

Trigonometric and Hyperbolic Polya type inequalities

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Abstract

Here based on trigonometric and hyperbolic type Taylor formulae we derive Polya type inequalities in a number of cases.

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1 Main Results

We present a collection of Polya's type inequalities.

Theorem 1 Let $f \in C^2([a, b], K)$, where $K = \mathbb{R}$ or \mathbb{C} , such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We set

$$M_1 := \max \left\{ \|f'' + f\|_{\infty, [a, \frac{a+b}{2}]}, \|f'' + f\|_{\infty, [\frac{a+b}{2}, b]} \right\}, \quad (1)$$

$$M_2 := \max \left\{ \|f'' + f\|_{L_1([a, \frac{a+b}{2}])}, \|f'' + f\|_{L_1([\frac{a+b}{2}, b])} \right\}, \quad (2)$$

and

$$M_3 := \max \left\{ \|f'' + f\|_{L_q([a, \frac{a+b}{2}])}, \|f'' + f\|_{L_q([\frac{a+b}{2}, b])} \right\}. \quad (3)$$

Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \leq \min \left\{ \begin{array}{l} \frac{(b-a)^3}{8} M_1, \\ \frac{(b-a)^2}{4} M_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{1+\frac{1}{p}} (p+1)^{\frac{1}{p}} (2+\frac{1}{p})} M_3 \end{array} \right\}. \quad (4)$$

Proof. Here $f \in C^2([a, b], K)$, such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1$. By Corollary 3.4 of [1], we have

$$f(x) = \int_a^x (f''(t) + f(t)) \sin(x-t) dt, \quad (5)$$

and

$$f(x) = \int_x^b (f''(t) + f(t)) \sin(x-t) dt. \quad (6)$$

By using $|\sin x| \leq |x|$, $\forall x \in \mathbb{R}$, we obtain

$$\begin{aligned} |f(x)| &\leq \int_a^x |f''(t) + f(t)| |\sin(x-t)| dt \leq \\ &\int_a^x |f''(t) + f(t)| (x-t) dt \leq \left(\int_a^x (x-t) dt \right) \|f'' + f\|_{\infty, [a, \frac{a+b}{2}]} = \\ &\frac{(x-a)^2}{2} \|f'' + f\|_{\infty, [a, \frac{a+b}{2}]}, \quad \forall x \in \left[a, \frac{a+b}{2} \right]. \end{aligned} \quad (7)$$

Also it holds

$$\begin{aligned} |f(x)| &\leq (x-a) \left(\int_a^x |f''(t) + f(t)| dt \right) \leq \\ &(x-a) \|f'' + f\|_{L_1([a, \frac{a+b}{2}])}, \quad \forall x \in \left[a, \frac{a+b}{2} \right]. \end{aligned} \quad (8)$$

Furthermore, by Hölder's inequality, we have $(p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1)$

$$\begin{aligned} |f(x)| &\leq \left(\int_a^x |f''(t) + f(t)|^q dt \right)^{\frac{1}{q}} \left(\int_a^x (x-t)^p dt \right)^{\frac{1}{p}} \leq \\ &\|f'' + f\|_{L_q([a, \frac{a+b}{2}])} \frac{(x-a)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}}, \quad \forall x \in \left[a, \frac{a+b}{2} \right]. \end{aligned} \quad (9)$$

We have found that

$$|f(x)| \leq \left\{ \begin{array}{l} \frac{(x-a)^2}{2} \|f'' + f\|_{\infty, [a, \frac{a+b}{2}]}, \\ (x-a) \|f'' + f\|_{L_1([a, \frac{a+b}{2}])}, \\ \frac{(x-a)^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}} \|f'' + f\|_{L_q([a, \frac{a+b}{2}])}, \quad p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \end{array} \right\}, \quad (10)$$

$\forall x \in [a, \frac{a+b}{2}]$.

Similarly acting, we get that

$$|f(x)| = \left| \int_x^b (f''(t) + f(t)) \sin(x-t) dt \right| \leq$$

$$\int_x^b |f''(t) + f(t)| |\sin(t-x)| dt \leq \quad (11)$$

$$\int_x^b |f''(t) + f(t)| (t-x) dt,$$

and

$$|f(x)| \leq \left\{ \begin{array}{l} \frac{(b-x)^2}{2} \|f'' + f\|_{\infty, [\frac{a+b}{2}, b]}, \\ (b-x) \|f'' + f\|_{L_1([\frac{a+b}{2}, b])}, \\ \frac{(b-x)^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}} \|f'' + f\|_{L_q([\frac{a+b}{2}, b])}, \quad p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \end{array} \right\}, \quad (12)$$

$\forall x \in [\frac{a+b}{2}, b]$.

