\operatorname{Re}(B^*A) - |B|^2 \geq 0$. Then for all $T, V \in \mathcal{B}(H)$*

$$\omega(V\nabla(B,A)T) \leq \frac{1}{2}\left[\left\|\frac{||A|T|^2 + ||A|V^*|^2}{2}\right\| + \omega\left(V|A|^2 T\right)\right]. \tag{3.6}$$

Proof. Observe that

$$\begin{aligned}\left\langle ||A|T|^2 x, x\right\rangle^{1/2}\left\langle ||A|V^*|^2 x, x\right\rangle^{1/2} &\leq \frac{1}{2}\left(\left\langle ||A|T|^2 x, x\right\rangle + \left\langle ||A|V^*|^2 x, x\right\rangle\right)\\ &= \left\langle \frac{||A|T|^2 + ||A|V^*|^2}{2}x, x\right\rangle\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

By using (3.5) we get

$$|\langle V\nabla(B,A)Tx, x\rangle| \leq \frac{1}{2}\left[\left\langle \frac{||A|T|^2 + ||A|V^*|^2}{2}x, x\right\rangle + \left|\left\langle V|A|^2 Tx, x\right\rangle\right|\right],$$

for $x \in H$, $\|x\| = 1$.

If we take the supremum over $\|x\| = 1$, then we get

$$\begin{aligned}
\omega\left(V\nabla\left(B,A\right)T\right) &= \sup_{\|x\|=1}\left|\left\langle V\nabla\left(B,A\right)Tx,x\right\rangle\right| \\
&\leq \frac{1}{2}\sup_{\|x\|=1}\left[\left\langle \frac{\left|\left|A\right|T\right|^{2}+\left|\left|A\right|V^{*}\right|^{2}}{2}x,x\right\rangle + \left|\left\langle V\left|A\right|^{2}Tx,x\right\rangle\right|\right] \\
&\leq \frac{1}{2}\left[\sup_{\|x\|=1}\left\langle \frac{\left|\left|A\right|T\right|^{2}+\left|\left|A\right|V^{*}\right|^{2}}{2}x,x\right\rangle + \sup_{\|x\|=1}\left|\left\langle V\left|A\right|^{2}Tx,x\right\rangle\right|\right] \\
&= \frac{1}{2}\left[\left\|\frac{\left|\left|A\right|T\right|^{2}+\left|\left|A\right|V^{*}\right|^{2}}{2}\right\| + \omega\left(V\left|A\right|^{2}T\right)\right],
\end{aligned}$$

which proves (3.6). □

Corollary 6. *Let* $B \in \mathcal{B}(H)$ *such that* $\nabla(B) := 2\operatorname{Re}(B^{*}) - |B|^{2} \geq 0$. *Then for all* $T, V \in \mathcal{B}(H)$,

$$\omega\left(V\nabla\left(B\right)T\right) \leq \frac{1}{2}\left[\left\|\frac{|T|^{2}+|V^{*}|^{2}}{2}\right\| + \omega\left(VT\right)\right].$$

Remark 5. *If we take* $B = P$, *a projection in* H, *in Corollary 6, then we recapture a result from Corollary 1,* [8]

$$\omega\left(VPT\right) \leq \frac{1}{2}\left[\left\|\frac{|T|^{2}+|V^{*}|^{2}}{2}\right\| + \omega\left(VT\right)\right].$$

Corollary 7. *Let* $A \in \mathcal{B}(H)$ *such that* $\nabla(A) := 2\operatorname{Re}(A) - I \geq 0$. *Then for all* $T, V \in \mathcal{B}(H)$,

$$\omega\left(V\nabla\left(A\right)T\right) \leq \frac{1}{2}\left[\left\|\frac{\left|\left|A\right|T\right|^{2}+\left|\left|A\right|V^{*}\right|^{2}}{2}\right\| + \omega\left(V\left|A\right|^{2}T\right)\right].$$

Let $C \in \mathcal{B}(H)$ such that $\operatorname{Re} C \geq 0$. Then for all $T, V \in \mathcal{B}(H)$

$$\omega\left(V\left(\operatorname{Re} C\right)T\right) \leq \frac{1}{4}\left[\left\|\frac{\left|\left|I+C\right|T\right|^{2}+\left|\left|I+C\right|V^{*}\right|^{2}}{2}\right\| + \frac{1}{2}\omega\left(V\left|I+C\right|^{2}T\right)\right].$$

If $Q \geq 0$, then by taking $C = Q$, we get

$$\omega\left(VQT\right) \leq \frac{1}{4}\left[\left\|\frac{\left|\left(I+Q\right)T\right|^{2}+\left|\left(I+Q\right)V^{*}\right|^{2}}{2}\right\| + \frac{1}{2}\omega\left(V\left(I+Q\right)^{2}T\right)\right].$$

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