Trigonometric and hyperbolic Korovkin theory

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Abstract

Based on trigonometric and hyperbolic Taylor's type formulae we establish related Shisha-Mond form inequalities leading to interesting Korovkin theorems. We deal with the high order of approximation of positive linear operators to the unit operator. The results are quantitative via the modulus of continuity. We finish with applications to Bernstein operators.

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1 Introduction

In this article mainly we are motivated by the following result.

Theorem 1 *(P.P. Korovkin [3], (1960), p. 14) Let* $[a,b]$ *be a closed interval in* $\mathbb{R}$ *and* $(L_n)_{n\in\mathbb{N}}$ *be a sequence of positive linear operators mapping* $C([a,b])$ *into itself. Suppose that* $(L_n f)$ *converges uniformly to* f *for the three test functions* $f = 1,\ x,\ x^2$. *Then* $(L_n f)$ *converges uniformly to* f *on* $[a,b]$ *for all functions* $f \in C([a,b])$.

Let $f \in C([a,b])$ and $0 \leq h \leq b-a$. The first modulus of continuity of f at h is given by

$$\omega_1(f,h) = \sup_{\substack{x,y\in[a,b] \\ |x-y|\leq h}} |f(x) - f(y)|.$$

If $h > b-a$, then we define $\omega_1(f,h) = \omega_1(f,b-a)$.

Another motivation is the following.

Theorem 2 *(Shisha and Mond [4], (1968)) Let* $[a,b] \subset \mathbb{R}$ *a closed interval. Let* $\{L_n\}_{n\in\mathbb{N}}$ *be a sequence of positive linear operators acting on* $C([a,b])$ *into*

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itself. For $n = 1, ...,$ suppose $L_n(1)$ is bounded. Let $f \in C([a,b])$. Then for $n = 1, 2, ...,$ we have

(i)

$$\|L_n f - f\|_\infty \leq \|f\|_\infty \|L_n 1 - 1\|_\infty + \|L_n 1 + 1\|_\infty \omega_1(f, \mu_n)$$

where

$$\mu_n = \left\| L_n\left((t-x)\right)^2(x) \right\|_\infty^{\frac{1}{2}}$$

and $\|\cdot\|_\infty$ stands for the sup-norm over $[a,b]$.

One can easily see, for $n = 1, 2, ...$

(ii)

$$\mu_n^2 \leq \left\| L_n\left(t^2; x\right) - x^2 \right\|_\infty + 2c \|L_n(t; x) - x\|_\infty + c^2 \|L_n(1; x) - 1\|_\infty,$$

where $c = \max(|a|, |b|)$.

Thus, given the Korovkin assumptions (see Theorem 1) as $n \to \infty$ we get by (ii) that $\mu_n \to 0$, and by (i) that $\|L_n f - f\|_\infty \to 0$ for any $f \in C([a,b])$. That is one derives the Korovkin conclusion in a quantitative way and with rates of convergence.

Here by the use of trigonometric and hyperbolic Taylor's formulae we derive a higher order of quantitative approximation for sequences of positive linear operators converging to the unit operator. This is along with the related Shisha-Mond type inequalities and corresponding Korovkin theorems.

2 Main Results

2.1 Trigonometric approximation by positive linear operators

We present the following quantitative results.

Theorem 3 *Let $f \in C^2([a,b])$, $x_0 \in [a,b]$, and a sequence of positive linear operators $(L_n)_{n \in \mathbb{N}}$ from $C([a,b])$ into itself, $r > 0$, and $D_3(x_0) := \left(L_n\left(|\cdot - x_0|^3\right)(x_0)\right)^{\frac{1}{3}}$.*

Then

(i)

$$|L_n(f)(x_0) - f(x_0)| \leq |f(x_0)| |L_n(1)(x_0) - 1| +$$

$$|f'(x_0)| |L_n(\sin(\cdot - x_0))(x_0)| + 2|f''(x_0)| \left(L_n\left(\sin^2\left(\frac{\cdot - x_0}{2}\right)\right)(x_0)\right)$$

$$+\frac{\omega_1(f'' + f, rD_3(x_0))}{2} D_3^2(x_0) \left[(L_n(1)(x_0))^{\frac{1}{3}} + \frac{1}{3r}\right], \quad (1)$$

(ii)

$$\|L_n(f)-f\|_\infty \leq \|f\|_\infty \|L_n(1)-1\|_\infty +$$

$$\|f'\|_\infty \|L_n(\sin(\cdot - x_0))(x_0)\|_\infty + 2\|f''\|_\infty \left\| L_n\left(\sin^2\left(\frac{\cdot - x_0}{2}\right)\right)(x_0)\right\|_\infty$$

$$+\frac{\omega_1\left(f''+f, r\|D_3\|_\infty\right)}{2}\|D_3\|_\infty^2\left[\|L_n(1)\|_\infty^{\frac{1}{3}}+\frac{1}{3r}\right], \tag{2}$$

