GENERAL INEQUALITIES FOR THE NUMERICAL RADIUS OF OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a complex Hilbert space. Assume that f and g are nonnegative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. In this paper we show among others that, if A, B, $T\in\mathcal{B}(H)$ and $r\geq 1$, then

$$\omega^{2r}(BTA) \leq \frac{1}{2}\left(\|f(|T|)A\|^{2r}\|g(|T^*|)B^*\|^{2r}+\left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|^r\right).$$

In particular, for $f(t)=t^\lambda$, $g(t)=t^{1-\lambda}$ with $\lambda\in[0,1]$, we derive

$$\omega^{2r}(BTA) \leq \frac{1}{2}\left(\left\||T|^\lambda A\right\|^{2r}\left\||T^*|^{1-\lambda}B^*\right\|^{2r}+\left\|\left||T^*|^{1-\lambda}B^*\right|^2\left||T|^\lambda A\right|^2\right\|^r\right)$$

for $r\geq 1$.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T)=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x\in H$ one has

$$|\langle Tx,x\rangle|\leq w(T)\|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T:H\to H$, i.e.,

(i) $\omega(T)\geq 0$ for any $T\in B(H)$ and $\omega(T)=0$ if and only if $T=0$;
(ii) $\omega(\lambda T)=|\lambda|\,\omega(T)$ for any $\lambda\in\mathbb{C}$ and $T\in B(H)$;
(iii) $\omega(T+V)\leq\omega(T)+\omega(V)$ for any $T,V\in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T)\leq\|T\|\leq 2\omega(T) \tag{1.3}$$

for any $T\in B(H)$.

F. Kittaneh, in 2003 [6], showed that for any operator $T\in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T)\leq\frac{1}{2}\left(\|T\|+\left\|T^2\right\|^{1/2}\right). \tag{1.4}$$

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Utilizing the Cartesian decomposition for operators, F. Kittaneh in [7] improved the inequality (1.3) as follows:

$$\frac{1}{4}\|T^*T+TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T+TT^*\| \tag{1.5}$$

for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [3]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{1.6}$$

and

$$\omega^{2r}(T) \leq \left\|\alpha |T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|, \tag{1.7}$$

where $\alpha \in (0,1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (1.6) that

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{1.8}$$

and from (1.7) that

$$\omega^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [2] and [1].

In 1988, F. Kittaneh obtained the following generalization of Schwarz inequality [5]:

Theorem 1. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. For any $T \in \mathcal{B}(H)$*

$$|\langle Tx, y\rangle| \leq \|f(|T|)x\| \|g(|T^*|)y\| \tag{1.10}$$

for all x, $y \in H$.

If we take $f(t) = t^\lambda$, $g(t) = t^{1-\lambda}$ with $\lambda \in [0,1]$, then we obtain *Kato's inequality* [4]

$$|\langle Tx, y\rangle| \leq \left\||T|^\lambda x\right\| \left\||T^*|^{1-\lambda} y\right\| \tag{1.11}$$

for all x, $y \in H$.

Motivated by the above results, in this paper we obtain among others the following inequalities for the numerical radius

$$\begin{aligned}&\omega^{2r}(BTA)\\ &\leq \frac{1}{2}\left(\|f(|T|)A\|^{2r}\|g(|T^*|)B^*\|^{2r} + \left\||g(|T^*|)B^*|^2 |f(|T|)A|^2\right\|^r\right),\end{aligned}$$

where f and g are non-negative functions on $[0,\infty)$ that are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$, A, B, $T \in \mathcal{B}(H)$ and $r \geq 1$.

2. Main Results

We start with the following result:

Theorem 2. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. For any A, B, $T\in\mathcal{B}(H)$ we have*

$$\|BTA\| \le \|f(|T|)A\| \, \|Bg(|T^*|)\| \tag{2.1}$$

and

$$\omega(BTA) \le \frac{1}{2}\left\| |f(|T|)A|^2 + |g(|T^*|)B^*|^2 \right\|. \tag{2.2}$$

Also,

$$\omega^2(BTA) \le \frac{1}{2}\left[\|f(|T|)A\|^2 \|g(|T^*|)B^*\|^2 + \left\| |g(|T^*|)B^*|^2 |f(|T|)A|^2 \right\| \right]. \tag{2.3}$$

Proof. Observe that by (1.10) we have

$$\begin{aligned} |\langle Tx,y\rangle|^2 &\le \|f(|T|)x\|^2 \|g(|T^*|)y\|^2 \\ &= \langle f(|T|)x, f(|T|)x\rangle \langle g(|T^*|)y, g(|T^*|)y\rangle \\ &= \left\langle f^2(|T|)x,x\right\rangle \left\langle g^2(|T^*|)y,y\right\rangle \end{aligned}$$

for all x, $y\in H$.

If we take Ax instead of x and B^*y instead of y, then we get

$$\begin{aligned} |\langle TAx,B^*y\rangle|^2 &\le \left\langle f^2(|T|)Ax,Ax\right\rangle \left\langle g^2(|T^*|)B^*y,B^*y\right\rangle \\ &= \left\langle A^*f^2(|T|)Ax,x\right\rangle \left\langle Bg^2(|T^*|)B^*y,y\right\rangle \\ &= \left\langle (f(|T|)Ax)^* f(|T|)Ax,x\right\rangle \left\langle (g(|T^*|)B^*)^* g(|T^*|)B^*y,y\right\rangle \\ &= \left\langle |f(|T|)A|^2 x,x\right\rangle \left\langle |g(|T^*|)B^*|^2 y,y\right\rangle, \end{aligned}$$

namely

$$|\langle BTAx,y\rangle|^2 \le \left\langle |f(|T|)A|^2 x,x\right\rangle \left\langle |g(|T^*|)B^*|^2 y,y\right\rangle \tag{2.4}$$

for all x, $y\in H$.