Consequently, we obtain

$$\int_a^{\frac{a+b}{2}} |f(x)| dx \leq \left\{ \begin{array}{l} \frac{(b-a)^3}{16} \|f'' + f\|_{\infty, [a, \frac{a+b}{2}]}, \\ \frac{(b-a)^2}{8} \|f'' + f\|_{L_1([a, \frac{a+b}{2}])}, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{2+\frac{1}{p}}(p+1)^{\frac{1}{p}}(2+\frac{1}{p})} \|f'' + f\|_{L_q([a, \frac{a+b}{2}])}, \quad p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1. \end{array} \right\} \quad (13)$$

Similarly, we derive that

$$\int_{\frac{a+b}{2}}^b |f(x)| dx \leq \left\{ \begin{array}{l} \frac{(b-a)^3}{16} \|f'' + f\|_{\infty, [\frac{a+b}{2}, b]}, \\ \frac{(b-a)^2}{8} \|f'' + f\|_{L_1([\frac{a+b}{2}, b])}, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{2+\frac{1}{p}}(p+1)^{\frac{1}{p}}(2+\frac{1}{p})} \|f'' + f\|_{L_q([\frac{a+b}{2}, b])}, \quad p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1. \end{array} \right\} \quad (14)$$

We have that

$$\int_a^b |f(x)| dx = \int_a^{\frac{a+b}{2}} |f(x)| dx + \int_{\frac{a+b}{2}}^b |f(x)| dx \leq$$

$$\left\{ \begin{array}{l} \frac{(b-a)^3}{16} \left[\|f'' + f\|_{\infty, [a, \frac{a+b}{2}]} + \|f'' + f\|_{\infty, [\frac{a+b}{2}, b]} \right], \\ \frac{(b-a)^2}{8} \left[\|f'' + f\|_{L_1([a, \frac{a+b}{2}])} + \|f'' + f\|_{L_1([\frac{a+b}{2}, b])} \right], \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{2+\frac{1}{p}}(p+1)^{\frac{1}{p}}(2+\frac{1}{p})} \left[\|f'' + f\|_{L_q([a, \frac{a+b}{2}])} + \|f'' + f\|_{L_q([\frac{a+b}{2}, b])} \right]. \end{array} \right\} \leq \quad (15)$$

$$\left\{ \begin{array}{l} \frac{(b-a)^3}{8} M_1, \\ \frac{(b-a)^2}{4} M_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}(2+\frac{1}{p})} M_3 \end{array} \right\}. \quad (16)$$

The claim is proved. ■

We continue with

Theorem 2 All as in Theorem 1. Denote

$$M_1^* := \max \left\{ \|f'' - f\|_{\infty, [a, \frac{a+b}{2}]}, \|f'' - f\|_{\infty, [\frac{a+b}{2}, b]} \right\}, \quad (17)$$

$$M_2^* := \max \left\{ \|f'' - f\|_{L_1([a, \frac{a+b}{2}])}, \|f'' - f\|_{L_1([\frac{a+b}{2}, b])} \right\}, \quad (18)$$

and

$$M_3^* := \max \left\{ \|f'' - f\|_{L_q([a, \frac{a+b}{2}])}, \|f'' - f\|_{L_q([\frac{a+b}{2}, b])} \right\}. \quad (19)$$

Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \leq \min \cosh(b-a) \times \left\{ \begin{array}{l} \frac{(b-a)^3}{8} M_1^*, \\ \frac{(b-a)^2}{4} M_2^*, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}(2+\frac{1}{p})} M_3^* \end{array} \right\}. \quad (20)$$

Proof. As similar to Theorem 1 it is omitted. It based on Corollary 3.5 of [1]. Also we use that $|\sinh x| \leq \cosh(b-a)|x|$, $\forall x \in [-(b-a), b-a]$, by the mean value theorem. ■

It follows

Theorem 3 Let $f \in C^4([a, b], K)$, where $K = \mathbb{R}$ or \mathbb{C} , such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, 2, 3$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We set

$$A_1 := \max \left\{ \|f^{(4)} - f\|_{\infty, [a, \frac{a+b}{2}]}, \|f^{(4)} - f\|_{\infty, [\frac{a+b}{2}, b]} \right\}, \quad (21)$$

$$A_2 := \max \left\{ \|f^{(4)} - f\|_{L_1([a, \frac{a+b}{2}])}, \|f^{(4)} - f\|_{L_1([\frac{a+b}{2}, b])} \right\}, \quad (22)$$

and

$$A_3 := \max \left\{ \|f^{(4)} - f\|_{L_q([a, \frac{a+b}{2}])}, \|f^{(4)} - f\|_{L_q([\frac{a+b}{2}, b])} \right\}. \quad (23)$$

Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \leq \min \frac{(\cosh(b-a) + 1)}{4} \times \left\{ \begin{array}{l} \frac{(b-a)^3}{4} A_1, \\ \frac{(b-a)^2}{2} A_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}(2+\frac{1}{p})} A_3 \end{array} \right\}. \quad (24)$$

Proof. As similar to Theorem 1 it is omitted. It is based on Corollary 3.6 of [1]. ■

We continue with

Theorem 4 Let $f \in C^4([a, b], K)$, where $K = \mathbb{R}$ or \mathbb{C} , such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, 2, 3$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let also $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha^2 - \beta^2) \neq 0$. We set

$$B_1 := \max \left\{ \left\| f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{\infty, [a, \frac{a+b}{2}]}, \right. \\ \left. \left\| f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{\infty, [\frac{a+b}{2}, b]} \right\}, \quad (25)$$

$$B_2 := \max \left\{ \left\| f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{L_1([a, \frac{a+b}{2}])}, \right. \\ \left. \left\| f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{L_1([\frac{a+b}{2}, b])} \right\}, \quad (26)$$

and

$$B_3 := \max \left\{ \left\| f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{L_q([a, \frac{a+b}{2}])}, \right. \\ \left. \left\| f^{(4)} + (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{L_q([\frac{a+b}{2}, b])} \right\}. \quad (27)$$

Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \leq \\ \min \frac{1}{|\beta^2 - \alpha^2|} \times \begin{cases} \frac{(b-a)^3}{4} B_1, \\ \frac{(b-a)^2}{2} B_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{\frac{1}{p}} (p+1)^{\frac{1}{p}} (2+\frac{1}{p})} B_3 \end{cases}. \quad (28)$$

Proof. As similar to Theorem 1 it is omitted. It is based on Corollary 3.7 of [1]. ■

We finish with

Theorem 5 All as in Theorem 4. However here, instead of B_1, B_2, B_3 , we set

$$D_1 := \max \left\{ \left\| f^{(4)} - (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{\infty, [a, \frac{a+b}{2}]}, \right. \\ \left. \left\| f^{(4)} - (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{\infty, [\frac{a+b}{2}, b]} \right\}, \quad (29)$$

$$D_2 := \max \left\{ \left\| f^{(4)} - (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{L_1([a, \frac{a+b}{2}])}, \right. \\ \left. \left\| f^{(4)} - (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{L_1([\frac{a+b}{2}, b])} \right\}, \quad (30)$$

and

$$D_3 := \max \left\{ \left\| f^{(4)} - (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{L_q([a, \frac{a+b}{2}])}, \right. \\ \left. \left\| f^{(4)} - (\alpha^2 + \beta^2) f'' + \alpha^2 \beta^2 f \right\|_{L_q([\frac{a+b}{2}, b])} \right\}. \quad (31)$$

Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \leq \\ \min \frac{\cosh(b-a)}{|\beta^2 - \alpha^2|} \times \begin{cases} \frac{(b-a)^3}{4} D_1, \\ \frac{(b-a)^2}{2} D_2, \\ \frac{(b-a)^{2+\frac{1}{p}}}{2^{\frac{1}{p}} (p+1)^{\frac{1}{p}} (2+\frac{1}{p})} D_3 \end{cases}. \quad (32)$$

Proof. As similar to Theorem 1 it is omitted. It is based on Corollary 3.9 of [1]. ■

References

- [1] Ali Hasan Ali and Zsolt Páles, *Taylor-type expansions in terms of exponential polynomials*, *Mathematical Inequalities & Applications*, 25(4) (2022), 1123-1141.