(iii) if $L_n(1)=1$, and $f'(x_0)=f''(x_0)=0$ for some $x_0\in[a,b]$, we obtain

$$|L_n(f)(x_0)-f(x_0)| \leq$$

$$\frac{\omega_1\left(f''+f, rD_3(x_0)\right)}{2}D_3^2(x_0)\left[(L_n(1)(x_0))^{\frac{1}{3}}+\frac{1}{3r}\right], \tag{3}$$

and

(iv)

$$|L_n(f)(x_0)-f(x_0)L_n(1)(x_0)-f'(x_0)L_n(\sin(\cdot - x_0))(x_0)$$

$$-2f''(x_0)L_n\left(\sin^2\left(\frac{\cdot - x_0}{2}\right)\right)(x_0)\bigg| \leq$$

$$\frac{\omega_1\left(f''+f, rD_3(x_0)\right)}{2}D_3^2(x_0)\left[(L_n(1)(x_0))^{\frac{1}{3}}+\frac{1}{3r}\right]. \tag{4}$$

Proof. Here $f\in C^2([a,b])$, with $x, x_0\in[a,b]$. By [2], we have that

$$f(x)-f(x_0)=f'(x_0)\sin(x-x_0)+2f''(x_0)\sin^2\left(\frac{x-x_0}{2}\right)+$$

$$\int_{x_0}^{x}\left[(f''(t)+f(t))-(f''(x_0)+f(x_0))\right]\sin(x-t)\,dt. \tag{5}$$

Denote by

$$R(x,x_0):=\int_{x_0}^{x}\left[(f''(t)+f(t))-(f''(x_0)+f(x_0))\right]\sin(x-t)\,dt. \tag{6}$$

Clearly, we have that

$$|(f''(t)+f(t))-(f''(x_0)+f(x_0))| \leq$$

$$\omega_1(f''+f,h)\left(1+\frac{|t-x_0|}{h}\right), \quad \text{all } t\in[a,b],\ h>0. \tag{7}$$

Let $x\geq x_0$, then

$$|R(x,x_0)| \leq \int_{x_0}^{x}|(f''(t)+f(t))-(f''(x_0)+f(x_0))|\,|\sin(x-t)|\,dt \overset{(7)}{\leq} \tag{8}$$

(using $|\sin x| \leq |x|$, $\forall\ x \in \mathbb{R}$)

$$\omega_1 \left(f'' + f, h\right) \int_{x_0}^{x} \left(1 + \frac{(t - x_0)}{h}\right) (x - t)\, dt =$$

$$\omega_1 \left(f'' + f, h\right) \left[\int_{x_0}^{x} (x - t)\, dt + \frac{1}{h} \int_{x_0}^{x} (x - t)(t - x_0)\, dt\right] =$$

$$\omega_1 \left(f'' + f, h\right) \left[\frac{(x - x_0)^2}{2} + \frac{(x - x_0)^3}{6h}\right]. \tag{9}$$

So, if $x \geq x_0$, we got that

$$|R(x, x_0)| \leq \frac{\omega_1 \left(f'' + f, h\right)}{2} \left[(x - x_0)^2 + \frac{(x - x_0)^3}{3h}\right], \quad h > 0. \tag{10}$$

Let now $x < x_0$, then

$$|R(x, x_0)| = \left|\int_{x}^{x_0} \left[\left(f''(t) + f(t)\right) - \left(f''(x_0) + f(x_0)\right)\right] \sin(x - t)\, dt\right| \leq$$

$$\int_{x}^{x_0} \left|\left(f''(t) + f(t)\right) - \left(f''(x_0) + f(x_0)\right)\right| |\sin(x - t)|\, dt \leq \tag{11}$$

$$\omega_1 \left(f'' + f, h\right) \int_{x}^{x_0} \left(1 + \frac{(x_0 - t)}{h}\right) (t - x)\, dt =$$

$$\omega_1 \left(f'' + f, h\right) \left[\int_{x}^{x_0} (t - x)\, dt + \frac{1}{h} \int_{x}^{x_0} (x_0 - t)(t - x)\, dt\right] =$$

$$\omega_1 \left(f'' + f, h\right) \left[\frac{(x_0 - x)^2}{2} + \frac{(x_0 - x)^3}{6h}\right]. \tag{12}$$

So, if $x < x_0$, we got that

$$|R(x, x_0)| \leq \frac{\omega_1 \left(f'' + f, h\right)}{2} \left[(x_0 - x)^2 + \frac{(x_0 - x)^3}{3h}\right]. \tag{13}$$

Thus, it holds

$$|R(x, x_0)| \leq \frac{\omega_1 \left(f'' + f, h\right)}{2} \left[(x - x_0)^2 + \frac{|x - x_0|^3}{3h}\right], \quad h > 0, \tag{14}$$

$\forall\ x,\ x_0 \in [a, b]$.

Let a sequence of positive linear operators $(L_n)_{n \in \mathbb{N}}$ from $C([a, b])$ into itself.

We notice that

$$|R(x, x_0)| \leq |R(x, x_0)|, \quad \text{iff } -|R(x, x_0)| \leq R(x, x_0) \leq |R(x, x_0)|,$$

and

$$-L_n\left(\left|R\left(x,x_0\right)\right|\right) \leq L_n\left(R\left(x,x_0\right)\right) \leq L_n\left(\left|R\left(x,x_0\right)\right|\right).$$

That is

$$\left|L_n\left(R\left(x,x_0\right)\right)\right| \leq L_n\left(\left|R\left(x,x_0\right)\right|\right). \tag{15}$$

By (5) we can write

$$f\left(\cdot\right)-f\left(x_0\right)=f'\left(x_0\right)\sin\left(\cdot-x_0\right)+2f''\left(x_0\right)\sin^2\left(\frac{\cdot-x_0}{2}\right)+R\left(\cdot,x_0\right). \tag{16}$$

Thus, it holds

$$L_n\left(f\right)\left(x_0\right)-f\left(x_0\right)L_n\left(1\right)\left(x_0\right)=$$

$$f'\left(x_0\right)L_n\left(\sin\left(\cdot-x_0\right)\right)\left(x_0\right)+2f''\left(x_0\right)L_n\left(\sin^2\left(\frac{\cdot-x_0}{2}\right)\right)\left(x_0\right)+L_n\left(R\left(\cdot,x_0\right)\right)\left(x_0\right), \tag{17}$$

$\forall\ x_0\in[a,b]$.

Consequently we have

$$\left|L_n\left(f\right)\left(x_0\right)-f\left(x_0\right)L_n\left(1\right)\left(x_0\right)\right| \overset{(17)}{\leq}$$

$$\left|f'\left(x_0\right)\right|\left|L_n\left(\sin\left(\cdot-x_0\right)\right)\left(x_0\right)\right|+2\left|f''\left(x_0\right)\right|\left(L_n\left(\sin^2\left(\frac{\cdot-x_0}{2}\right)\right)\left(x_0\right)\right)$$

$$+L_n\left(\left|R\left(\cdot,x_0\right)\right|\right)\leq \tag{18}$$

(by (14))

$$\left|f'\left(x_0\right)\right|\left|L_n\left(\sin\left(\cdot-x_0\right)\right)\left(x_0\right)\right|+2\left|f''\left(x_0\right)\right|\left(L_n\left(\sin^2\left(\frac{\cdot-x_0}{2}\right)\right)\left(x_0\right)\right)+$$

$$\frac{\omega_1\left(f''+f,h\right)}{2}\left[L_n\left(\left(\cdot-x_0\right)^2\right)\left(x_0\right)+\frac{L_n\left(\left|\cdot-x_0\right|^3\right)\left(x_0\right)}{3h}\right], \tag{19}$$

$\forall\ x_0\in[a,b]$.

We observe that

$$\left|L_n\left(f\right)\left(x_0\right)-f\left(x_0\right)\right|=$$

$$\left|L_n\left(f\right)\left(x_0\right)-f\left(x_0\right)-f\left(x_0\right)L_n\left(1\right)x_0+f\left(x_0\right)L_n\left(1\right)\left(x_0\right)\right|=$$

$$\left|\left(L_n\left(f\right)\left(x_0\right)-f\left(x_0\right)L_n\left(1\right)\left(x_0\right)\right)+f\left(x_0\right)\left(L_n\left(1\right)\left(x_0\right)-1\right)\right|\leq$$

$$\left|L_n\left(f\right)\left(x_0\right)-f\left(x_0\right)L_n\left(1\right)\left(x_0\right)\right|+\left|f\left(x_0\right)\right|\left|L_n\left(1\right)\left(x_0\right)-1\right|. \tag{20}$$

We have found that (by (19), (20))

$$\left|L_n\left(f\right)\left(x_0\right)-f\left(x_0\right)\right|\leq\left|f\left(x_0\right)\right|\left|L_n\left(1\right)\left(x_0\right)-1\right|+$$

$$|f'(x_0)|\,|L_n(\sin(\cdot - x_0))(x_0)| + 2\,|f''(x_0)|\left(L_n\left(\sin^2\left(\frac{\cdot - x_0}{2}\right)\right)(x_0)\right)$$

$$+\frac{\omega_1(f''+f,h)}{2}\left[L_n\left((\cdot - x_0)^2\right)(x_0) + \frac{L_n\left(|\cdot - x_0|^3\right)(x_0)}{3h}\right] \le \tag{21}$$

$$|f(x_0)|\,|L_n(1)(x_0)-1| + |f'(x_0)|\,L_n(|\cdot - x_0|)(x_0) + \frac{|f''(x_0)|}{2}L_n\left((\cdot - x_0)^2\right)(x_0)$$

$$+\frac{\omega_1(f''+f,h)}{2}\left[L_n\left((\cdot - x_0)^2\right)(x_0) + \frac{L_n\left(|\cdot - x_0|^3\right)(x_0)}{3h}\right] =: (\psi), \tag{22}$$

$\forall\ x_0 \in [a,b]$, $h > 0$.

By Riesz representation theorem and Hölder's inequality we get

$$L_n\left((\cdot - x_0)^2\right)(x_0) \le \left(L_n\left(|\cdot - x_0|^3\right)(x_0)\right)^{\frac{2}{3}}\left(L_n(1)(x_0)\right)^{\frac{1}{3}}, \tag{23}$$

$\forall\ x_0 \in [a,b]$.

Let $r > 0$, and take

$$h := r\left(L_n\left(|\cdot - x_0|^3\right)(x_0)\right)^{\frac{1}{3}} = rD_3(x_0), \tag{24}$$

where

$$D_3(x_0) := \left(L_n|\cdot - x_0|^3(x_0)\right)^{\frac{1}{3}}. \tag{25}$$

That is

$$\left(L_n(\cdot - x_0)^2\right)(x_0) \le D_3^2(x_0)\left(L_n(1)(x_0)\right)^{\frac{1}{3}}. \tag{26}$$

Momentarily we assume that $D_3(x_0) > 0$. Therefore it holds

$$\left[L_n\left((\cdot - x_0)^2\right)(x_0) + \frac{L_n\left(|\cdot - x_0|^3\right)(x_0)}{3h}\right] \le$$

$$\left[D_3^2(x_0)\left(L_n(1)(x_0)\right)^{\frac{1}{3}} + \frac{D_3^3(x_0)}{3rD_3(x_0)}\right] = \tag{27}$$

$$D_3^2(x_0)\left[\left(L_n(1)(x_0)\right)^{\frac{1}{3}} + \frac{1}{3r}\right].$$

We continue as follows

$$|L_n(f)(x_0) - f(x_0)| \le (\psi) \overset{(27)}{\le} |f(x_0)|\,|L_n(1)(x_0)-1| +$$

$$|f'(x_0)|\,L_n(|\cdot - x_0|)(x_0) + \frac{|f''(x_0)|}{2}L_n\left((\cdot - x_0)^2\right)(x_0) +$$

$$\frac{\omega_1\left(f''+f, rD_3\left(x_0\right)\right)}{2} D_3^2\left(x_0\right)\left[\left(L_n\left(1\right)\left(x_0\right)\right)^{\frac{1}{3}}+\frac{1}{3r}\right]. \tag{28}$$

Notice also in general that

$$L_n\left(\left|\cdot - x_0\right|\right)\left(x_0\right) \leq \left(L_n\left(\left(\cdot - x_0\right)^2\right)\left(x_0\right)\right)^{\frac{1}{2}}\left(L_n\left(1\right)\left(x_0\right)\right)^{\frac{1}{2}} \leq \tag{29}$$

$$D_3\left(x_0\right)\left(L_n\left(1\right)\left(x_0\right)\right)^{\frac{2}{3}}.$$

In case of $D_3\left(x_0\right)=0$, by (22), we have that

$$\left|L_n\left(f\right)\left(x_0\right)-f\left(x_0\right)\right| \leq \left|f\left(x_0\right)\right|\left|L_n\left(1\right)\left(x_0\right)-1\right|. \tag{30}$$

By Riesz representation theorem we have that

$$L_n\left(f\right)\left(x_0\right)=\int_{[a,b]} f\left(t\right) d\mu_{x_0}\left(t\right), \tag{31}$$

where μ_{x_0} is a positive finite Borel measure such that $\mu_{x_0}\left([a,b]\right)=M>0$. Notice that $M=L_n\left(1\right)\left(x_0\right)$.

So, here we have that

$$D_3^3\left(x_0\right)=L_n\left(\left|\cdot - x_0\right|^3\right)\left(x_0\right)=\int_{[a,b]}\left|t-x_0\right|^3 d\mu_{x_0}\left(t\right)=0, \tag{32}$$

which implies $t=x_0$, a.e, consequently we have that $\mu_{x_0}=\delta_{x_0}M$, where δ_{x_0} is the Dirac measure at x_0.

Thus

$$L_n\left(f\right)\left(x_0\right)=f\left(x_0\right)M=f\left(x_0\right)L_n\left(1\right)\left(x_0\right). \tag{33}$$

Hence we derive

$$\left|L_n\left(f\right)\left(x_0\right)-f\left(x_0\right)\right|=\left|f\left(x_0\right)L_n\left(1\right)\left(x_0\right)-f\left(x_0\right)\right|=\left|f\left(x_0\right)\right|\left|L_n\left(1\right)\left(x_0\right)-1\right|. \tag{34}$$

So (30) becomes equality. Thus, (28) is valid always.

By [1], p. 388, $L_n\left(\left|\cdot - x_0\right|^3\right)\left(x_0\right)$ is continuous in $x_0 \in [a,b]$.

The theorem is proved. ■

It follows a Korovkin type theorem.

Theorem 4 *Here all as in Theorem 3. Assume that* $L_n\left(1\right)\to 1$, *and* $L_n\left(\left|\cdot - x_0\right|^3\right)\left(x_0\right)\to 0$, *both uniformly convergent, as* $n\to\infty$. *Then* $L_n\left(f\right)\to f$, *uniformly convergent, as* $n\to\infty$, $\forall\ f\in C^2\left([a,b]\right)$.

Proof. By (2); using (23), (29), and that every uniformly convergent sequence of bounded functions is uniformly bounded. Also that $\omega_1\left(f''+f, r\left\|D_3\right\|_\infty\right)\to 0$, as $n\to\infty$. ■

2.2 Hyperbolic approximation by positive linear operators

We give the following related results.

Theorem 5 *Let* $f \in C^2([a,b])$, $x_0 \in [a,b]$, *and a sequence of positive linear operators* $(L_n)_{n\in\mathbb{N}}$ *from* $C([a,b])$ *into itself,* $r>0$, *and* $D_3(x_0) := \left(L_n\left(|\cdot - x_0|^3\right)(x_0)\right)^{\frac{1}{3}}$. *Then*

(i)

$$|L_n(f)(x_0) - f(x_0)| \leq |f(x_0)| \, |L_n(1)(x_0) - 1| +$$

$$|f'(x_0)| \, |L_n(\sinh(\cdot - x_0))(x_0)| + 2|f''(x_0)| \left(L_n\left(\sinh^2\left(\frac{\cdot - x_0}{2}\right)\right)(x_0)\right)$$

$$+\frac{\cosh(b-a)\,\omega_1(f''-f, rD_3(x_0))}{2} D_3^2(x_0)\left[(L_n(1)(x_0))^{\frac{1}{3}} + \frac{1}{3r}\right], \tag{35}$$

(ii)

$$\|L_n(f) - f\|_\infty \leq \|f\|_\infty \|L_n(1) - 1\|_\infty +$$

$$\|f'\|_\infty \|L_n(\sinh(\cdot - x_0))(x_0)\|_\infty + 2\|f''\|_\infty \left\|L_n\left(\sinh^2\left(\frac{\cdot - x_0}{2}\right)\right)(x_0)\right\|_\infty$$

$$+\frac{\cosh(b-a)\,\omega_1(f''-f, r\|D_3\|_\infty)}{2} \|D_3\|_\infty^2 \left[\|L_n(1)\|_\infty^{\frac{1}{3}} + \frac{1}{3r}\right], \tag{36}$$

(iii) if $L_n(1) = 1$, *and* $f'(x_0) = f''(x_0) = 0$ *for some* $x_0 \in [a,b]$, *we obtain*

$$|L_n(f)(x_0) - f(x_0)| \leq$$

$$\frac{\cosh(b-a)\,\omega_1(f''-f, rD_3(x_0))}{2} D_3^2(x_0)\left[(L_n(1)(x_0))^{\frac{1}{3}} + \frac{1}{3r}\right], \tag{37}$$

and

(iv)

$$\Big|L_n(f)(x_0) - f(x_0)L_n(1)(x_0) - f'(x_0)L_n(\sinh(\cdot - x_0))(x_0)$$

$$-2f''(x_0)L_n\left(\sinh^2\left(\frac{\cdot - x_0}{2}\right)\right)(x_0)\Big| \leq$$

$$\frac{\cosh(b-a)\,\omega_1(f''-f, rD_3(x_0))}{2} D_3^2(x_0)\left[(L_n(1)(x_0))^{\frac{1}{3}} + \frac{1}{3r}\right]. \tag{38}$$

Proof. Here $f \in C^2([a,b])$, with $x, x_0 \in [a,b]$. By [2], we have that

$$f(x) - f(x_0) = f'(x_0)\sinh(x - x_0) + 2f''(x_0)\sinh^2\left(\frac{x - x_0}{2}\right) +$$

$$\int_{x_0}^{x} [(f''(t) - f(t)) - (f''(x_0) - f(x_0))] \sinh(x - t)\, dt. \tag{39}$$

By the mean value theorem we derive that

$$|\sinh x| \leq \cosh(b-a)|x|, \ \ \forall\, x \in [-(b-a), b-a]. \tag{40}$$

Denote by

$$\overline{R}(x,x_0) := \int_{x_0}^{x} \left[(f''(t) - f(t)) - (f''(x_0) - f(x_0)) \right] \sinh(x-t)\, dt. \tag{41}$$

Clearly, we have that

$$|(f''(t) - f(t)) - (f''(x_0) - f(x_0))| \leq$$

$$\omega_1(f''-f,h)\left(1 + \frac{|t-x_0|}{h}\right), \ \ \text{all } t \in [a,b],\ h>0. \tag{42}$$

Let $x \geq x_0$, then

$$\left|\overline{R}(x,x_0)\right| \leq \int_{x_0}^{x} |(f''(t)-f(t)) - (f''(x_0)-f(x_0))|\, |\sinh(x-t)|\, dt \overset{\text{(by (41), (42))}}{\leq}$$

$$\omega_1(f''-f,h)\cosh(b-a)\int_{x_0}^{x}\left(1+\frac{(t-x_0)}{h}\right)(x-t)\,dt =$$

$$\cosh(b-a)\,\omega_1(f''-f,h)\left[\int_{x_0}^{x}(x-t)\,dt + \frac{1}{h}\int_{x_0}^{x}(x-t)(t-x_0)\,dt\right] =$$

$$\cosh(b-a)\,\omega_1(f''-f,h)\left[\frac{(x-x_0)^2}{2} + \frac{(x-x_0)^3}{6h}\right]. \tag{43}$$

So, if $x \geq x_0$, we got that

$$\left|\overline{R}(x,x_0)\right| \leq \frac{\cosh(b-a)\,\omega_1(f''-f,h)}{2}\left[(x-x_0)^2 + \frac{(x-x_0)^3}{3h}\right], \ \ h>0. \tag{44}$$

In case of $x < x_0$, we find similarly, that

$$\left|\overline{R}(x,x_0)\right| \leq \frac{\cosh(b-a)\,\omega_1(f''-f,h)}{2}\left[(x_0-x)^2 + \frac{(x_0-x)^3}{3h}\right]. \tag{45}$$

Thus, it holds

$$\left|\overline{R}(x,x_0)\right| \leq \frac{\cosh(b-a)\,\omega_1(f''-f,h)}{2}\left[(x-x_0)^2 + \frac{|x-x_0|^3}{3h}\right], \ \ h>0, \tag{46}$$

$\forall\, x,\ x_0 \in [a,b]$.

Let a sequence of positive linear operators $(L_n)_{n\in\mathbb{N}}$ from $C([a,b])$ into itself.

Clearly we have

$$\left|L_n\left(\overline{R}(x,x_0)\right)\right| \leq L_n\left(\left|\overline{R}(x,x_0)\right|\right). \tag{47}$$

By (39) we can write

$$f(\cdot)-f(x_0)=f'(x_0)\sinh(\cdot-x_0)+2f''(x_0)\sinh^2\left(\frac{\cdot-x_0}{2}\right)+\overline{R}(\cdot,x_0). \tag{48}$$

Thus, it holds

$$L_n(f)(x_0)-f(x_0)L_n(1)(x_0)=$$

$$f'(x_0)L_n(\sinh(\cdot-x_0))(x_0)+2f''(x_0)L_n\left(\sinh^2\left(\frac{\cdot-x_0}{2}\right)\right)(x_0)$$

$$+L_n\left(\overline{R}(\cdot,x_0)\right)(x_0), \tag{49}$$

$\forall\ x_0\in[a,b]$.

Consequently we have

$$|L_n(f)(x_0)-f(x_0)L_n(1)(x_0)| \overset{(49)}{\leq}$$

$$|f'(x_0)||L_n(\sinh(\cdot-x_0))(x_0)|+2|f''(x_0)|\left|L_n\left(\sinh^2\left(\frac{\cdot-x_0}{2}\right)\right)(x_0)\right|$$

$$+L_n\left(\left|\overline{R}(\cdot,x_0)\right|\right)\leq \tag{50}$$

$$|f'(x_0)||L_n(\sinh(\cdot-x_0))(x_0)|+2|f''(x_0)|\left|L_n\left(\sinh^2\left(\frac{\cdot-x_0}{2}\right)\right)(x_0)\right|+$$

$$\frac{\cosh(b-a)\,\omega_1(f''-f,h)}{2}\left[L_n\left((\cdot-x_0)^2\right)(x_0)+\frac{L_n\left(|\cdot-x_0|^3\right)(x_0)}{3h}\right], \tag{51}$$

$\forall\ x_0\in[a,b]$.

As in (20), we again have that

$$|L_n(f)(x_0)-f(x_0)|\leq$$

$$|(L_n(f)(x_0)-f(x_0)L_n(1)(x_0))|+|f(x_0)||L_n(1)(x_0)-1|.$$

It turns out that (by (51), (20))

$$|L_n(f)(x_0)-f(x_0)|\leq|f(x_0)||L_n(1)(x_0)-1|+$$

$$|f'(x_0)||L_n(\sinh(\cdot-x_0))(x_0)|+2|f''(x_0)|\left(L_n\left(\sinh^2\left(\frac{\cdot-x_0}{2}\right)\right)(x_0)\right)$$

$$+\frac{\cosh(b-a)\,\omega_1(f''-f,h)}{2}\left[L_n\left((\cdot-x_0)^2\right)(x_0)+\frac{L_n\left(|\cdot-x_0|^3\right)(x_0)}{3h}\right]\leq \tag{52}$$

$$|f(x_0)|\,|L_n(1)(x_0)-1| + \cosh(b-a)\,|f'(x_0)|\,L_n(|\cdot - x_0|)(x_0) +$$
$$\frac{\cosh^2(b-a)\,|f''(x_0)|}{2} L_n\left((\cdot - x_0)^2\right)(x_0) +$$
$$\frac{\cosh(b-a)\,\omega_1(f''-f,h)}{2}\left[L_n\left((\cdot - x_0)^2\right)(x_0) + \frac{L_n\left(|\cdot - x_0|^3\right)(x_0)}{3h}\right] =: (\xi), \tag{53}$$

$\forall\ x_0 \in [a,b]$, $h > 0$.

As in (23) we have

$$L_n\left((\cdot - x_0)^2\right)(x_0) \le \left(L_n(|\cdot - x_0|)^3 (x_0)\right)^{\frac{2}{3}} (L_n(1)(x_0))^{\frac{1}{3}}, \tag{54}$$

$\forall\ x_0 \in [a,b]$.

Let $r > 0$, and take

$$h := r\left(L_n\left(|\cdot - x_0|^3\right)(x_0)\right)^{\frac{1}{3}} = rD_3(x_0), \tag{55}$$

where

$$D_3(x_0) := \left(L_n\left(|\cdot - x_0|^3\right)(x_0)\right)^{\frac{1}{3}}. \tag{56}$$

As in (26) we have

$$\left(L_n(\cdot - x_0)^2\right)(x_0) \le D_3^2(x_0)\,(L_n(1)(x_0))^{\frac{1}{3}}. \tag{57}$$

At the moment we assume that $D_3(x_0) > 0$. As in (27) we get

$$\left[L_n\left((\cdot - x_0)^2\right)(x_0) + \frac{L_n\left(|\cdot - x_0|^3\right)(x_0)}{3h}\right] \le$$
$$D_3^2(x_0)\left[(L_n(1)(x_0))^{\frac{1}{3}} + \frac{1}{3r}\right]. \tag{58}$$

It follows that

$$|L_n(f)(x_0) - f(x_0)| \le (\xi) \overset{(58)}{\le} |f(x_0)|\,|L_n(1)(x_0)-1| +$$
$$\cosh(b-a)\,|f'(x_0)|\,L_n(|\cdot - x_0|)(x_0) + \frac{\cosh^2(b-a)\,|f''(x_0)|}{2} L_n\left((\cdot - x_0)^2\right)(x_0)$$
$$+\frac{\cosh(b-a)\,\omega_1(f''-f, rD_3(x_0))}{2} D_3^2(x_0)\left[(L_n(1)(x_0))^{\frac{1}{3}} + \frac{1}{3r}\right]. \tag{59}$$

As in (29) we get that

$$L_n(|\cdot - x_0|)(x_0) \le D_3(x_0)\,(L_n(1)(x_0))^{\frac{2}{3}}. \tag{60}$$

In case of $D_3(x_0) = 0$, by (53), we have that

$$|L_n(f)(x_0) - f(x_0)| \leq |f(x_0)| |L_n(1)(x_0) - 1| . \tag{61}$$

The last is valid as equality as in the proof of Theorem 3.

The proof of this theorem now is completed. ∎

Next comes a Korovkin type result.

Theorem 6 *All as in Theorem 5. Assume that* $L_n(1) \to 1$, *and* $L_n\left(|\cdot - x_0|^3\right)(x_0) \to 0$, *both uniformly convergent, as* $n \to \infty$. *Then* $L_n(f) \to f$, *uniformly convergent, as* $n \to \infty$, $\forall\ f \in C^2([a,b])$.

Proof. By (36); using (54), (60), and that every uniformly convergent sequence of bounded functions is uniformly bounded. Also that $\omega_1\left(f'' - f, r\|D_3\|_\infty\right) \to 0$, as $n \to \infty$. ∎

3 Applications

Let $f \in C([0,1])$, $x \in [0,1]$, and the Bernstein polynomials

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} . \tag{62}$$

We have that $B_n(1) = 1$, and

$$B_n\left((\cdot - x)^2\right)(x) = \frac{x(1-x)}{n} \leq \frac{1}{4n}. \tag{63}$$

Clearly it holds

$$B_n\left(|\cdot - x|^3\right)(x) \leq B_n\left((\cdot - x)^2\right)(x) \leq \frac{1}{4n}, \tag{64}$$

$\forall\ n \in \mathbb{N}$.

We have that

$$D_3^*(x_0) := \left(B_n\left(|\cdot - x_0|^3\right)(x)\right)^{\frac{1}{3}} \leq \frac{1}{\sqrt[3]{4n}}, \tag{65}$$

$\forall\ x_0 \in [0,1]$, $n \in \mathbb{N}$.

By (29) we get that

$$B_n(|\cdot - x_0|)(x) \leq \frac{1}{\sqrt[3]{4n}},\ \forall\ x_0 \in [0,1],\ n \in \mathbb{N}. \tag{66}$$

Consequently we obtain

$$\left\|B_n\left((\cdot - x_0)^2\right)(x_0)\right\|_\infty \leq \frac{1}{4n}, \tag{67}$$

and

$$\|D_3^*\| \leq \frac{1}{\sqrt[3]{4n}}, \tag{68}$$

with

$$\|B_n(|\cdot - x_0|)(x_0)\|_\infty \leq \frac{1}{\sqrt[3]{4n}}, \quad \forall\ n \in \mathbb{N}. \tag{69}$$

By (28) and the above we derive

Proposition 7 *Let $f \in C^2([0,1])$, $r > 0$. Then*

$$\|B_n(f) - f\|_\infty \leq \frac{\|f'\|_\infty}{\sqrt[3]{4n}} + \frac{\|f''\|_\infty}{8n} + \frac{\omega_1\left(f'' + f, \frac{r}{\sqrt[3]{4n}}\right)}{2(4n)^{\frac{2}{3}}}\left[1 + \frac{1}{3r}\right]. \tag{70}$$

By (59) and the above we obtain

Proposition 8 *Let $f \in C^2([0,1])$, $r > 0$. Then*

$$\|B_n(f) - f\|_\infty \leq$$

$$\cosh 1\left[\frac{\|f'\|_\infty}{\sqrt[3]{4n}} + \frac{\cosh 1\,\|f''\|_\infty}{8n} + \frac{\omega_1\left(f'' - f, \frac{r}{\sqrt[3]{4n}}\right)}{2(4n)^{\frac{2}{3}}}\left[1 + \frac{1}{3r}\right]\right]. \tag{71}$$

From (70) and/or (71), we obtain that $B_n(f) \to f$, uniformly, as $n \to \infty$, $\forall$ $f \in C^2([0,1])$.

We also get (use of (28))

Corollary 9 *Let $f \in C^2([0,1])$, $r > 0$, with $f'(x_0) = f''(x_0) = 0$, for some $x_0 \in [0,1]$. Then*

$$|B_n(f)(x_0) - f(x_0)| \leq \frac{\omega_1\left(f'' + f, \frac{r}{\sqrt[3]{4n}}\right)}{2(4n)^{\frac{2}{3}}}\left(1 + \frac{1}{3r}\right). \tag{72}$$

We finally give (use of (59))

Corollary 10 *All as in Corollary 9. Then*

$$|B_n(f)(x_0) - f(x_0)| \leq \frac{\cosh 1\omega_1\left(f'' - f, \frac{r}{\sqrt[3]{4n}}\right)}{2(4n)^{\frac{2}{3}}}\left(1 + \frac{1}{3r}\right). \tag{73}$$

By (72), (73), we have $\lim_{n\to\infty} B_n(f)(x_0) = f(x_0)$, pointwise, at the high speed $\frac{1}{n}$.

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