Therefore

$$\begin{aligned} \|BTA\|^2 &= \sup_{\|x\|=\|y\|=1} |\langle BTAx,y\rangle|^2 \\ &\le \sup_{\|x\|=\|y\|=1} \left[\left\langle |f(|T|)A|^2 x,x\right\rangle \left\langle |g(|T^*|)B^*|^2 y,y\right\rangle \right] \\ &= \sup_{\|x\|=1} \left\langle |f(|T|)A|^2 x,x\right\rangle \sup_{\|y\|=1} \left\langle |g(|T^*|)B^*|^2 y,y\right\rangle \\ &= \left\| |f(|T|)A|^2 \right\| \left\| |g(|T^*|)B^*|^2 \right\| = \|f(|T|)A\|^2 \|g(|T^*|)B^*\|^2 \\ &= \|f(|T|)A\|^2 \|Bg(|T^*|)\|^2, \end{aligned}$$

which is equivalent to (2.1).

From (2.4) we also have

$$|\langle BTAx,x\rangle|^2 \le \left\langle |f(|T|)A|^2 x,x\right\rangle \left\langle |g(|T^*|)B^*|^2 x,x\right\rangle \tag{2.5}$$

for all $x\in H$.

By the *A-G inequality*, we also have

$$\begin{aligned}
|\langle BTAx,x\rangle| &\leq \left\langle |f(|T|)A|^2 x,x\right\rangle^{1/2} \left\langle |g(|T^*|)B^*|^2 x,x\right\rangle^{1/2} \\
&\leq \frac{1}{2}\left[\left\langle |f(|T|)A|^2 x,x\right\rangle + \left\langle |g(|T^*|)B^*|^2 x,x\right\rangle\right] \\
&= \left\langle \left(\frac{|f(|T|)A|^2 + |g(|T^*|)B^*|^2}{2}\right) x,x\right\rangle
\end{aligned}$$

for all $x \in H$.

By taking the supremum, we get

$$\begin{aligned}
\omega(BTA) &= \sup_{\|x\|=1} |\langle BTAx,x\rangle| \\
&\leq \sup_{\|x\|=1} \left\langle \left(\frac{|f(|T|)A|^2 + |g(|T^*|)B^*|^2}{2}\right) x,x\right\rangle \\
&= \left\| \frac{|f(|T|)A|^2 + |g(|T^*|)B^*|^2}{2} \right\|,
\end{aligned}$$

which proves (2.2).

Let $x \in H$, $\|x\| = 1$, then by Buzano's inequality, we recall that

$$\frac{1}{2}\left[\|u\|\|v\| + |\langle u,v\rangle|\right] \geq |\langle u,e\rangle\langle e,v\rangle|$$

holds for any u, v, $e \in H$ with $\|e\| = 1$, we derive

$$\begin{aligned}
&\left\langle |f(|T|)A|^2 x,x\right\rangle \left\langle x, |g(|T^*|)B^*|^2 x\right\rangle \\
&\leq \frac{1}{2}\left[\left\| |f(|T|)A|^2 x\right\| \left\| |g(|T^*|)B^*|^2 x\right\| + \left\langle |f(|T|)A|^2 x, |g(|T^*|)B^*|^2 x\right\rangle\right] \\
&= \frac{1}{2}\left[\left\| |f(|T|)A|^2 x\right\| \left\| |g(|T^*|)B^*|^2 x\right\| + \left\langle |g(|T^*|)B^*|^2 |f(|T|)A|^2 x,x\right\rangle\right].
\end{aligned}$$

By making use of (2.5) we get

$$\begin{aligned}
(2.6)\quad &|\langle BTAx,x\rangle|^2 \\
&\leq \frac{1}{2}\left[\left\| |f(|T|)A|^2 x\right\| \left\| |g(|T^*|)B^*|^2 x\right\| + \left\langle |g(|T^*|)B^*|^2 |f(|T|)A|^2 x,x\right\rangle\right]
\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

If we take the supremum, then we obtain

$$\begin{aligned}\omega^2(BTA) &= \sup_{\|x\|=1}|\langle BTAx,x\rangle| \\ &\le \frac{1}{2}\sup_{\|x\|=1}\left[\left\||f(|T|)A|^2x\right\|\left\||g(|T^*|)B^*|^2x\right\| \right.\\ &\left.+\left\langle |g(|T^*|)B^*|^2|f(|T|)A|^2x,x\right\rangle\right] \\ &\le \frac{1}{2}\left[\sup_{\|x\|=1}\left\||f(|T|)A|^2x\right\|\sup_{\|x\|=1}\left\||g(|T^*|)B^*|^2x\right\| \right.\\ &\left.+\sup_{\|x\|=1}\left\langle |g(|T^*|)B^*|^2|f(|T|)A|^2x,x\right\rangle\right] \\ &= \frac{1}{2}\left[\|f(|T|)A\|^2\|g(|T^*|)B^*\|^2+\left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|\right],\end{aligned}$$

which proves (2.3). □

Remark 1. *If we take $f(t)=t^{\lambda}$, $g(t)=t^{1-\lambda}$ with $\lambda\in[0,1]$ in Theorem 2, then we get for any A, B, $T\in\mathcal{B}(H)$,*

$$\|BTA\|\le\left\||T|^{\lambda}A\right\|\left\|B|T^*|^{1-\lambda}\right\|$$

and

$$\omega(BTA)\le\frac{1}{2}\left\|\left||T|^{\lambda}A\right|^2+\left||T^*|^{1-\lambda}B^*\right|^2\right\|. \tag{2.7}$$

Also,

$$\omega^2(BTA)\le\frac{1}{2}\left[\left\||T|^{\lambda}A\right\|^2\left\||T^*|^{1-\lambda}B^*\right\|^2+\left\|\left||T^*|^{1-\lambda}B^*\right|^2\left||T|^{\lambda}A\right|^2\right\|\right]. \tag{2.8}$$

Moreover, if we take $\lambda=1/2$, then we get

$$\|BTA\|\le\left\||T|^{1/2}A\right\|\left\|B|T^*|^{1/2}\right\|$$

and

$$\omega(BTA)\le\frac{1}{2}\left\|\left||T|^{1/2}A\right|^2+\left||T^*|^{1/2}B^*\right|^2\right\|.$$

Also,

$$\omega^2(BTA)\le\frac{1}{2}\left[\left\||T|^{1/2}A\right\|^2\left\||T^*|^{1/2}B^*\right\|^2+\left\|\left||T^*|^{1/2}B^*\right|^2\left||T|^{1/2}A\right|^2\right\|\right].$$

Theorem 3. *Assume that the conditions of Theorem 2 are satisfied. If $r>0$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\ge1$, then*

$$\omega^{2r}(BTA)\le\left\|\frac{1}{p}|f(|T|)A|^{2pr}+\frac{1}{q}|g(|T^*|)B^*|^{2qr}\right\|. \tag{2.9}$$

If $r\ge1$, then

$$\begin{aligned}&\omega^{2r}(BTA) \\ &\le\frac{1}{2}\left(\|f(|T|)A\|^{2r}\|g(|T^*|)B^*\|^{2r}+\left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|^r\right).\end{aligned} \tag{2.10}$$

If $r \geq 1$, p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* pr, $qr \geq 2$, *then also*

$$\omega^{2r}(BTA) \leq \frac{1}{2}\left(\left\| \frac{1}{p}\left|f(|T|)A\right|^{2pr} + \frac{1}{q}\left|g(|T^*|)B^*\right|^{2qr}\right\| + \left\| \left|g(|T^*|)B^*\right|^2 \left|f(|T|)A\right|^2 \right\|^r\right). \tag{2.11}$$

Proof. If we take the power $r > 0$ in 2.5, then we get, by Young and McCarthy inequalities that

$$\begin{aligned}
|\langle BTAx, x\rangle|^{2r} &\leq \left\langle \left|f(|T|)A\right|^2 x, x\right\rangle^r \left\langle \left|g(|T^*|)B^*\right|^2 x, x\right\rangle^r \\
&\leq \frac{1}{p}\left\langle \left|f(|T|)A\right|^2 x, x\right\rangle^{pr} + \frac{1}{q}\left\langle \left|g(|T^*|)B^*\right|^2 x, x\right\rangle^{qr} \\
&\leq \frac{1}{p}\left\langle \left|f(|T|)A\right|^{2pr} x, x\right\rangle + \frac{1}{q}\left\langle \left|g(|T^*|)B^*\right|^{2qr} x, x\right\rangle \\
&= \left\langle \left[\frac{1}{p}\left|f(|T|)A\right|^{2pr} + \frac{1}{q}\left|g(|T^*|)B^*\right|^{2qr}\right] x, x\right\rangle
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$\begin{aligned}
\omega^{2r}(BTA) &= \sup_{\|x\|=1} |\langle BTAx, x\rangle|^{2r} \\
&\leq \sup_{\|x\|=1} \left\langle \left[\frac{1}{p}\left|f(|T|)A\right|^{2pr} + \frac{1}{q}\left|g(|T^*|)B^*\right|^{2qr}\right] x, x\right\rangle \\
&= \left\| \frac{1}{p}\left|f(|T|)A\right|^{2pr} + \frac{1}{q}\left|g(|T^*|)B^*\right|^{2qr}\right\|,
\end{aligned}$$

which proves (2.9).

By taking the power $r \geq 1$ in (2.6) and using the convexity of the power function, we get

$$\begin{aligned}
&|\langle BTAx, x\rangle|^{2r} \\
&\leq \left[\frac{\left\| \left|f(|T|)A\right|^2 x\right\| \left\| \left|g(|T^*|)B^*\right|^2 x\right\| + \left\langle \left|g(|T^*|)B^*\right|^2 \left|f(|T|)A\right|^2 x, x\right\rangle}{2}\right]^r \\
&\leq \frac{\left\| \left|f(|T|)A\right|^2 x\right\|^r \left\| \left|g(|T^*|)B^*\right|^2 x\right\|^r + \left\langle \left|g(|T^*|)B^*\right|^2 \left|f(|T|)A\right|^2 x, x\right\rangle^r}{2}
\end{aligned} \tag{2.12}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\|=1$, then we get that

$$\begin{aligned}
&\omega^{2r}(BTA) \\
&\leq \sup_{\|x\|=1}\left(\frac{1}{2}\left\||f(|T|)A|^{2}x\right\|^{r}\left\||g(|T^{*}|)B^{*}|^{2}x\right\|^{r}\right. \\
&\left.+\frac{1}{2}\left\langle|g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}x,x\right\rangle^{r}\right) \\
&\leq \frac{1}{2}\sup_{\|x\|=1}\left\||f(|T|)A|^{2}x\right\|^{r}\sup_{\|x\|=1}\left\||g(|T^{*}|)B^{*}|^{2}x\right\|^{r} \\
&+\frac{1}{2}\sup_{\|x\|=1}\left\langle|g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}x,x\right\rangle^{r} \\
&=\frac{1}{2}\left(\left\||f(|T|)A|^{2}\right\|^{r}\left\||g(|T^{*}|)B^{*}|^{2}\right\|^{r}+\left\||g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}\right\|^{r}\right) \\
&=\frac{1}{2}\left(\|f(|T|)A\|^{2r}\|g(|T^{*}|)B^{*}\|^{2r}+\left\||g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}\right\|^{r}\right),
\end{aligned}$$

which proves (2.10).

Also, observe that

$$\begin{aligned}
&\left\||f(|T|)A|^{2}x\right\|^{r}\left\||g(|T^{*}|)B^{*}|^{2}x\right\|^{r} \\
&\leq \frac{1}{p}\left\||f(|T|)A|^{2}x\right\|^{pr}+\frac{1}{q}\left\||g(|T^{*}|)B^{*}|^{2}x\right\|^{qr} \\
&=\frac{1}{p}\left\||f(|T|)A|^{2}x\right\|^{2\frac{pr}{2}}+\frac{1}{q}\left\||g(|T^{*}|)B^{*}|^{2}x\right\|^{2\frac{qr}{2}} \\
&=\frac{1}{p}\left\langle|f(|T|)A|^{4}x,x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle|g(|T^{*}|)B^{*}|^{4}x,x\right\rangle^{\frac{qr}{2}} \\
&\leq \frac{1}{p}\left\langle|f(|T|)A|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle|g(|T^{*}|)B^{*}|^{2qr}x,x\right\rangle \\
&=\left\langle\left(\frac{1}{p}|f(|T|)A|^{2pr}+\frac{1}{q}|g(|T^{*}|)B^{*}|^{2qr}\right)x,x\right\rangle,
\end{aligned}$$

then

$$\begin{aligned}
&\frac{\left\||f(|T|)A|^{2}x\right\|^{r}\left\||g(|T^{*}|)B^{*}|^{2}x\right\|^{r}+\left\langle|g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}x,x\right\rangle^{r}}{2} \\
&\leq \frac{1}{2}\left\langle\left(\frac{1}{p}|f(|T|)A|^{2pr}+\frac{1}{q}|g(|T^{*}|)B^{*}|^{2qr}\right)x,x\right\rangle \\
&+\frac{1}{2}\left\langle|g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}x,x\right\rangle^{r}
\end{aligned}$$

and by (2.12) we get

$$\begin{aligned}
|\langle BTAx,x\rangle|^{2r} \leq{}& \frac{1}{2}\left\langle\left(\frac{1}{p}|f(|T|)A|^{2pr}+\frac{1}{q}|g(|T^{*}|)B^{*}|^{2qr}\right)x,x\right\rangle \\
&+\frac{1}{2}\left\langle|g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}x,x\right\rangle^{r}
\end{aligned}$$

or $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, we derive (2.11). □

Remark 2. *Consider $f(t)=t^{\lambda}$, $g(t)=t^{1-\lambda}$ with $\lambda\in[0,1]$ in Theorem 3. If $r>0$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\geq 1$, then*

$$\omega^{2r}(BTA)\leq\left\Vert \frac{1}{p}\left\vert |T|^{\lambda}A\right\vert^{2pr}+\frac{1}{q}\left\vert |T^{*}|^{1-\lambda}B^{*}\right\vert^{2qr}\right\Vert. \tag{2.13}$$

In particular,

$$\omega^{2r}(BTA)\leq\left\Vert \frac{1}{p}\left\vert |T|^{1/2}A\right\vert^{2pr}+\frac{1}{q}\left\vert |T^{*}|^{1/2}B^{*}\right\vert^{2qr}\right\Vert.$$

If $r\geq 1$, then

$$\begin{aligned}&\omega^{2r}(BTA)\\&\leq\frac{1}{2}\left(\left\Vert |T|^{\lambda}A\right\Vert^{2r}\left\Vert |T^{*}|^{1-\lambda}B^{*}\right\Vert^{2r}+\left\Vert \left\vert |T^{*}|^{1-\lambda}B^{*}\right\vert^{2}\left\vert |T|^{\lambda}A\right\vert^{2}\right\Vert^{r}\right).\end{aligned} \tag{2.14}$$

In particular,

$$\begin{aligned}&\omega^{2r}(BTA)\\&\leq\frac{1}{2}\left(\left\Vert |T|^{1/2}A\right\Vert^{2r}\left\Vert |T^{*}|^{1/2}B^{*}\right\Vert^{2r}+\left\Vert \left\vert |T^{*}|^{1/2}B^{*}\right\vert^{2}\left\vert |T|^{1/2}A\right\vert^{2}\right\Vert^{r}\right).\end{aligned}$$

If $r\geq 1$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\geq 2$, then also

$$\begin{aligned}\omega^{2r}(BTA)\leq\frac{1}{2}&\left(\left\Vert \frac{1}{p}\left\vert |T|^{\lambda}A\right\vert^{2pr}+\frac{1}{q}\left\vert |T^{*}|^{1-\lambda}B^{*}\right\vert^{2qr}\right\Vert\right.\\&\left.+\left\Vert \left\vert |T^{*}|^{1-\lambda}B^{*}\right\vert^{2}\left\vert |T|^{\lambda}A\right\vert^{2}\right\Vert^{r}\right).\end{aligned} \tag{2.15}$$

In particular,

$$\begin{aligned}\omega^{2r}(BTA)\leq\frac{1}{2}&\left(\left\Vert \frac{1}{p}\left\vert |T|^{1/2}A\right\vert^{2pr}+\frac{1}{q}\left\vert |T^{*}|^{1/2}B^{*}\right\vert^{2qr}\right\Vert\right.\\&\left.+\left\Vert \left\vert |T^{*}|^{1/2}B^{*}\right\vert^{2}\left\vert |T|^{1/2}A\right\vert^{2}\right\Vert^{r}\right).\end{aligned}$$

Remark 3. *If we take $p=q=2$ and assume that $r\geq\frac{1}{2}$, then from (2.9) we get*

$$\omega^{2r}(BTA)\leq\frac{1}{2}\left\Vert |f(|T|)A|^{4r}+|g(|T^{*}|)B^{*}|^{4r}\right\Vert,$$

which for $r=\frac{1}{2}$ gives

$$\omega(BTA)\leq\frac{1}{2}\left\Vert |f(|T|)A|^{2}+|g(|T^{*}|)B^{*}|^{2}\right\Vert,$$

while for $r=1$ gives

$$\omega^{2}(BTA)\leq\frac{1}{2}\left\Vert |f(|T|)A|^{4}+|g(|T^{*}|)B^{*}|^{4}\right\Vert.$$

If we take $r=1$ in (2.9), then we get

$$\omega^{2}(BTA)\leq\left\Vert \frac{1}{p}|f(|T|)A|^{2p}+\frac{1}{q}|g(|T^{*}|)B^{*}|^{2q}\right\Vert$$

for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

If we take $r=1$ *in (2.10) then we get*

$$\omega^2(BTA) \le \frac{1}{2}\left(\|f(|T|)A\|^2\|g(|T^*|)B^*\|^2 + \left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|\right),$$

while for $r=2$,

$$\omega^4(BTA) \le \frac{1}{2}\left(\|f(|T|)A\|^8\|g(|T^*|)B^*\|^8 + \left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|^2\right).$$

Also, if we take $p=q=2$ *and* $r\ge 1$ *in (2.11), then we get*

$$\omega^{2r}(BTA) \le \frac{1}{2}\left(\frac{1}{2}\left\||f(|T|)A|^{4r} + |g(|T^*|)B^*|^{4r}\right\| + \left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|^r\right).$$

In particular, for $r=1$ *we derive*

$$\omega^{2}(BTA) \le \frac{1}{2}\left(\frac{1}{2}\left\||f(|T|)A|^{4} + |g(|T^*|)B^*|^{4}\right\| + \left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|\right).$$

Moreover, if $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and take* $r=2$ *in (2.11), then we derive the inequality*

$$\omega^{4}(BTA) \le \frac{1}{2}\left(\left\|\frac{1}{p}|f(|T|)A|^{4p} + \frac{1}{q}|g(|T^*|)B^*|^{4q}\right\| + \left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|^2\right),$$

which for $p=q=2$ *provides*

$$\omega^{4}(BTA) \le \frac{1}{2}\left(\frac{1}{2}\left\||f(|T|)A|^{8} + |g(|T^*|)B^*|^{8}\right\| + \left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|^2\right).$$

If in Remark 3 we take $f(t)=t^{\lambda}$, $g(t)=t^{1-\lambda}$ with $\lambda\in[0,1]$ then we can get other similar inequalities. The details are omitted.

Theorem 4. *With the assumptions of Theorem 2, we have for* $r\ge 1$ *that*

$$\omega^2(BTA) \le \left\|(1-\alpha)|f(|T|)A|^{2r} + \alpha|g(|T^*|)B^*|^{2r}\right\|^{1/r} \times \|f(|T|)A\|^{2\alpha}\|g(|T^*|)B^*\|^{2(1-\alpha)} \tag{2.16}$$

for all $\alpha\in[0,1]$.

Also, we have

$$\omega^2(BTA) \le \left\|(1-\alpha)|f(|T|)A|^{2r} + \alpha|g(|T^*|)B^*|^{2r}\right\|^{1/r} \times \left\|\alpha|f(|T|)A|^{2r} + (1-\alpha)|g(|T^*|)B^*|^{2r}\right\|^{1/r} \tag{2.17}$$

for all $\alpha\in[0,1]$ *and* $r\ge 1$.

Proof. From (2.5) we have for all $\alpha \in [0,1]$ that

$$\begin{aligned}
\left|\langle BTAx, x\rangle\right|^2 &\leq \left\langle \left|f\left(|T|\right)A\right|^2 x, x\right\rangle \left\langle \left|g\left(|T^*|\right)B^*\right|^2 x, x\right\rangle \\
&= \left\langle \left|f\left(|T|\right)A\right|^2 x, x\right\rangle^{1-\alpha} \left\langle \left|g\left(|T^*|\right)B^*\right|^2 x, x\right\rangle^{\alpha} \\
&\times \left\langle \left|f\left(|T|\right)A\right|^2 x, x\right\rangle^{\alpha} \left\langle \left|g\left(|T^*|\right)B^*\right|^2 x, x\right\rangle^{1-\alpha} \\
&\leq \left[(1-\alpha)\left\langle \left|f\left(|T|\right)A\right|^2 x, x\right\rangle + \alpha \left\langle \left|g\left(|T^*|\right)B^*\right|^2 x, x\right\rangle\right] \\
&\times \left\langle \left|f\left(|T|\right)A\right|^2 x, x\right\rangle^{\alpha} \left\langle \left|g\left(|T^*|\right)B^*\right|^2 x, x\right\rangle^{1-\alpha}
\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

If we take the power $r \geq 1$, then we get by the convexity of power r that

$$\begin{aligned}
\left|\langle BTAx, x\rangle\right|^{2r} &\leq \left[(1-\alpha)\left\langle \left|f\left(|T|\right)A\right|^2 x, x\right\rangle + \alpha \left\langle \left|g\left(|T^*|\right)B^*\right|^2 x, x\right\rangle\right]^r \\
&\times \left\langle \left|f\left(|T|\right)A\right|^2 x, x\right\rangle^{r\alpha} \left\langle \left|g\left(|T^*|\right)B^*\right|^2 x, x\right\rangle^{r(1-\alpha)} \\
&\leq \left[(1-\alpha)\left\langle \left|f\left(|T|\right)A\right|^2 x, x\right\rangle^r + \alpha \left\langle \left|g\left(|T^*|\right)B^*\right|^2 x, x\right\rangle^r\right] \\
&\times \left\langle \left|f\left(|T|\right)A\right|^2 x, x\right\rangle^{r\alpha} \left\langle \left|g\left(|T^*|\right)B^*\right|^2 x, x\right\rangle^{r(1-\alpha)}
\end{aligned} \tag{2.18}$$

for all $x \in H$, $\|x\| = 1$.

If we use McCarthy inequality for power $r \geq 1$, then we get

$$\begin{aligned}
&(1-\alpha)\left\langle \left|f\left(|T|\right)A\right|^2 x, x\right\rangle^r + \alpha \left\langle \left|g\left(|T^*|\right)B^*\right|^2 x, x\right\rangle^r \\
&\leq (1-\alpha)\left\langle \left|f\left(|T|\right)A\right|^{2r} x, x\right\rangle + \alpha \left\langle \left|g\left(|T^*|\right)B^*\right|^{2r} x, x\right\rangle \\
&= \left\langle \left[(1-\alpha)\left|f\left(|T|\right)A\right|^{2r} + \alpha \left|g\left(|T^*|\right)B^*\right|^{2r}\right] x, x\right\rangle
\end{aligned}$$

and by (2.18)

$$\begin{aligned}
\left|\langle BTAx, x\rangle\right|^{2r} &\leq \left\langle \left[(1-\alpha)\left|f\left(|T|\right)A\right|^{2r} + \alpha \left|g\left(|T^*|\right)B^*\right|^{2r}\right] x, x\right\rangle \\
&\times \left\langle \left|f\left(|T|\right)A\right|^2 x, x\right\rangle^{r\alpha} \left\langle \left|g\left(|T^*|\right)B^*\right|^2 x, x\right\rangle^{r(1-\alpha)}
\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

If we take the supremum over $\|x\|=1$, then we get

$$\begin{aligned}\omega^{2r}(BTA) &= \sup_{\|x\|=1} |\langle BTAx, x\rangle|^{2r} \\ &\le \sup_{\|x\|=1} \left\langle \left[(1-\alpha)|f(|T|)A|^{2r} + \alpha |g(|T^*|)B^*|^{2r}\right] x, x\right\rangle \\ &\times \sup_{\|x\|=1} \left\langle |f(|T|)A|^2 x, x\right\rangle^{r\alpha} \sup_{\|x\|=1} \left\langle |g(|T^*|)B^*|^2 x, x\right\rangle^{r(1-\alpha)} \\ &= \left\|(1-\alpha)|f(|T|)A|^{2r} + \alpha |g(|T^*|)B^*|^{2r}\right\| \\ &\times \left\||f(|T|)A|^2\right\|^{r\alpha} \left\||g(|T^*|)B^*|^2\right\|^{r(1-\alpha)} \\ &= \left\|(1-\alpha)|f(|T|)A|^{2r} + \alpha |g(|T^*|)B^*|^{2r}\right\| \\ &\times \|f(|T|)A\|^{2r\alpha} \|g(|T^*|)B^*\|^{2r(1-\alpha)},\end{aligned}$$

which proves (2.16).

We also have

$$\begin{aligned}|\langle BTAx, x\rangle|^2 &\le \left\langle |f(|T|)A|^2 x, x\right\rangle^{1-\alpha} \left\langle |g(|T^*|)B^*|^2 x, x\right\rangle^{\alpha} \\ &\times \left\langle |f(|T|)A|^2 x, x\right\rangle^{\alpha} \left\langle |g(|T^*|)B^*|^2 x, x\right\rangle^{1-\alpha} \\ &\le \left[(1-\alpha)\left\langle |f(|T|)A|^2 x, x\right\rangle + \alpha \left\langle |g(|T^*|)B^*|^2 x, x\right\rangle\right] \\ &\times \left[\alpha\left\langle |f(|T|)A|^2 x, x\right\rangle + (1-\alpha) \left\langle |g(|T^*|)B^*|^2 x, x\right\rangle\right],\end{aligned}$$

for all $x \in H$, $\|x\|=1$.

This implies in the same way that

$$\begin{aligned}|\langle BTAx, x\rangle|^{2r} &\le \left\langle \left[(1-\alpha)|f(|T|)A|^{2r} + \alpha |g(|T^*|)B^*|^{2r}\right] x, x\right\rangle \\ &\times \left\langle \left[\alpha|f(|T|)A|^{2r} + (1-\alpha) |g(|T^*|)B^*|^{2r}\right] x, x\right\rangle\end{aligned}$$

for all $x \in H$, $\|x\|=1$, which proves (2.17). □

Remark 4. *Consider $f(t)=t^{\lambda}$, $g(t)=t^{1-\lambda}$ with $\lambda\in[0,1]$ in Theorem 4, then*

$$\begin{aligned}\omega^2(BTA) &\le \left\|(1-\alpha)\left||T|^{\lambda}A\right|^{2r} + \alpha\left||T^*|^{1-\lambda}B^*\right|^{2r}\right\|^{1/r} \\ &\times \left\||T|^{\lambda}A\right\|^{2\alpha} \left\||T^*|^{1-\lambda}B^*\right\|^{2(1-\alpha)}\end{aligned}$$

and

$$\begin{aligned}\omega^2(BTA) &\le \left\|(1-\alpha)\left||T|^{\lambda}A\right|^{2r} + \alpha\left||T^*|^{1-\lambda}B^*\right|^{2r}\right\|^{1/r} \\ &\times \left\|\alpha\left||T|^{\lambda}A\right|^{2r} + (1-\alpha)\left||T^*|^{1-\lambda}B^*\right|^{2r}\right\|^{1/r}\end{aligned}$$

for all $\alpha\in[0,1]$ and $r\ge 1$.

In particular, for $\lambda = 1/2$ we obtain

$$\omega^2(BTA) \le \left\| (1-\alpha)\left| |T|^{1/2} A \right|^{2r} + \alpha \left| |T^*|^{1/2} B^* \right|^{2r} \right\|^{1/r} \times \left\| |T|^{1/2} A \right\|^{2\alpha} \left\| |T^*|^{1/2} B^* \right\|^{2(1-\alpha)}$$

and

$$\omega^2(BTA) \le \left\| (1-\alpha)\left| |T|^{1/2} A \right|^{2r} + \alpha \left| |T^*|^{1/2} B^* \right|^{2r} \right\|^{1/r} \times \left\| \alpha \left| |T|^{1/2} A \right|^{2r} + (1-\alpha) \left| |T^*|^{1/2} B^* \right|^{2r} \right\|^{1/r}$$

for all $\alpha \in [0,1]$ and $r \ge 1$.

Corollary 1. *With the assumptions of Theorem 2 we have*

$$\omega^2(BTA) \le \frac{1}{2^{1/r}} \left\| |f(|T|)A|^{2r} + |g(|T^*|)B^*|^{2r} \right\|^{1/r} \|f(|T|)A\| \, \|g(|T^*|)B^*\|$$

for all $r \ge 1$.

Also, we have

$$\omega(BTA) \le \frac{1}{2^{1/r}} \left\| |f(|T|)A|^{2r} + |g(|T^*|)B^*|^{2r} \right\|^{1/r}$$

for all $r \ge 1$.

In particular, we have

$$\omega^2(BTA) \le \frac{1}{2} \left\| |f(|T|)A|^{2} + |g(|T^*|)B^*|^{2} \right\| \|f(|T|)A\| \, \|g(|T^*|)B^*\|$$

and

$$\omega(BTA) \le \frac{1}{2} \left\| |f(|T|)A|^{2} + |g(|T^*|)B^*|^{2} \right\|.$$

If we take $f(t) = t^\lambda$, $g(t) = t^{1-\lambda}$ with $\lambda \in [0,1]$ in Corollary 1, then we get

$$\omega^2(BTA) \le \frac{1}{2^{1/r}} \left\| \left| |T|^\lambda A \right|^{2r} + \left| |T^*|^{1-\lambda} B^* \right|^{2r} \right\|^{1/r} \left\| |T|^\lambda A \right\| \left\| |T^*|^{1-\lambda} B^* \right\|$$

for all $r \ge 1$.

Also, we have

$$\omega(BTA) \le \frac{1}{2^{1/r}} \left\| \left| |T|^\lambda A \right|^{2r} + \left| |T^*|^{1-\lambda} B^* \right|^{2r} \right\|^{1/r}$$

for all $r \ge 1$.

In particular, we obtain

$$\omega^2(BTA) \le \frac{1}{2} \left\| \left| |T|^\lambda A \right|^{2} + \left| |T^*|^{1-\lambda} B^* \right|^{2} \right\| \left\| |T|^\lambda A \right\| \left\| |T^*|^{1-\lambda} B^* \right\| \quad (2.19)$$

and

$$\omega(BTA) \le \frac{1}{2} \left\| \left| |T|^\lambda A \right|^{2} + \left| |T^*|^{1-\lambda} B^* \right|^{2} \right\|.$$

3. Some Further Bounds

In this section we provide some simpler upper bounds for the numerical radius of the product of three operators:

Proposition 1. *For any A, B, $T \in \mathcal{B}(H)$, we have*

$$\omega(BTA) \le \frac{1}{2}\left\| \left| |T|^{\lambda} A \right|^2 + \left| |T^*|^{1-\lambda} B^* \right|^2 \right\| \le \frac{1}{2}\left\| \|T\|^{2\lambda} |A|^2 + \|T\|^{2(1-\lambda)} |B^*|^2 \right\| \le \frac{1}{2}\|T\|^2 \left\| |A|^2 + |B^*|^2 \right\| \tag{3.1}$$

and, for $\lambda = 1/2$,

$$\omega(BTA) \le \frac{1}{2}\left\| \left| |T|^{1/2} A \right|^2 + \left| |T^*|^{1/2} B^* \right|^2 \right\| \le \frac{1}{2}\|T\| \left\| |A|^2 + |B^*|^2 \right\|.$$

Also

$$\begin{aligned} &\omega^2(BTA) \\ &\le \frac{1}{2}\left[\left\| |T|^{\lambda} A \right\|^2 \left\| |T^*|^{1-\lambda} B^* \right\|^2 + \left\| \left| |T^*|^{1-\lambda} B^* \right|^2 \left| |T|^{\lambda} A \right|^2 \right\| \right] \\ &\le \frac{1}{2}\left[\left\| |T|^{\lambda} A \right\| \left\| |T^*|^{1-\lambda} B^* \right\| \left(\left\| |T|^{\lambda} A \right\| \left\| |T^*|^{1-\lambda} B^* \right\| + \|T\| \|AB\| \right) \right] \\ &\le \frac{1}{2}\|T\|^2 \|A\| \|B\| \left(\|A\| \|B\| + \|AB\| \right) \end{aligned} \tag{3.2}$$

for all $\lambda \in [0,1]$.

Proof. Observe that, since $|T| \le \|T\| I$ and $|T^*| \le \|T\| I$, then

$$\begin{aligned} \left| |T|^{\lambda} A \right|^2 + \left| |T^*|^{1-\lambda} B^* \right|^2 &= A^* |T|^{2\lambda} A + B |T^*|^{2(1-\lambda)} B^* \\ &\le \|T\|^{2\lambda} A^*A + \|T\|^{2(1-\lambda)} BB^* \\ &\le \|T\|^2 (A^*A + BB^*), \end{aligned}$$

which implies that

$$\begin{aligned} \left\| \left| |T|^{\lambda} A \right|^2 + \left| |T^*|^{1-\lambda} B^* \right|^2 \right\| &\le \left\| \|T\|^{2\lambda} A^*A + \|T\|^{2(1-\lambda)} BB^* \right\| \\ &\le \|T\|^2 \left\| \left| |A|^2 + |B^*|^2 \right| \right\| \end{aligned}$$

and by (2.7) we get (3.1).

Further, observe that

$$\left| |T^*|^{1-\lambda} B^* \right|^2 \left| |T|^{\lambda} A \right|^2 = B |T^*|^{1-\lambda} |T^*|^{1-\lambda} B^* A^* |T|^{\lambda} |T|^{\lambda} A.$$

By taking the norm, we get

$$\begin{aligned} \left\| \left| |T^*|^{1-\lambda} B^* \right|^2 \left| |T|^{\lambda} A \right|^2 \right\| &\le \left\| B |T^*|^{1-\lambda} \right\| \left\| |T^*|^{1-\lambda} \right\| \|B^*A^*\| \left\| |T|^{\lambda} \right\| \left\| |T|^{\lambda} A \right\| \\ &= \left\| |T^*|^{1-\lambda} B^* \right\| \|T\|^{1-\lambda} \|AB\| \|T\|^{\lambda} \left\| |T|^{\lambda} A \right\| \\ &= \left\| |T|^{\lambda} A \right\| \left\| |T^*|^{1-\lambda} B^* \right\| \|T\| \|AB\|. \end{aligned}$$

Therefore

$$\begin{aligned}
&\left\| |T|^{\lambda} A \right\|^{2} \left\| |T^{*}|^{1-\lambda} B^{*} \right\|^{2} + \left\| \left| |T^{*}|^{1-\lambda} B^{*} \right|^{2} \left| |T|^{\lambda} A \right|^{2} \right\| \\
&\leq \left\| |T|^{\lambda} A \right\|^{2} \left\| |T^{*}|^{1-\lambda} B^{*} \right\|^{2} + \left\| |T|^{\lambda} A \right\| \left\| |T^{*}|^{1-\lambda} B^{*} \right\| \|T\| \|AB\| \\
&= \left\| |T|^{\lambda} A \right\| \left\| |T^{*}|^{1-\lambda} B^{*} \right\| \left(\left\| |T|^{\lambda} A \right\| \left\| |T^{*}|^{1-\lambda} B^{*} \right\| + \|T\| \|AB\| \right) \\
&\leq \|T\|^{\lambda} \|T\|^{1-\lambda} \|A\| \|B\| \left(\|T\|^{\lambda} \|T\|^{1-\lambda} \|A\| \|B\| + \|T\| \|AB\| \right) \\
&= \|T\| \|A\| \|B\| \left(\|T\| \|A\| \|B\| + \|T\| \|AB\| \right) \\
&= \|T\|^{2} \|A\| \|B\| \left(\|A\| \|B\| + \|T\| \|AB\| \right),
\end{aligned}$$

which proves (3.2). □

We also have:

Proposition 2. *For any A, B, $T \in \mathcal{B}(H)$, we have*

$$\omega^{2r}(BTA) \leq \frac{1}{2} \|T\|^{2r} \|A\|^{r} \|B\|^{r} \left(\|A\|^{r} \|B\|^{r} + \|AB\|^{r} \right) \tag{3.3}$$

for $r \geq 1$.

In particular, for $r = 2$ we obtain

$$\omega^{4}(BTA) \leq \frac{1}{2} \|T\|^{4} \|A\|^{2} \|B\|^{2} \left(\|A\|^{2} \|B\|^{2} + \|AB\|^{2} \right).$$

Also, for $\lambda \in [0,1]$

$$\begin{aligned}
\omega^{2}(BTA) &\leq \frac{1}{2} \|T\| \left\| \|T\|^{2\lambda} |A|^{2} + \|T\|^{2(1-\lambda)} |B^{*}|^{2} \right\| \|A\| \|B\| \\
&\leq \frac{1}{2} \|T\|^{3} \|A\| \|B\| \left\| |A|^{2} + |B^{*}|^{2} \right\|
\end{aligned} \tag{3.4}$$

and for $\lambda = 1/2$ we get

$$\omega^{2}(BTA) \leq \frac{1}{2} \|T\|^{2} \|A\| \|B\| \left\| |A|^{2} + |B^{*}|^{2} \right\|.$$

Proof. As in the proof of Proposition 1 we have by (2.14) that

$$\begin{aligned}
&\omega^{2r}(BTA) \\
&\leq \frac{1}{2}\left(\left\||T|^{\lambda} A\right\|^{2r}\left\||T^*|^{1-\lambda} B^*\right\|^{2r}+\left\|\left||T^*|^{1-\lambda} B^*\right|^2\left||T|^{\lambda} A\right|^2\right\|^r\right) \\
&= \frac{1}{2}\left\|\left||T|^{\lambda} A\right|^2\right\|^r\left\|\left||T^*|^{1-\lambda} B^*\right|^2\right\|^r \\
&+\frac{1}{2}\left\|B|T^*|^{1-\lambda}|T^*|^{1-\lambda} B^* A^*|T|^{\lambda}|T|^{\lambda} A\right\|^r \\
&= \frac{1}{2}\left\|A^*|T|^{2\lambda} A\right\|^r\left\|B|T^*|^{2(1-\lambda)} B^*\right\|^r \\
&+\frac{1}{2}\left\|B|T^*|^{1-\lambda}|T^*|^{1-\lambda} B^* A^*|T|^{\lambda}|T|^{\lambda} A\right\|^r \\
&\leq \frac{1}{2}\|T\|^{2\lambda r}\|A\|^{2r}\|T\|^{2(1-\lambda) r}\|B\|^{2r} \\
&+\frac{1}{2}\|B\|^r\|T\|^{2(1-\lambda) r}\|B^* A^*\|^r\|T\|^{2\lambda r}\|A\|^r \\
&= \frac{1}{2}\|T\|^{2r}\|A\|^{2r}\|B\|^{2r}+\frac{1}{2}\|B\|^r\|T\|^{2r}\|AB\|^r\|A\|^r \\
&= \frac{1}{2}\|T\|^{2r}\|A\|^r\|B\|^r\left(\|A\|^r\|B\|^r+\|AB\|^r\right),
\end{aligned}$$

which proves (3.3).

Also, from (2.19) we get

$$\begin{aligned}
\omega^2(BTA) &\leq \frac{1}{2}\left\|\left||T|^{\lambda} A\right|^2+\left||T^*|^{1-\lambda} B^*\right|^2\right\|\left\||T|^{\lambda} A\right\|\left\||T^*|^{1-\lambda} B^*\right\| \\
&\leq \frac{1}{2}\left\|\|T\|^{2\lambda}|A|^2+\|T\|^{2(1-\lambda)}|B^*|^2\right\|\|T\|^{\lambda}\|A\|\|T\|^{1-\lambda}\|B\| \\
&= \frac{1}{2}\|T\|\left\|\|T\|^{2\lambda}|A|^2+\|T\|^{2(1-\lambda)}|B^*|^2\right\|\|A\|\|B\| \\
&\leq \frac{1}{2}\|T\|^3\left\||A|^2+|B^*|^2\right\|\|A\|\|B\|,
\end{aligned}$$

which proves (3.4). □

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa