POWER INEQUALITIES FOR THE NUMERICAL RADIUS IN TERMS OF GENERALIZED ALUTHGE TRANSFORM OF OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a complex Hilbert space. In this paper we show among others that, if S, $V\in\mathcal{B}(H)$, $r\geq 1$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\geq 2$, then

$$\omega^{2r}(SV)\leq\frac{1}{2}\left(\left\|\frac{1}{p}|S^*|^{2pr}+\frac{1}{q}|V|^{2qr}\right\|+\omega^r\left(|V|^2|S^*|^2\right)\right).$$

Moreover, if $T=U|T|$ is the *polar decomposition* of the bounded linear operator T with U a partial isometry, then

$$\omega^{2r}(\Delta_t(T))\leq\frac{1}{2}\left(\left\|\frac{1}{p}|T|^{2tpr}+\frac{1}{q}|T|^{2(1-t)qr}\right\|+\|T\|^{2r}\right)$$

where $\Delta_t(T):=|T|^t U|T|^{1-t}$, $t\in[0,1]$ is the *generalized Aluthge transform.*

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T)=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x\in H$ one has

$$|\langle Tx,x\rangle|\leq w(T)\|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T:H\to H$, i.e.,

(i) $\omega(T)\geq 0$ for any $T\in B(H)$ and $\omega(T)=0$ if and only if $T=0$;
(ii) $\omega(\lambda T)=|\lambda|\,\omega(T)$ for any $\lambda\in\mathbb{C}$ and $T\in B(H)$;
(iii) $\omega(T+V)\leq\omega(T)+\omega(V)$ for any $T,V\in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T)\leq\|T\|\leq 2\omega(T) \tag{1.3}$$

for any $T\in B(H)$.

F. Kittaneh, in 2003 [13], showed that for any operator $T\in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T)\leq\frac{1}{2}\left(\|T\|+\left\|T^2\right\|^{1/2}\right). \tag{1.4}$$

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

Key words and phrases. Bounded operators, Aluthge transform, Dougal transform, Partial isometry, Numerical radius.

RGMIA Res. Rep. Coll. **26** (2023), Art. 26, 14 pp. Received 19/04/23

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [14] improved the inequality (1.3) as follows:

$$\frac{1}{4}\left\| T^{*}T+TT^{*}\right\| \leq \omega^{2}\left(T\right) \leq \frac{1}{2}\left\| T^{*}T+TT^{*}\right\| \tag{1.5}$$

for any operator $T\in B\left(H\right)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [11]:

If for an operator $T\in B\left(H\right)$ we denote $\left\vert T\right\vert :=\left(T^{*}T\right)^{1/2}$, then

$$\omega^{r}\left(T\right) \leq \frac{1}{2}\left\| \left\vert T\right\vert^{2\alpha r}+\left\vert T^{*}\right\vert^{2\left(1-\alpha\right) r}\right\| \tag{1.6}$$

and

$$\omega^{2r}\left(T\right) \leq \left\| \alpha\left\vert T\right\vert^{2r}+\left(1-\alpha\right) \left\vert T^{*}\right\vert^{2r}\right\| , \tag{1.7}$$

where $\alpha\in\left(0,1\right)$ and $r\geq1.$

If we take $\alpha=\frac{1}{2}$ and $r=1$ we get from (1.6) that

$$\omega\left(T\right) \leq \frac{1}{2}\left\| \left\vert T\right\vert +\left\vert T^{*}\right\vert \right\| \tag{1.8}$$

and from (1.7) that

$$\omega^{2}\left(T\right) \leq \frac{1}{2}\left\| \left\vert T\right\vert^{2}+\left\vert T^{*}\right\vert^{2}\right\| . \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [10] and [5].

Let $T=U\left\vert T\right\vert$ be the *polar decomposition* of the bounded linear operator T with U a partial isometry. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T}:=\left\vert T\right\vert^{1/2}U\left\vert T\right\vert^{1/2}$, see [1].

The following properties of $\widetilde{T}$ are as follows:

(i) $\left\| \widetilde{T}\right\| \leq\left\| T\right\| ,$

(ii) $\omega\left(\widetilde{T}\right) \leq\omega\left(T\right) ,$

(iii) $r\left(\widetilde{T}\right) =\omega\left(T\right) ,$

(iv) $\omega\left(\widetilde{T}\right) \leq\left\| T^{2}\right\|^{1/2}\left(\leq\left\| T\right\| \right)$, [15].

Utilizing this transform T. Yamazaki, [15] obtained in 2007 the following refinement of Kittaneh's inequality (1.4):

$$\omega\left(T\right) \leq\frac{1}{2}\left(\left\| T\right\| +\omega\left(\widetilde{T}\right) \right) \leq\frac{1}{2}\left(\left\| T\right\| +\left\| T^{2}\right\|^{1/2}\right) \tag{1.10}$$

for any operator $T\in B\left(H\right)$.

We remark that if $\widetilde{T}=0$, then obviously $w\left(T\right) =\frac{1}{2}\left\| T\right\|$.

For $t\in\left(0,1\right)$

$$\Delta_{t}\left(T\right) :=\left\vert T\right\vert^{t}U\left\vert T\right\vert^{1-t}$$

is the *generalized Aluthge transform* introduced in by Cho and Tanahashi in [9].

Abu-Omar and Kittaneh [2] improved on inequality (1.10) using generalized Aluthge transform to prove that

$$\omega\left(T\right) \leq\frac{1}{2}\left(\left\| T\right\| +\min_{t\in\left[0,1\right] }\omega\left(\Delta_{t}\left(T\right) \right) \right) . \tag{1.11}$$

For $t=1$ this also gives the following result for the *Dougal transform*

$$\omega(T)\le\frac{1}{2}\left(\|T\|+\omega\left(\widehat{T}\right)\right). \tag{1.12}$$

Also, if we put $|T|^0=I$, then $\Delta_0(T):=U|T|=T$.

In [4] Bunia et al. also proved that

$$\omega(T)\le\min_{t\in[0,1]}\left\{\frac{1}{2}\omega(\Delta_t(T))+\frac{1}{4}\left(\|T\|^{2t}+\|T\|^{2(1-t)}\right)\right\}, \tag{1.13}$$

which for $t=1/2$ gives (1.10) as well.

Motivated by the above results, in this paper we show among others that, if S, $V\in\mathcal{B}(H)$, $r\ge1$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\ge2$, then

$$\omega^{2r}(SV)\le\frac{1}{2}\left(\left\|\frac{1}{p}|S^*|^{2pr}+\frac{1}{q}|V|^{2qr}\right\|+\omega^r\left(|V|^2|S^*|^2\right)\right).$$

Moreover, we have

$$\omega^{2r}(\Delta_t(T))\le\frac{1}{2}\left(\left\|\frac{1}{p}|T|^{2tpr}+\frac{1}{q}|T|^{2(1-t)qr}\right\|+\|T\|^{2r}\right)$$

for $t\in[0,1]$

2. Main Results

We recall the following vector inequality for positive operators $A\ge0$, obtained by C. A. McCarthy in [8]

$$\langle Ax,x\rangle^p\le\langle A^px,x\rangle,\ p\ge1$$

for $x\in H$, $\|x\|=1$ and Buzano's inequality [7],

$$\frac{1}{2}\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right]\ge|\langle x,e\rangle\langle e,y\rangle| \tag{2.1}$$

that holds for any x, y, $e\in H$ with $\|e\|=1$.

Also recall the following result for operator matrices obtained by F. Kittaneh in [12]:

Lemma 1. *Let A, B, $C\in\mathcal{B}(H)$ with A, $B\ge0$. Then the operator matrix*

$$\begin{bmatrix}A & C^*\\ C & B\end{bmatrix}\in\mathcal{B}(H\oplus H)$$

is positive, if and only if

$$|\langle Cx,y\rangle|^2\le\langle Ax,x\rangle\langle By,y\rangle$$

for all x, $y\in H$.

We need the following results that are of interest in themselves:

Lemma 2. *Let A, B, $C\in\mathcal{B}(H)$ with A, $B\ge0$. If the operator matrix*

$$\begin{bmatrix}A & C^*\\ C & B\end{bmatrix}\in B(H\oplus H)$$

is positive then for $r>0$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\ge1$,

$$\omega^{2r}(C)\le\left\|\frac{1}{p}A^{pr}+\frac{1}{q}B^{qr}\right\|. \tag{2.2}$$

If $r \geq 1$, *then*

$$\omega^{2r}(C) \leq \frac{1}{2}\left[\|A\|^{r}\|B\|^{r}+\omega^{r}(BA)\right]. \tag{2.3}$$

If $r \geq 1$, p, $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr \geq 2$, *then also*

$$\omega^{2r}(C) \leq \frac{1}{2}\left(\left\|\frac{1}{p}A^{pr}+\frac{1}{q}B^{qr}\right\|+\omega^{r}(BA)\right). \tag{2.4}$$

Proof. Let $x \in H$ with $\|x\|=1$. Then by Lemma 1 we get

$$|\langle Cx, x\rangle|^{2} \leq \langle Ax, x\rangle\langle Bx, x\rangle.$$

If we take the power $r > 0$, we get, by Young and McCarthy inequalities that

$$\begin{aligned}|\langle Cx, x\rangle|^{2r} &\leq \langle Ax, x\rangle^{r}\langle Bx, x\rangle^{r} \leq \frac{1}{p}\langle Ax, x\rangle^{pr}+\frac{1}{q}\langle Bx, x\rangle^{qr}\\ &\leq \frac{1}{p}\langle A^{pr}x, x\rangle+\frac{1}{q}\langle B^{qr}x, x\rangle=\left\langle\left(\frac{1}{p}A^{pr}+\frac{1}{q}B^{qr}\right)x, x\right\rangle\end{aligned}$$

for $x \in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, then we get that

$$\begin{aligned}\omega^{2r}(C) &= \sup_{\|x\|=1}|\langle Cx, x\rangle|^{2r} \leq \sup_{\|x\|=1}\left\langle\left(\frac{1}{p}A^{pr}+\frac{1}{q}B^{qr}\right)x, x\right\rangle\\ &= \left\|\frac{1}{p}A^{pr}+\frac{1}{q}B^{qr}\right\|,\end{aligned}$$

which proves (2.2).

Further, using Buzano's inequality, we have

$$|\langle Cx, x\rangle|^{2} \leq \langle Ax, x\rangle\langle Bx, x\rangle \leq \frac{\|Ax\|\|Bx\|+|\langle Ax, Bx\rangle|}{2}.$$

By taking the power $r \geq 1$ and using the convexity of the power function, we get

$$\begin{aligned}|\langle Cx, x\rangle|^{2r} &\leq \left(\frac{\|Ax\|\|Bx\|+|\langle Ax, Bx\rangle|}{2}\right)^{r}\\ &\leq \frac{\|Ax\|^{r}\|Bx\|^{r}+|\langle Ax, Bx\rangle|^{r}}{2}=\frac{\|Ax\|^{r}\|Bx\|^{r}+|\langle BAx, x\rangle|^{r}}{2}\end{aligned} \tag{2.5}$$

for $x \in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, then we get that

$$\begin{aligned}\omega^{2r}(C) &= \sup_{\|x\|=1}|\langle Cx, x\rangle|^{2r} \leq \sup_{\|x\|=1}\left(\frac{\|Ax\|^{r}\|Bx\|^{r}+|\langle BAx, x\rangle|^{r}}{2}\right)\\ &\leq \frac{1}{2}\left(\sup_{\|x\|=1}\{\|Ax\|^{r}\|Bx\|^{r}\}+\sup_{\|x\|=1}|\langle BAx, x\rangle|^{r}\right)\\ &\leq \frac{1}{2}\left(\sup_{\|x\|=1}\|Ax\|^{r}\sup_{\|x\|=1}\|Bx\|^{r}+\sup_{\|x\|=1}|\langle BAx, x\rangle|^{r}\right)\\ &= \frac{1}{2}\left(\|A\|^{r}\|B\|^{r}+\omega^{r}(BA)\right)\end{aligned}$$

and the inequality (2.3) is proved.

From (2.5) we also have

$$\begin{aligned}
|\langle Cx,x\rangle|^{2r} &\le \frac{1}{2}\left(\|Ax\|^{r}\|Bx\|^{r}+|\langle Ax,Bx\rangle|^{r}\right)\\
&\le \frac{1}{2}\left(\frac{1}{p}\|Ax\|^{pr}+\frac{1}{q}\|Bx\|^{qr}+|\langle Ax,Bx\rangle|^{r}\right)\\
&= \frac{1}{2}\left(\frac{1}{p}\|Ax\|^{2\frac{pr}{2}}+\frac{1}{q}\|Bx\|^{2\frac{qr}{2}}+|\langle Ax,Bx\rangle|^{r}\right)\\
&= \frac{1}{2}\left(\frac{1}{p}\left\langle A^{2}x,x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle B^{2}x,x\right\rangle^{\frac{qr}{2}}+|\langle Ax,Bx\rangle|^{r}\right)\\
&\le \frac{1}{2}\left(\frac{1}{p}\langle A^{pr}x,x\rangle+\frac{1}{q}\langle B^{qr}x,x\rangle+|\langle Ax,Bx\rangle|^{r}\right)\\
&= \frac{1}{2}\left(\left\langle\left(\frac{1}{p}A^{pr}+\frac{1}{q}B^{qr}\right)x,x\right\rangle+|\langle Ax,Bx\rangle|^{r}\right)
\end{aligned}$$

and by taking the supremum over $\|x\|=1$, we derive (2.4). □

Remark 1. *With the assumptions in Lemma 2, if we take $p=q=2$ and assume that $r\ge\frac{1}{2}$, then from (2.2) we get*

$$\omega^{2r}(C)\le\frac{1}{2}\left\|A^{2r}+B^{2r}\right\|,$$

which for $r=\frac{1}{2}$ gives

$$\omega(C)\le\frac{1}{2}\|A+B\|,$$

while for $r=1$ gives the result from [3]

$$\omega^{2}(C)\le\frac{1}{2}\left\|A^{2}+B^{2}\right\|.$$

If we take in (2.2) $r=1$, then we get

$$\omega^{2}(C)\le\left\|\frac{1}{p}A^{p}+\frac{1}{q}B^{q}\right\|$$

for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

If we take $r=1$ in (2.3) then we get the result from [3]

$$\omega^{2}(C)\le\frac{1}{2}\left[\|A\|\|B\|+\omega(BA)\right],$$

while for $r=2$,

$$\omega^{4}(C)\le\frac{1}{2}\left[\|A\|^{2}\|B\|^{2}+\omega^{2}(BA)\right].$$

Also, if we take $p=q=2$ and $r\ge1$ in (2.4), then we get

$$\omega^{2r}(C)\le\frac{1}{2}\left(\frac{1}{2}\left\|A^{2r}+B^{2r}\right\|+\omega^{r}(BA)\right).$$

In particular, for $r=1$ we derive the result obtained in [3]

$$\omega^{2}(C)\le\frac{1}{2}\left(\frac{1}{2}\left\|A^{2}+B^{2}\right\|+\omega(BA)\right).$$

Moreover, if $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and take* $r=2$ *in (2.4), then we derive the inequality*

$$\omega^{4}(C)\leq\frac{1}{2}\left(\left\|\frac{1}{p}A^{2p}+\frac{1}{q}B^{2q}\right\|+\omega^{2}(BA)\right),$$

which for $p=q=2$ *provides*

$$\omega^{4}(C)\leq\frac{1}{2}\left(\frac{1}{2}\left\|A^{4}+B^{4}\right\|+\omega^{2}(BA)\right).$$

We also have:

Theorem 1. *Let* $S,\ V\in\mathcal{B}(H)$, *then for* $r>0,\ p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr\geq1$,

$$\omega^{2r}(SV)\leq\left\|\frac{1}{p}|S^{*}|^{2pr}+\frac{1}{q}|V|^{2qr}\right\|. \tag{2.6}$$

If $r\geq1$, *then*

$$\omega^{2r}(SV)\leq\frac{1}{2}\left[\|S\|^{2r}\|V\|^{2r}+\omega^{r}\left(|V|^{2}|S^{*}|^{2}\right)\right]. \tag{2.7}$$

If $r\geq1,\ p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $pr,\ qr\geq2$, *then also*

$$\omega^{2r}(SV)\leq\frac{1}{2}\left(\left\|\frac{1}{p}|S^{*}|^{2pr}+\frac{1}{q}|V|^{2qr}\right\|+\omega^{r}\left(|V|^{2}|S^{*}|^{2}\right)\right). \tag{2.8}$$

Proof. Observe that the operator matrix

$$\begin{bmatrix} SS^{*} & SV \\ V^{*}S^{*} & V^{*}V \end{bmatrix}\in\mathcal{B}(H\oplus H)$$

is positive. Then by Lemma 2 for $A=|S^{*}|^{2}$, $B=|V|^{2}$ and $C=V^{*}S^{*}$ we get the desired inequalities (2.6)-(2.8). □

Remark 2. *With the assumptions of Theorem 1, if we take* $p=q=2$ *and assume that* $r\geq\frac{1}{2}$, *then from (2.6) we get*

$$\omega^{2r}(SV)\leq\frac{1}{2}\left\||S^{*}|^{4r}+|V|^{4r}\right\|.$$

If we take $r=\frac{1}{2}$, *then we obtain the known result*

$$\omega(SV)\leq\frac{1}{2}\left\||S^{*}|^{2}+|V|^{2}\right\|,$$

while for $r=1$, *the result from* [3]

$$\omega^{2}(SV)\leq\frac{1}{2}\left\||S^{*}|^{4}+|V|^{4}\right\|.$$

If we take in (2.6) $r=1$, *then we get*

$$\omega^{2}(SV)\leq\left\|\frac{1}{p}|S^{*}|^{2p}+\frac{1}{q}|V|^{2q}\right\|,$$

where $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

If we take $r=1$ *in (2.7), then we get, see also* [3]

$$\omega^{2}(SV)\leq\frac{1}{2}\left[\|S\|^{2}\|V\|^{2}+\omega\left(|V|^{2}|S^{*}|^{2}\right)\right],$$

while for $r = 2$,

$$\omega^4(SV) \le \frac{1}{2}\left[\|S\|^4\|V\|^4 + \omega^2\left(|V|^2|S^*|^2\right)\right].$$

Also, if we take $p = q = 2$ *and* $r \ge 1$ *in (2.8), then we get*

$$\omega^{2r}(SV) \le \frac{1}{2}\left(\frac{1}{2}\left\||S^*|^{4r} + |V|^{4r}\right\| + \omega^r\left(|V|^2|S^*|^2\right)\right).$$

In particular, for $r = 1$ *we derive the result obtained in* [3]

$$\omega^2(SV) \le \frac{1}{2}\left(\frac{1}{2}\left\||S^*|^4 + |V|^4\right\| + \omega\left(|V|^2|S^*|^2\right)\right).$$

Moreover, if $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and take* $r = 2$ *in (2.8), then we derive the inequality*

$$\omega^4(SV) \le \frac{1}{2}\left(\left\|\frac{1}{p}|S^*|^{4p} + \frac{1}{q}|V|^{4q}\right\| + \omega^2\left(|V|^2|S^*|^2\right)\right),$$

which for $p = q = 2$ *provides*

$$\omega^4(SV) \le \frac{1}{2}\left(\frac{1}{2}\left\||S^*|^8 + \frac{1}{q}|V|^8\right\| + \omega^2\left(|V|^2|S^*|^2\right)\right).$$

We also have:

Lemma 3. *Let* $A,\ B,\ C \in \mathcal{B}(H)$ *with* $A,\ B \ge 0$. *If the operator matrix*

$$\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \in B(H \oplus H)$$

is positive then for $\alpha \in [0,1]$ *and* $r \ge 1$,

$$\omega^2(C) \le \|(1-\alpha)A^r + \alpha B^r\|^{1/r}\|A\|^{\alpha}\|B\|^{1-\alpha} \tag{2.9}$$

and

$$\omega^2(C) \le \|(1-\alpha)A^r + \alpha B^r\|^{1/r}\|\alpha A^r + (1-\alpha)B^r\|^{1/r}. \tag{2.10}$$

Proof. From Lemma 1 we have for $\alpha \in [0,1]$ that

$$\begin{aligned}|\langle Cx,x\rangle|^2 &\le \langle Ax,x\rangle\langle Bx,x\rangle = \langle Ax,x\rangle^{1-\alpha}\langle Bx,x\rangle^{\alpha}\langle Ax,x\rangle^{\alpha}\langle Bx,x\rangle^{1-\alpha} \\ &\le [(1-\alpha)\langle Ax,x\rangle + \alpha\langle Bx,x\rangle]\langle Ax,x\rangle^{\alpha}\langle Bx,x\rangle^{1-\alpha}\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

If we take the power $r \ge 1$, then we get by the convexity of power r

$$\begin{aligned}|\langle Cx,x\rangle|^{2r} &\le [(1-\alpha)\langle Ax,x\rangle + \alpha\langle Bx,x\rangle]^r\langle Ax,x\rangle^{r\alpha}\langle Bx,x\rangle^{r(1-\alpha)} \\ &\le [(1-\alpha)\langle Ax,x\rangle^r + \alpha\langle Bx,x\rangle^r]\langle Ax,x\rangle^{r\alpha}\langle Bx,x\rangle^{r(1-\alpha)}.\end{aligned} \tag{2.11}$$

If we use McCarthy inequality for power $r \ge 1$, then we get

$$\begin{aligned}(1-\alpha)\langle Ax,x\rangle^r + \alpha\langle Bx,x\rangle^r &\le (1-\alpha)\langle A^r x,x\rangle + \alpha\langle B^r x,x\rangle \\ &= \langle[(1-\alpha)A^r + \alpha B^r]x,x\rangle\end{aligned}$$

and by (2.11), we obtain

$$\begin{aligned}|\langle Cx,x\rangle|^{2r} &\le [(1-\alpha)\langle Ax,x\rangle + \alpha\langle Bx,x\rangle]^r\langle Ax,x\rangle^{r\alpha}\langle Bx,x\rangle^{r(1-\alpha)} \\ &\le \langle[(1-\alpha)A^r + \alpha B^r]x,x\rangle\langle Ax,x\rangle^{r\alpha}\langle Bx,x\rangle^{r(1-\alpha)}\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

If we take the supremum over $\|x\| = 1$, then we get

$$\begin{aligned}\omega^{2r}(C) &= \sup_{\|x\|=1} |\langle Cx, x\rangle|^{2r} \\ &\leq \sup_{\|x\|=1} \left\{ \langle [(1-\alpha) A^r + \alpha B^r] x, x\rangle \langle Ax, x\rangle^{r\alpha} \langle Bx, x\rangle^{r(1-\alpha)} \right\} \\ &\leq \sup_{\|x\|=1} \{\langle [(1-\alpha) A^r + \alpha B^r] x, x\rangle\} \sup_{\|x\|=1} \langle BAx, x\rangle^{r\alpha} \sup_{\|x\|=1} \langle Bx, x\rangle^{r(1-\alpha)} \\ &= \|(1-\alpha) A^r + \alpha B^r\| \, \|A\|^{r\alpha} \, \|B\|^{r(1-\alpha)},\end{aligned}$$

which is equivalent to (2.9).

Similarly

$$\begin{aligned}|\langle Cx, x\rangle|^2 &\leq \langle Ax, x\rangle \langle Bx, x\rangle = \langle Ax, x\rangle^{1-\alpha} \langle Bx, x\rangle^{\alpha} \langle Ax, x\rangle^{\alpha} \langle Bx, x\rangle^{1-\alpha} \\ &\leq [(1-\alpha)\langle Ax, x\rangle + \alpha \langle Bx, x\rangle] [\alpha \langle Ax, x\rangle + (1-\alpha)\langle Bx, x\rangle],\end{aligned}$$

which gives that

$$|\langle Cx, x\rangle|^{2r} \leq \langle [(1-\alpha) A^r + \alpha B^r] x, x\rangle \langle [\alpha A^r + (1-\alpha) B^r] x, x\rangle$$

for all $x \in H$, $\|x\| = 1$. This proves (2.10). □

Corollary 1. *With the assumption of Lemma 3, we have*

$$\omega^2(C) \leq \frac{1}{2^{1/r}} \|A^r + B^r\|^{1/r} \|A\|^{1/2} \|B\|^{1/2} \tag{2.12}$$

and

$$\omega(C) \leq \frac{1}{2^{1/r}} \|A^r + B^r\|^{1/r}. \tag{2.13}$$

We have:

Theorem 2. *Let S, $V \in \mathcal{B}(H)$, then for $\alpha \in [0,1]$ and $r \geq 1$,*

$$\omega^2(SV) \leq \left\|(1-\alpha)|S^*|^{2r} + \alpha |V|^{2r}\right\|^{1/r} \|S\|^{2\alpha} \|V\|^{2(1-\alpha)} \tag{2.14}$$

and

$$\omega^2(SV) \leq \left\|(1-\alpha)|S^*|^{2r} + \alpha |V|^{2r}\right\|^{1/r} \left\|\alpha |S^*|^{2r} + (1-\alpha)|V|^{2r}\right\|^{1/r}. \tag{2.15}$$

In particular,

$$\omega^2(SV) \leq \frac{1}{2^{1/r}} \left\||S^*|^{2r} + |V|^{2r}\right\|^{1/r} \|S\| \|V\| \tag{2.16}$$

and

$$\omega(SV) \leq \frac{1}{2^{1/r}} \left\||S^*|^{2r} + |V|^{2r}\right\|^{1/r}. \tag{2.17}$$

Proof. We take in Lemma 3 $A = |S^*|^2$, $B = |V|^2$ and $C = V^*S^*$ to get the desired inequalities. □

3. Applications for Generalized Aluthge Transform

We have the following inequalities for one operator:

Theorem 3. *Let $T \in \mathcal{B}(H)$ and $t \in [0,1]$, then*

$$\omega^{2r}(T) \leq \left\| \frac{1}{p} |T^*|^{2tpr} + \frac{1}{q} |T|^{2(1-t)qr} \right\|, \tag{3.1}$$

for $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$.

Also, for $r \geq 1$,

$$\omega^{2r}(T) \leq \frac{1}{2} \left[\|T\|^{2r} + \omega^r \left(|T|^{2(1-t)} |T^*|^{2t} \right) \right]. \tag{3.2}$$

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then

$$\omega^{2r}(T) \leq \frac{1}{2} \left(\left\| \frac{1}{p} |T^*|^{2tpr} + \frac{1}{q} |T|^{2(1-t)qr} \right\| + \omega^r \left(|T|^{2(1-t)} |T^*|^{2t} \right) \right). \tag{3.3}$$

Moreover, if $\alpha \in [0,1]$ and $r \geq 1$, then

$$\omega^2(T) \leq \left\| (1-\alpha) |T^*|^{2tr} + \alpha |T|^{2(1-t)r} \right\|^{1/r} \|T\|^{2[t\alpha + (1-t)(1-\alpha)]}. \tag{3.4}$$

Proof. If we take $S = U|T|^t$ and $V = |T|^{1-t}$ in (2.6) and observe that $SV = U|T| = T$,

$$|S^*|^2 = SS^* = U|T|^t |T|^t U^* = U|T|^{2t} U^* = |T^*|^{2t},$$

then

$$\omega^{2r}(T) \leq \left\| \frac{1}{p} |T^*|^{2tpr} + \frac{1}{q} |T|^{2(1-t)qr} \right\|,$$

which proves (3.1).

The same choice of S and V in (2.7) gives

$$\omega^{2r}(T) \leq \frac{1}{2} \left[\left\| U|T|^t \right\|^{2r} \|T\|^{2(1-t)r} + \omega^r \left(|T|^{2(1-t)} |T^*|^{2t} \right) \right]. \tag{3.5}$$

Observe that

$$|S|^2 = S^*S = |T|^t U^*U |T|^t = |T|^t |T|^t = |T|^{2t}$$

since U is an isometry on $\operatorname{ran}(|T|)$. Then $\left\| U|T|^t \right\|^{2r} = \|T\|^{2rt}$ and by (3.5) we get (3.2).

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then by (2.8) we get (3.3).

Further, if we use (2.14) for $S = U|T|^t$ and $V = |T|^{1-t}$ we also get for $\alpha \in [0,1]$ and $r \geq 1$ that

$$\begin{aligned} \omega^2(T) &\leq \left\| (1-\alpha) |T^*|^{2tr} + \alpha |T|^{2(1-t)r} \right\|^{1/r} \|T\|^{2t\alpha} \|T\|^{2(1-t)(1-\alpha)} \\ &= \left\| (1-\alpha) |T^*|^{2tr} + \alpha |T|^{2(1-t)r} \right\|^{1/r} \|T\|^{2[t\alpha + (1-t)(1-\alpha)]}, \end{aligned}$$

which proves (3.4). □

Corollary 2. *Let $T \in \mathcal{B}(H)$, then*

$$\omega^{2r}(T) \leq \left\| \frac{1}{p} |T^*|^{pr} + \frac{1}{q} |T|^{qr} \right\|, \tag{3.6}$$

for $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$.

Also, for $r\geq 1$

$$\omega^{2r}(T)\leq \frac{1}{2}\left[\|T\|^{2r}+\omega^{r}(|T||T^{*}|)\right]. \tag{3.7}$$

If $r\geq 1$, $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $pr,\ qr\geq 2$, *then*

$$\omega^{2r}(T)\leq \frac{1}{2}\left(\left\|\frac{1}{p}|T^{*}|^{pr}+\frac{1}{q}|T|^{qr}\right\|+\omega^{r}(|T||T^{*}|)\right). \tag{3.8}$$

Moreover, if $\alpha\in[0,1]$ *and* $r\geq 1$, *then*

$$\omega^{2}(T)\leq \left\|(1-\alpha)|T^{*}|^{r}+\alpha|T|^{r}\right\|^{1/r}\|T\|. \tag{3.9}$$

Remark 3. *If we take* $r=1$ *in (3.6), then we get*

$$\omega^{2}(T)\leq \left\|\frac{1}{p}|T^{*}|^{p}+\frac{1}{q}|T|^{q}\right\|,$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *In particular, for* $p=q=2$ *we get*

$$\omega^{2}(T)\leq \frac{1}{2}\left\||T^{*}|^{2}+|T|^{2}\right\|.$$

If we take $r=1$ *in (3.2), then we get*

$$\omega^{2}(T)\leq \frac{1}{2}\left[\|T\|^{2}+\omega\left(|T|^{2(1-t)}|T^{*}|^{2t}\right)\right].$$

If $r=1$ *and* $p=q=2$, *then by (3.8) we get*

$$\omega^{2}(T)\leq \frac{1}{2}\left(\frac{1}{2}\left\||T^{*}|^{2}+|T|^{2}\right\|+\omega(|T||T^{*}|)\right).$$

If we take $r=2$ *and* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *in (3.8), then we obtain*

$$\omega^{4}(T)\leq \frac{1}{2}\left(\left\|\frac{1}{p}|T^{*}|^{2p}+\frac{1}{q}|T|^{2q}\right\|+\omega^{2}(|T||T^{*}|)\right),$$

which for $p=q=2$ *gives*

$$\omega^{4}(T)\leq \frac{1}{2}\left(\frac{1}{2}\left\||T^{*}|^{4}+|T|^{4}\right\|+\omega^{2}(|T||T^{*}|)\right).$$

We also have:

Theorem 4. *Let* $T\in\mathcal{B}(H)$ *and* $t\in[0,1]$, *then*

$$\omega^{2r}(\Delta_t(T))\leq \left\|\frac{1}{p}\left|U^{*}|T|^{t}\right|^{2pr}+\frac{1}{q}|T|^{2(1-t)qr}\right\| \tag{3.10}$$

for $r>0$, $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $pr,\ qr\geq 1$.

Also

$$\omega^{2r}(\Delta_t(T))\leq \frac{1}{2}\left[\|T\|^{2r}+\omega^{r}\left(|T|^{2-t}UU^{*}|T|^{t}\right)\right] \tag{3.11}$$

for $r\geq 1$.

Moreover,

$$\omega^{2r}(\Delta_t(T)) \leq \frac{1}{2}\left(\left\|\frac{1}{p}\left|U^{*}|T|^{t}\right|^{2pr}+\frac{1}{q}|T|^{2(1-t)qr}\right\|+\omega^{r}\left(|T|^{2-t}UU^{*}|T|^{t}\right)\right) \tag{3.12}$$

for $r \geq 1$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pr, qr \geq 2$.

Also, for $\alpha \in [0,1]$ *and* $r \geq 1$,

$$\omega^2(\Delta_t(T)) \leq \left\| (1-\alpha) \left| U^* |T|^t \right|^{2pr} + \alpha |T|^{2(1-t)qr} \right\|^{1/r} \|T\|^{2[t\alpha+(1-t)(1-\alpha)]}. \tag{3.13}$$

Proof. If we take $S = |T|^t U$ and $V = |T|^{1-t}$ in (2.6) and observe that $SV = |T|^t U |T|^{1-t} = \Delta_t(T)$, then we get

$$\omega^{2r}(\Delta_t(T)) \leq \left\| \frac{1}{p} \left| U^* |T|^t \right|^{2pr} + \frac{1}{q} |T|^{2(1-t)qr} \right\|.$$

With the same choice and by using (2.7) we derive

$$\begin{aligned}
\omega^{2r}(\Delta_t(T)) &\leq \frac{1}{2}\left[\left\| |T|^t U \right\|^{2r} \|T\|^{2(1-t)r} + \omega^r \left(|T|^{2(1-t)} |T|^t UU^* |T|^t \right) \right] \\
&= \frac{1}{2}\left[\left\| |T|^t U \right\|^{2r} \|T\|^{2(1-t)r} + \omega^r \left(|T|^{2-t} UU^* |T|^t \right) \right] \\
&\leq \frac{1}{2}\left[\|T\|^{2tr} \|T\|^{2(1-t)r} + \omega^r \left(|T|^{2-t} UU^* |T|^t \right) \right] \\
&= \frac{1}{2}\left[\|T\|^{2r} + \omega^r \left(|T|^{2-t} UU^* |T|^t \right) \right],
\end{aligned}$$

which proves (3.11).

If $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$, then from (2.8) we get (3.12).

If we take $S = |T|^t U$ and $V = |T|^{1-t}$ in (2.14) then for $\alpha \in [0,1]$ and $r \geq 1$,

$$\begin{aligned}
&\omega^2(\Delta_t(T)) \\
&\leq \left\| (1-\alpha) \left| U^* |T|^t \right|^{2pr} + \alpha |T|^{2(1-t)qr} \right\|^{1/r} \left\| |T|^t U \right\|^{2\alpha} \|T\|^{2(1-t)(1-\alpha)} \\
&\leq \left\| (1-\alpha) \left| U^* |T|^t \right|^{2pr} + \alpha |T|^{2(1-t)qr} \right\|^{1/r} \|T\|^{2t\alpha} \|T\|^{2(1-t)(1-\alpha)} \\
&= \left\| (1-\alpha) \left| U^* |T|^t \right|^{2pr} + \alpha |T|^{2(1-t)qr} \right\|^{1/r} \|T\|^{2[t\alpha+(1-t)(1-\alpha)]},
\end{aligned}$$

which proves (3.13). □

For $t = 1/2$ we obtain the following inequalities for the Aluthge transform $\widetilde{T}$.

Corollary 3. *Let* $T \in \mathcal{B}(H)$, *then*

$$\omega^{2r}\left(\widetilde{T}\right) \leq \left\| \frac{1}{p} \left| U^* |T|^{1/2} \right|^{2pr} + \frac{1}{q} |T|^{qr} \right\| \tag{3.14}$$

for $r > 0$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pr, qr \geq 1$.

Also

$$\omega^{2r}\left(\widetilde{T}\right) \leq \frac{1}{2}\left[\|T\|^{2r} + \omega^r \left(|T|^{3/2} UU^* |T|^{1/2} \right) \right] \tag{3.15}$$

for $r \geq 1$.

Moreover,

$$\omega^{2r}\left(\widetilde{T}\right) \leq \frac{1}{2}\left(\left\| \frac{1}{p} \left| U^* |T|^{1/2} \right|^{2pr} + \frac{1}{q} |T|^{qr} \right\| + \omega^r \left(|T|^{3/2} UU^* |T|^{1/2} \right) \right) \tag{3.16}$$

for $r\geq 1$, p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr\geq 2$.

Also, for $\alpha\in[0,1]$ *and* $r\geq 1$,

$$\omega^2\left(\widetilde{T}\right)\leq\left\|(1-\alpha)\left|U^*|T|^{1/2}\right|^{2pr}+\alpha|T|^{qr}\right\|^{1/r}\|T\|. \tag{3.17}$$

For $t=1$ we also obtain the following results for the Dougal transform $\widehat{T}$.

Corollary 4. *Let* $T\in\mathcal{B}(H)$, *then*

$$\omega^{2r}\left(\widehat{T}\right)\leq\left\|\frac{1}{p}\left|\left(\widehat{T}\right)^*\right|^{2pr}+\frac{1}{q}I\right\|$$

for $r>0$, p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr\geq 1$.

Also

$$\omega^{2r}\left(\widehat{T}\right)\leq\frac{1}{2}\left[\|T\|^{2r}+\left\|\widehat{T}\right\|^{2r}\right]$$

for $r\geq 1$.

Moreover,

$$\omega^{2r}\left(\widehat{T}\right)\leq\frac{1}{2}\left(\left\|\frac{1}{p}\left|\left(\widehat{T}\right)^*\right|^{2pr}+\frac{1}{q}I\right\|+\left\|\widehat{T}\right\|^{2r}\right)$$

for $r\geq 1$, p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr\geq 2$.

Also, for $\alpha\in[0,1]$ *and* $r\geq 1$,

$$\omega^2\left(\widehat{T}\right)\leq\left\|(1-\alpha)\left|\left(\widehat{T}\right)^*\right|^{2pr}+\alpha I\right\|^{1/r}\|T\|^{2\alpha}.$$

For $t=0$ we also get:

Corollary 5. *Let* $T\in\mathcal{B}(H)$, *then*

$$\omega^{2r}(T)\leq\left\|\frac{1}{p}|U^*|^{2pr}+\frac{1}{q}|T|^{2qr}\right\|$$

for $r>0$, p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr\geq 1$.

Also

$$\omega^{2r}(T)\leq\frac{1}{2}\left[\|T\|^{2r}+\omega^r\left(|T|^2UU^*\right)\right]$$

for $r\geq 1$.

Moreover,

$$\omega^{2r}(T)\leq\frac{1}{2}\left(\left\|\frac{1}{p}|U^*|^{2pr}+\frac{1}{q}|T|^{2qr}\right\|+\omega^r\left(|T|^2UU^*\right)\right)$$

for $r\geq 1$, p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr\geq 2$.

Also, for $\alpha\in[0,1]$ *and* $r\geq 1$,

$$\omega^2(T)\leq\left\|(1-\alpha)|U^*|^{2pr}+\alpha|T|^{2qr}\right\|^{1/r}\|T\|^{2(1-\alpha)}.$$

We also have the following upper bounds for the numerical radius of generalized Aluthge transform:

Theorem 5. *Let $T \in \mathcal{B}(H)$ and $t \in [0,1]$, then*

$$\omega^{2r}(\Delta_t(T)) \le \left\| \frac{1}{p}|T|^{2tpr} + \frac{1}{q}|T|^{2(1-t)qr} \right\| \tag{3.18}$$

for $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 1$.

If $r \ge 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 2$, then

$$\omega^{2r}(\Delta_t(T)) \le \frac{1}{2}\left(\left\| \frac{1}{p}|T|^{2tpr} + \frac{1}{q}|T|^{2(1-t)qr} \right\| + \|T\|^{2r} \right). \tag{3.19}$$

Also, for $\alpha \in [0,1]$ and $r \ge 1$,

$$\omega^{2}(\Delta_t(T)) \le \left\| (1-\alpha)|T|^{2tr} + \alpha |T|^{2(1-t)r} \right\|^{1/r} \|T\|^{2[t\alpha + (1-t)(1-\alpha)]}. \tag{3.20}$$

Proof. If we take $S = |T|^t$ and $V = U|T|^{1-t}$ and observe that $SV = |T|^t U |T|^{1-t} = \Delta_t(T)$,

$$|V|^{2qr} = (V^*V)^{qr} = \left(|T|^{1-t} U^*U |T|^{1-t}\right)^{qr} = |T|^{2(1-t)qr}$$

then by (2.6) we get (3.18).

If $r \ge 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 2$, then by the same choices in (2.8) we obtain

$$\begin{aligned} \omega^{2r}(\Delta_t(T)) &\le \frac{1}{2}\left(\left\| \frac{1}{p}|T|^{2tpr} + \frac{1}{q}|T|^{2(1-t)qr} \right\| + \omega^r\left(|T|^{2(1-t)}|T|^{2t}\right) \right) \\ &= \frac{1}{2}\left(\left\| \frac{1}{p}|T|^{2tpr} + \frac{1}{q}|T|^{2(1-t)qr} \right\| + \|T\|^{2r} \right), \end{aligned}$$

which proves (3.19).

For $\alpha \in [0,1]$ and $r \ge 1$, then by (2.14) we get

$$\begin{aligned} \omega^{2}(\Delta_t(T)) &\le \left\| (1-\alpha)|T|^{2tr} + \alpha |T|^{2(1-t)r} \right\|^{1/r} \|T\|^{2t\alpha} \|T\|^{2(1-t)(1-\alpha)} \\ &= \left\| (1-\alpha)|T|^{2tr} + \alpha |T|^{2(1-t)r} \right\|^{1/r} \|T\|^{2[t\alpha + (1-t)(1-\alpha)]}, \end{aligned}$$

which proves (3.20). □

Remark 4. *If we take $t = 1/2$ in (3.18) then we get*

$$\omega^{2r}\left(\widetilde{T}\right) \le \left\| \frac{1}{p}|T|^{pr} + \frac{1}{q}|T|^{qr} \right\| \tag{3.21}$$

for $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 1$.

If $r \ge 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 2$, then by (3.19) we obtain

$$\omega^{2r}\left(\widetilde{T}\right) \le \frac{1}{2}\left(\left\| \frac{1}{p}|T|^{pr} + \frac{1}{q}|T|^{qr} \right\| + \|T\|^{2r} \right). \tag{3.22}$$

References

[1] A. Aluthge, Some generalized theorems on p-hyponormal operators, *Integral Equations Operator Theory* **24** (1996), 497-501.

[2] A. Abu-Omar and F. Kittaneh, A numerical radius inequality involving the generalized Aluthge transform, *Studia Math.* **216** (1) (2013) 69-75.

[3] P. Bunia, Improved bounds for the numerical radius via polar decomposition of operators, Preprint, `http://arxiv.org/abs/2303.03051v1`

[4] P. Bhunia, S. Bag, and K. Paul, Numerical radius inequalities and its applications in estimation of zeros of polynomials, *Linear Algebra and its Applications*, vol. **573** (2019) pp. 166–177.

[5] P. Bhunia , S. S. Dragomir , M. S. Moslehian , K. Paul, *Lectures on Numerical Radius Inequalities*, Springer Cham, 2022. `https://doi.org/10.1007/978-3-031-13670-2`.

[6] P. Bhunia, K. Feki and K. Paul, Numerical radius inequalities for products and sums of semi-Hilberian space operators, *Filomat* Vol **36** (2022), No. 4.

[7] M. L. Buzano, Generalizzazione della diseguaglianza di Cauchy-Schwarz. (Italian), *Rend. Sem. Mat. Univ. e Politech. Torino*, **31** (1971/73), 405–409 (1974).

[8] C. A. McCarthy, C_p, *Israel J. Math.* **5** (1967), 249–271.

[9] M. Cho and K. Tanahashi, Spectral relations for Aluthge transform, *Scientiae Mathematicae Japonicae*, **55** (1) (2002), 77-83.

[10] S. S. Dragomir, *Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces*, SpringerBriefs in Mathematics, 2013. `https://doi.org/10.1007/978-3-319-01448-7`.

[11] M. El-Haddad and F. Kittaneh, Numerical radius inequalities for Hilbert space operators. II, *Studia Math.* **182** (2007), No. 2, 133-140.

[12] F. Kittaneh, Notes on some inequalities for Hilbert space operators, *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 2, 283–293.

[13] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, *Studia Math.* **158** (2003), No. 1, 11-17.

[14] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, *Studia Math.*, **168** (2005), No. 1, 73-80.

[15] T. Yamazaki, On upper and lower bounds of the numerical radius and an equality condition, *Studia Math.* **178** (2007), No. 1, 83-89.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa