Brownian Motion Approximation by Parametrized and Deformed Neural Networks

George A. Anastassiou

Department of Mathematical Sciences
University of Memphis, Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Dimitra Kouloumpou

Section of Mathematics
Hellenic Naval Academy, Piraeus, 18539, Greece
dimkouloumpou@hna.gr

Abstract

The first author recently derived several approximation results by neural network operators see the new monograph [19]. There, the approximation methods derived from the parametrized and deformed neural networks induced by the $q-$deformed and $\lambda-$parametrized logistic and hyperbolic tangent activation functions. The results we apply here are univariate on a compact interval, regular and fractional. The outcome is the quantitative approximation of Brownian motion over the three dimensional sphere. We derive several Jackson type inequalities estimating the degree of convergence of our neural network operators to a general expectation function of Brownian motion. We give a detailed list of approximation applications regarding the expectation of well known functions of Brownian motion. Smoothness of our functions is taken into account producing higher speeds of approximation.

2020 AMS Subject Classification: 26A33, 41A17, 41A25, 60G15, 60G22

Keywords and Phrases: Neural network operators, Brownian motion, Expectation, Quantitative approximation.

1 Introduction

The first author in [1] and [2], see chapters $2-5$, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and 'Squasing' types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson

type inequalities. He treats there both the univariate and multivariate cases. The defining these operators 'bell-shaped' and 'squashing' functions are assumed to be compact support. Also the first author inspired by [23], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [4], [5], [6], [7] and [9], by treating both the univariate and multivariate cases. He did also the corresponding fractional cases [8], [10] and [14].
In [17], [19] the first author continued similar studies for Banach space valued functions for activation functions deriving from the $q-$deformed and $\lambda-$parametrized logistic and hyperbolic tangent sigmoid functions. The authors based and inspired by [27] perform here neural network quantitative approximations to Brownian motion over the three dimensional sphere. They present a series of Jackson type inequalities estimating the error of approximation to a general expectation function of the Brownian motion and its derivative. They produce regular and fractional calculus results. They finish with a lot of important applications.

2 About Parametrized and Deformed Neural Networks

2.1 About the q-deformed and λ-parametrized logistic activation function

Here we follow [21].
We consider here the q-deformed and λ-parametrized function acting as an activation function

$$\varphi_{q,\lambda}(x) = \frac{1}{1+qA^{-\lambda x}}, \quad x \in \mathbb{R}, \text{ where } q, \lambda > 0, \ A > 1. \tag{1}$$

This is an A-generalized logistic type function.

We easily observe that

$$\lim_{x \longrightarrow +\infty} \varphi_{q,\lambda}(x) = 1, \quad \lim_{x \longrightarrow -\infty} \varphi_{q,\lambda}(x) = 0. \tag{2}$$

Furthermore we have

$$\varphi_{q,\lambda}(x) = 1 - \varphi_{\frac{1}{q},\lambda}(-x). \tag{3}$$

We also have that

$$\varphi_{q,\lambda}(0) = \frac{1}{1+q} \text{ and } \varphi_{\frac{1}{q},\lambda}(0) = \frac{q}{1+q}. \tag{4}$$

We have

$$\varphi'_{q,\lambda}(x) = q\lambda(\ln A)\left(1+qA^{-\lambda x}\right)^{-2} A^{-\lambda x} > 0. \tag{5}$$

So that $\varphi_{q,\lambda}$ is a stricIty increasing function over $\mathbb{R}$. Furthermore it holds

$$\varphi''_{q,\lambda}(x) = q\lambda^2(\ln A)^2\left(A^{\lambda x} + q^2 A^{-\lambda x} + 2q\right)^{-2}\left(q^2 A^{-\lambda x} - A^{\lambda x}\right) \in C(\mathbb{R}). \tag{6}$$

$\varphi''_{q,\lambda}(x) > 0$, for $x < \frac{\log_A q}{\lambda}$ and there $\varphi_{q,\lambda}$ is concave up.

When $x > \frac{\log_A q}{\lambda}$, we have $\varphi''_{q,\lambda}(x) < 0$ and $\varphi_{q,\lambda}$ is concave down.

Of course

$$\varphi''_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = 0.$$

So, $\varphi_{q,\lambda}$ is a sigmoid function, see [18].

Consider the function

$$G_{q,\lambda}(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right), \quad x \in \mathbb{R}. \tag{7}$$

Then

$$G_{q,\lambda}(-x) = G_{\frac{1}{q},\lambda}(x), \quad \forall\, x \in \mathbb{R}. \tag{8}$$

We have that

$$G'_{q,\lambda}(x) = \frac{1}{2}\left(\varphi'_{q,\lambda}(x+1) - \varphi'_{q,\lambda}(x-1)\right). \tag{9}$$

$G'_{q,\lambda} > 0$, i.e. $G_{q,\lambda}$ is striclty increasing over $\left(-\infty, \frac{\log_A q}{\lambda} - 1\right)$. $G_{q,\lambda}$ is strictly decreasing over $\left(\frac{\log_A q}{\lambda}, +\infty\right)$. $G_{q,\lambda}$ is strictly concave down over $\left(\frac{\log_A q}{\lambda} - 1, \frac{\log_A q}{\lambda} + 1\right)$.

Overall $G_{q,\lambda}$ is a bell-shaped function over $\mathbb{R}$. $\frac{\log_A q}{\lambda}$ is the only critical number of $G_{q,\lambda}$ over $\mathbb{R}$. Therefore $G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right)$ is the maximum of $G_{q,\lambda}$.

The global maximul of $G_{q,\lambda}$ is

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{A^\lambda - 1}{2\left(A^\lambda + 1\right)}. \tag{10}$$

Finally we have that

$$\lim_{x \to +\infty} G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(+\infty) - \varphi_{q,\lambda}(+\infty)\right) = 0, \tag{11}$$

and

$$\lim_{x \to -\infty} G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(-\infty) - \varphi_{q,\lambda}(-\infty)\right) = 0. \tag{12}$$

Consequently the x-axis is the horizontal asymptote of $G_{q,\lambda}$. Of course $G_{q,\lambda}(x) > 0$, $\forall\, x \in \mathbb{R}$. We need

Theorem 1. *([22]) It holds*

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(x - i) = 1, \ \ \forall\, x \in \mathbb{R}, \forall\, q, \lambda > 0,\ A > 1. \tag{13}$$

Furthermore,

Theorem 2. *([22]) It holds*

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x)\, dx = 1, \quad \lambda, q > 0,\ A > 1. \tag{14}$$

So that $G_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda, q > 0,\ A > 1$.

We need the following result

Theorem 3. *([22]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then*

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{array}\right.}^{\infty} G_{q,\lambda}(nx - k) < \max\left\{q, \frac{1}{q}\right\} \frac{1}{A^{\lambda\left(n^{1-\alpha}-2\right)}} = \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \tag{15}$$

where $q, \lambda > 0,\ A > 1$; $\gamma := \max\left\{q, \frac{1}{q}\right\}$.

Let $\lceil\cdot\rceil$ the ceiling of the number, and $\lfloor\cdot\rfloor$ the integral part of the number.

We also need

Theorem 4. *([22]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q > 0$, $\lambda > 0$, $A > 1$, we consider the number $\lambda_q > z_0 > 0$ with $G_{q,\lambda}(z_0) = G_{q,\lambda}(0)$ and $\lambda_q > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx-k)} < max\left\{ \frac{1}{G_{q,\lambda}(\lambda_q)}, \frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)} \right\} =: K(q). \tag{16}$$

We finally mention

Remark 5. *([22]) (i) We have that*

$$\lim_{n\to+\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx-k) \neq 1, \quad \text{for at least some } x \in [a,b], \tag{17}$$

where $\lambda, q > 0$.

(ii) Let $[a,b] \subset \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx-k) \leq 1. \tag{18}$$

Definition 6. *Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the X-valued linear neural network operators*

$$L_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) G_{q,\lambda}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx-k)}, \quad x \in [a,b]. \tag{19}$$

Clearly here $L_n(f,x) \in C([a,b])$. We study here the pointwise and uniform convergence of $L_n(f,x)$ to $f(x)$ with rates.

For convenience, also we call

$$L_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) G_{q,\lambda}(nx-k), \tag{20}$$

$$L_n(f,x) := \frac{L_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx-k)}. \tag{21}$$

So that

$$L_n(f,x) - f(x) = \frac{L_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx-k)} - f(x) = \tag{22}$$

$$\frac{L_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)}.$$

Consequently, we derive that

$$|L_n(f,x) - f(x)| \leq K(q)\left|L_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)\right)\right| =$$

$$K(q)\left|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right) - f(x)\right)G_{q,\lambda}(nx-k)\right|. \tag{23}$$

We will estimate the right hand side of the last quantity.

For that we need, for $f \in C([a,b])$ the first modulus of continuity

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in[a,b]\\ |x-y|\leq\delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{24}$$

Similarly, it is defined ω_1 for $f \in C_{uB}(\mathbb{R})$ (uniformly continuous and bounded functions from $\mathbb{R}$ into $\mathbb{R}$), for $f \in C_B(\mathbb{R})$ (continuous and bounded real valued), and for $f \in C_u(\mathbb{R})$ (uniformly continuous).

The fact $f \in C([a,b])$ or $f \in C_u(\mathbb{R})$, is equivalent to $\lim_{\delta\to 0}\omega_1(f,\delta) = 0$, see [13].

We present a set of real valued neural network approximations to a function given with rates.

Theorem 7. *([21]) Let $f \in C([a,b])$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

i)

$$|L_n(f,x) - f(x)| \leq K(q)\left[\omega_1\left(f,\frac{1}{n^\alpha}\right) + 2\|f\|_\infty \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}\right] =: \rho, \tag{25}$$

and

ii)

$$\|L_n(f) - f\|_\infty \leq \rho. \tag{26}$$

We get that $\lim_{n\to\infty} L_n(f) = f$, pointwise and uniformly.

We need

Definition 8. *([11], [24]) Let $[a,b] \subset \mathbb{R}$, $\alpha > 0$; $m = \lceil\alpha\rceil \in \mathbb{N}$, ($\lceil\cdot\rceil$ is the ceiling of the number), $f : [a,b] \to \mathbb{R}$. We assume that $f^{(m)} \in L_1([a,b])$. We call the Caputo left fractional derivative of order α:*

$$(D_{*a}^\alpha f)(x) := \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt, \quad \forall\, x \in [a,b]. \tag{27}$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^\alpha f := f^{(m)}$ the ordinary real valued derivative and also set $D_{*a}^0 f := f$.

By [11], $\left(D_{*a}^{\alpha}f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha}f \in L_1([a,b])$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b])} < \infty$, then by [11], $D_{*a}^{\alpha}f \in C([a,b])$, hence $\left\|D_{*a}^{\alpha}f\right\| \in C([a,b])$.

We mention

Definition 9. *([12]) Let $[a,b] \subset \mathbb{R}$, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b])$, where $f : [a,b] \to \mathbb{R}$. We call the Caputo right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha}f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\,dz, \quad \forall\, x \in [a,b]. \tag{28}$$

We observe that $\left(D_{b-}^{m}f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0}f\right)(x) = f(x)$.

By [12], $\left(D_{b-}^{\alpha}f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha}f\right) \in L_1([a,b])$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [12], $D_{b-}^{\alpha}f \in C([a,b])$, hence $\left\|D_{b-}^{\alpha}f\right\| \in C([a,b])$. Next we present

Theorem 10. *([21])Let $0 < \alpha, \beta < 1$, $f \in C^1([a,b])$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

i)

$$|L_n(f,x) - f(x)| \leq$$

$$K(q)\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \tag{29}$$

and

ii)

$$\|L_n f - f\|_{\infty} \leq K(q)\frac{1}{\Gamma(\alpha+1)}\cdot$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.(b-a)^{\alpha}\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]} + \sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}. \tag{30}$$

2.2 About q-deformed and $\lambda-$parametrized hyperbolic tangent activation function $g_{q,\lambda}$

Here all this background comes from [16]. We use $g_{q,\lambda}$, see (29), and exhibit that it is a sigmoid activation function and we will present several of its properties related to the approximation by neural network operators. So, let us consider the function

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - qe^{-\lambda x}}{e^{\lambda x} + qe^{-\lambda x}}, \quad \lambda, q > 0, \ x \in \mathbb{R}. \tag{31}$$

We have that

$$g_{q,\lambda}(0) = \frac{1-q}{1+q}. \tag{32}$$

We also have

$$g_{q,\lambda}(-x) = -g_{\frac{1}{q},\lambda}(x), \ \ \forall\, x \in \mathbb{R}, \tag{33}$$

Furthermore

$$\lim_{x\to-\infty} g_{q,\lambda}(x) = -1, \text{ and } \lim_{x\to+\infty} g_{q,\lambda}(x) = +1 \tag{34}$$

We find that

$$g'_{q,\lambda}(x) = \frac{4q\lambda e^{2\lambda x}}{\left(e^{2\lambda x}+q\right)^2} > 0, \tag{35}$$

therefore $g_{q,\lambda}$ is striclty increasing.

Next we obtain ($x \in \mathbb{R}$)

$$g''_{q,\lambda}(x) = 8q\lambda^2 e^{2\lambda x}\left(\frac{q-e^{2\lambda x}}{\left(e^{2\lambda x}+q\right)^3}\right) \in C(\mathbb{R}). \tag{36}$$

So, in case of $x < \frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave up, with $g''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = 0$. And in case of $x > \frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave down.

Clearly, $g_{q,\lambda}$ is a shifted sigmoid function with $g_{q,\lambda}(0) = \frac{1-q}{1+q}$, and $g_{q,\lambda}(-x) = -g_{q^{-1},\lambda}(x)$, (a semi-odd function), see also [18].

By $1 > -1$, $x+1 > x-1$, we consider the activation function

$$M_{q.\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \tag{37}$$

$\forall x \in \mathbb{R}$; $q, \lambda > 0$. Notice that $M_{q,\lambda}(\pm\infty) = 0$, so the x-axis is horizontal asymptote. We have that

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \ \ \forall\, x \in \mathbb{R};\ \ q,\lambda > 0, \tag{38}$$

a deformed symmetry.

Furthermore $M_{q,\lambda}$ is strictly decreasing over $\left(\frac{\ln q}{2\lambda}+1, +\infty\right)$. $M_{q,\lambda}$ is concave down over $\left[\frac{\ln q}{2\lambda}-1, \frac{\ln q}{2\lambda}+1\right]$, and strictly concave down over $\left(\frac{\ln q}{2\lambda}-1, \frac{\ln q}{2\lambda}+1\right)$.

Consequently $M_{q,\lambda}$ has a bell-type shape over $\mathbb{R}$.

At $x = \frac{\ln q}{2\lambda}$, $M_{q,\lambda}$ achieves its global maximum, which is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2}, \quad \lambda > 0. \tag{39}$$

We mention

Theorem 11. *([20]) We have that*

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1, \ \ \forall\, x \in \mathbb{R},\ \forall\ \lambda, q > 0. \tag{40}$$

It follows

Theorem 12. *([20]) It holds*

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)\,dx = 1, \quad \lambda, q > 0. \tag{41}$$

So that $M_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda, q > 0$.

We need the following result

Theorem 13. *([20]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$; $q, \lambda > 0$. Then*

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{array}\right.}^{\infty} M_{q,\lambda}(nx-k) < \max\left\{q, \frac{1}{q}\right\} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} = T e^{-2\lambda n^{(1-\alpha)}}, \quad (42)$$

where $T := \max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Let $\lceil \cdot \rceil$ the ceiling of the number, and $\lfloor \cdot \rfloor$ the integral part of the number.

Theorem 14. *([20]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q > 0$, $\lambda > 0$, we consider the number $\lambda_q > z_0 > 0$ with $M_{q,\lambda}(z_0) = M_{q,\lambda}(0)$ and $\lambda_q > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)} < max\left\{\frac{1}{M_{q,\lambda}(\lambda_q)}, \frac{1}{M_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \Delta(q). \quad (43)$$

We also mention

Remark 15. *([20]) (i) We have that*

$$\lim_{n \to +\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \neq 1, \quad \text{for at least some } x \in [a,b], \quad (44)$$

where $\lambda, q > 0$.

(ii) Let $[a,b] \subset \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \leq 1. \quad (45)$$

Definition 16. *Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the real valued linear neural network operators*

$$H_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)}, \quad x \in [a,b]; q > 0, q \neq 1. \quad (46)$$

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. We study here the pointwise and uniform convergence of $H_n(f,x)$ to $f(x)$ with rates.

For convenience, also we call

$$H_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx-k), \quad (47)$$

that is

$$H_n(f,x) := \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)}. \tag{48}$$

So that

$$H_n(f,x) - f(x) = \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)} - f(x) = \tag{49}$$

$$\frac{H_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)}.$$

Consequently, we derive that

$$|H_n(f,x) - f(x)| \leq \Delta(q)\left|H_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)\right)\right| =$$

$$\Delta(q)\left|\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\left(f\left(\frac{k}{n}\right) - f(x)\right) M_{q,\lambda}(nx-k)\right|, \tag{50}$$

where $\Delta(q)$ as in (43). We will estimate the right hand side of the last quantity.

We present a set of real valued neural network approximations to a function given with rates.

Theorem 17. *([16])Let $f \in C([a,b])$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $q > 0$, $q \neq 1$, $x \in [a,b]$. Then*

i)

$$|H_n(f,x) - f(x)| \leq \Delta(q)\left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty T e^{-2\lambda n^{(1-\alpha)}}\right] =: \tau, \tag{51}$$

where $T := \max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

and

ii)

$$\|H_n(f) - f\|_\infty \leq \tau. \tag{52}$$

We get that $\lim_{n\to\infty} H_n(f) = f$, pointwise and uniformly.

Next present

Theorem 18. *([16]) Let $0 < \alpha, \beta < 1$, $q > 0$, $q \neq 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N}$: $n^{1-\beta} > 2$. Then*

i)

$$|H_n(f,x) - f(x)| \leq$$

$$\frac{\Delta(q)}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$Te^{-2\lambda n^{(1-\beta)}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Big\}, \tag{53}$$

and

ii)

$$\left\|H_n f-f\right\|_{\infty}\leq\frac{\Delta(q)}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.(b-a)^{\alpha}Te^{-2\lambda n^{(1-\beta)}}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}. \tag{54}$$

Here, $T:=\max\left\{q,\frac{1}{q}\right\}e^{4\lambda}$.

3 Combine 2.1 and 2.2

Let $a,b\in\mathbb{R}, f\in C([a,b])$. Let also $q,\lambda>0, A>1, \gamma=\max\left\{q,\frac{1}{q}\right\}$ as in previous sections. For the next theorems we call

$${}_1L_n(f,x):=L_n(f,x), x\in[a,b]$$

$${}_2L_n(f,x):=H_n(f,x), x\in[a,b].$$

Also we set

$$K_1(q)=K(q)$$

$$K_2(q)=\Delta(q).$$

Furthermore we set

$$\hat{\beta}_{1,n}(\lambda,\alpha)=A^{-\lambda\left(n^{1-\alpha}-2\right)}, n\in\mathbb{N}, 0<\alpha<1.$$

$$\hat{\beta}_{2,n}(\lambda,\alpha)=e^{4\lambda-2\lambda n^{1-\alpha}}, n\in\mathbb{N}, 0<\alpha<1.$$

We present

Theorem 19. *Let* $f\in C([a,b])$, $0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in[a,b]$. *Then for* $i=1,2$

i)

$$\left|{}_iL_n(f,x)-f(x)\right|\leq K_i(q)\left[\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+2\|f\|_{\infty}\gamma\hat{\beta}_{i,n}(\lambda,\alpha)\right]=:\rho_i, \tag{55}$$

and

ii)

$$\left\|{}_iL_n(f)-f\right\|_{\infty}\leq\rho_i. \tag{56}$$

We get that $\lim_{n\to\infty}{}_iL_n(f)=f$, *pointwise and uniformly.*

Proof. From Theorems 7 and 17. □

Next we present

Theorem 20. *Let* $0 < \alpha, \beta < 1$, $f \in C^1([a,b])$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then for* $i = 1, 2$

i)

$$|{}_iL_n(f,x) - f(x)| \leq$$

$$K_i(q) \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \gamma \hat{\beta}_{i,n}(\lambda, \beta) \left(\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \tag{57}$$

and

ii)

$$\|{}_iL_n f - f\|_\infty \leq K_i(q) \frac{1}{\Gamma(\alpha+1)} \cdot$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. (b-a)^\alpha \gamma \hat{\beta}_{i,n}(\lambda, \beta) \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \right) \right\}. \tag{58}$$

Proof. From Theorems 10 and 18. □

4 About Brownian Motion on 3 Dimensional Sphere

([27]) The Brownian motion on S^n is a diffusion (Markov) process $W_t, t \geq 0$, on S^n whose transition density is a function $P(t,x,y)$ on $(0,\infty) \times S^n \times S^n$ satisfying

$$\frac{\partial P}{\partial t} = \frac{1}{2} \Delta_n P, \tag{59}$$

and

$$P(t,x,y) \to \delta_x(y) \quad \text{as } t \to 0^+, \tag{60}$$

where Δ_n is the Laplace-Beltrami operator of S^n acting on the x-variables and $\delta_x(y)$ is the delta mass at x, i.e. $P(t,x,y)$ is the **heat kernel** of S^n. The heat kernel exists, it is unique, positive, and smooth in (t,x,y).

Remark 21. *The heat kernel* $P(t,x,y)$ *sutisfying the following properties*

1. *Symmetry:* $P(t,x,y) = P(t,y,x)$.
2. *The semigroup identity: For any* $s \in (0,t)$,

$$P(t,x,y) = \int_{S^n} P(s,x,z) P(t-s,z,y) d\mu(z), \tag{61}$$

where $d\mu$ *is the area measure element of* S^n.

3. *For all* $t > 0$ *and* $x \in S^n$

$$\int_{S^n} P(t,x,y) d\mu(y) = 1. \tag{62}$$

4. As $t \to \infty$, $P(t,x,y)$ approaches the uniform density on S^n, i.e.

$$\lim_{t\to\infty} P(t,x,y) = \frac{1}{A_n}, \tag{63}$$

where A_n is the area of the S^n with radius a. It is also well known that

$$A_n = \frac{2\pi^{\frac{n+1}{2}} a^n}{(\frac{n-1}{2})!}, \text{ for } n \text{ odd}$$

$$A_n = \frac{2^n(\frac{n}{2}-1)!\pi^{\frac{n}{2}} a^n}{(n-1)!}, \text{ for } n \text{ even.} \tag{64}$$

Finally, the symmetry of S^n implies that $P(t,x,y)$ depends only on t and $d(x,y)$, the distance between x and y. Thus in spherical coordinates it depends on t and the angle φ between x and y. Hence,

$$P(t,x,y) = p(t,\varphi),$$

where $p(t,\varphi)$ satisfies

$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta_n p = \frac{1}{2a^2}\left[(n-1)\cot\varphi \cdot \frac{\partial p}{\partial \varphi} + \frac{\partial^2 p}{\partial \varphi^2}\right] \tag{65}$$

and

$$\lim_{t\to 0^+} aA_{n-1}p(t,\varphi)\cdot \sin^{n-1}\varphi = \delta(\varphi). \tag{66}$$

Here $\delta(\cdot)$ is the standard Dirac delta function on $\mathbb{R}$.

4.1 Explicit form of the heat kernel of S^3

Proposition 22. *Let W_t, $t \ge 0$ be the Brownian motion on a 3-dimensional sphere S^3 of radius a. The transition density function $p(t,\varphi)$ of W_t is given by*

$$p(t,\varphi) = \frac{\exp\left(\frac{t}{2a^2}\right)}{(2\pi t)^{3/2}\sin\varphi}\sum_{n\in\mathbb{Z}}(\varphi+2n\pi)\exp\left(-\frac{(\varphi+2n\pi)^2a^2}{2t}\right), \tag{67}$$

where $\mathbb{Z}$ is the set of all integers. Equivalently

$$p(t,\varphi) = -\frac{i}{4\pi^2a^3\sin\varphi}\sum_{n\in\mathbb{Z}} n\exp\left(-\frac{t(n^2-1)}{2a^2} + i\varphi n\right) \tag{68}$$

and

$$p(t,\varphi) = \frac{1}{2\pi^2a^3\sin\varphi}\sum_{n\in\mathbb{N}} n\sin(n\varphi)\exp\left(-\frac{t(n^2-1)}{2a^2}\right). \tag{69}$$

Furthermore $p(t,\varphi)$ is analytic about $\varphi = 0$ and $\varphi = \pi$. In fact

$$p(t,0) = \lim_{\varphi\to 0^+} p(t,\varphi) = \frac{1}{2\pi^2a^3}\sum_{n\in\mathbb{N}} n^2\exp\left(-\frac{t(n^2-1)}{2a^2}\right) \tag{70}$$

and

$$p(t,\pi) = \lim_{\varphi\to\pi^-} p(t,\varphi) = \frac{1}{2\pi^2a^3}\sum_{n\in\mathbb{N}} n^2(-1)^n\exp\left(-\frac{t(n^2-1)}{2a^2}\right). \tag{71}$$

Theorem 23. *Consider function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $[0, \pi]$, i.e. there exists $M > 0$ such that $|g(\phi)| \leq M$, for every $\phi \in [0, \pi]$, and Lebesgue measurable on $\mathbb{R}$. Let also $W(t, \phi)$ be the Brownian motion on S^3. Then the expectation*

$$E\left(|g(W)|\right)(t) = \int_0^\pi |g(\phi)|\, p(t, \phi) d\phi$$

is continuous in t, and

$$E\left(|g(W)|\right)(t) \leq \pi M p\left(t_o, \phi_0\right), \tag{72}$$

where

$$p\left(t_0, \phi_0\right) = \max_{(t,\phi) \in [t_1, t_2] \times [0,\pi]} p\left(t, \phi\right), \text{ with } 0 < t_1 < t_2.$$

Here $p(t, \phi)$ is the transition density function of the Brownian motion W_t, $t \geq 0$ on S^3 given by (67).

Proof. It is known that the transition density function of the Brownian motion $W_t, t \geq 0$ on $S^3, p(t, \phi)$ is continuous in $(t, \phi) \in [t_1, t_2] \times [0, \pi], t_1 > 0$.
By the extreme value theorem there exists $(t_0, \phi_0) \in [t_1, t_2] \times [0, \pi]$ such that

$$p\left(t_0, \phi_0\right) = \max_{(t,\phi) \in [t_1, t_2] \times [0,\pi]} p\left(t, \phi\right).$$

So we have

$$0 \leq p\left(t, \phi\right) \leq p\left(t_0, \phi_0\right), \text{ for every } (t, \phi) \in [t_1, t_2] \times [0, \pi].$$

Let $N \in \mathbb{N}, t_N, t \in [t_1, t_2] : t_N \to t$, as $N \to \infty$.
Then, $p\left(t_N, \phi\right) \to p\left(t, \phi\right)$ for every $\phi \in [0, \pi]$.
The function $g : \mathbb{R} \to \mathbb{R}$, is bounded on $[0, \pi]$, i.e. there is a $M > 0$ such that $|g(\phi)| \leq M$, for every $\phi \in [0, \pi]$, and Lebesgue measurable on $\mathbb{R}$.
Furthermore we have that,

$$|g(\phi)|\, p(t_N, \phi) \to |g(\phi)|\, p\left(t, \phi\right), \text{ as } N \to \infty$$

and

$$|g(\phi)|\, p(t_N, \phi) \leq |g(\phi)|\, p\left(t_o, \phi_0\right), \text{ for all } \phi \in [0, \pi] \text{ and } N \in \mathbb{N}.$$

So, by dominated convergence theorem we get that

$$E\left(|g(W)|\right)(t_N) \to E\left(|g(W)|\right)(t) \text{ as } N \to \infty.$$

Thus $E\left(|g(W)|\right)(t)$ is proved to be continuous in t.
Moreover,
$|g(\phi)|\, p(t, \phi) \leq M p\left(t_o, \phi_0\right)$ for all $t \in [t_1, t_2]$ and $\phi \in [0, \pi]$.
Thus,

$$E\left(|g(W)|\right)(t) = \int_0^\pi |g(\phi)|\, p(t, \phi) d\phi \leq \pi M p\left(t_o, \phi_0\right).$$

□

Proposition 24. *Consider function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $[0, \pi]$ and Lebesgue measurable on $\mathbb{R}$. Let also $W(t, \phi)$ be the Brownian motion on S^3. Then the expectation*

$$E(|g(W)|)(t) = \int_0^\pi |g(\phi)| \, p(t, \phi) d\phi$$

is differentiable in t, and

$$\frac{\partial E(|g(W)|)}{\partial t} = \int_0^\pi |g(\phi)| \frac{\partial (p(t, \phi))}{\partial t} d\phi, \tag{73}$$

which is continuous in t.

Proof. As we said before, the transition density function of the Brownian motion $W_t, t \geq 0$ on $S^3, p(t, \phi)$ is continuous in $(t, \phi) \in [t_1, t_2] \times [0, \pi], t_1 > 0$.
We have

$$E(|g(W)|) = \int_0^\pi |g(\phi)| \, p(t, \phi) d\phi, \text{ for every } t \in [t_1, t_2].$$

We apply differentiation under the integral sign.
We notice

$$|g(\phi)| \frac{\partial p(t, \phi)}{\partial t} \leq M \left\| \frac{\partial p(t, \phi)}{\partial t} \right\|_{\infty, [t_1, t_2] \times [0, \pi]}.$$

Therefore there exists

$$\frac{\partial E(|g(W)|)}{\partial t} = \int_0^\pi |g(\phi)| \frac{\partial (p(t, \phi))}{\partial t} d\phi,$$

which is continuous in t (same proof as in Theorem 23). □

5 Main Results

We start with

Proposition 25. *Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $t \in [t_1, t_2]$, where $t_1 > 0$. Then for $i = 1, 2$*

i)

$$|_i L_n (E(|g(W)|))(t) - E(|g(W)|)(t)| \leq$$
$$K_i(q) \left[\omega_1 \left(E(|g(W)|), \frac{1}{n^\alpha} \right) + 2 \|E(|g(W)|)\|_\infty \gamma \hat{\beta}_{i,n}(\lambda, \alpha) \right] =: \rho_i, \tag{74}$$

and
ii)

$$\|_i L_n (E(|g(W)|)) - E(|g(W)|)\|_\infty \leq \rho_i. \tag{75}$$

We get that $\lim_{n \to \infty} {}_i L_n (E(|g(W)|)) = E(|g(W)|)$, pointwise and uniformly.

Proof. From Theorem 19. □
Next we present

Proposition 26. *Let $0 < \alpha, \beta < 1$, $t \in [t_1, t_2]$, where $t_1 > 0$ $n \in \mathbb{N} : n^{1-\beta} > 2$. Then for $i = 1, 2$*
i)

$$\left|{}_iL_n\left(E\left(|g(W)|\right)\right)(t) - E\left(|g(W)|\right)(t)\right| \leq$$

$$K_i\left(q\right)\frac{1}{\Gamma\left(\alpha+1\right)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}E\left(|g(W)|\right), \frac{1}{n^{\beta}}\right)_{[t_1,t]} + \omega_1\left(D_{*t}^{\alpha}E\left(|g(W)|\right), \frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\gamma\hat{\beta}_{i,n}\left(\lambda,\beta\right)\left(\left\|D_{t-}^{\alpha}E\left(|g(W)|\right)\right\|_{\infty,[t_1,t]}\left(t-t_1\right)^{\alpha} + \left\|D_{*t}^{\alpha}E\left(|g(W)|\right)\right\|_{\infty,[t,t_2]}\left(t_2-t\right)^{\alpha}\right)\right\}, \quad (76)$$

and
ii)

$$\left\|{}_iL_nE\left(|g(W)|\right) - E\left(|g(W)|\right)\right\|_{\infty} \leq K_i\left(q\right)\frac{1}{\Gamma\left(\alpha+1\right)}\cdot$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}E\left(|g(W)|\right), \frac{1}{n^{\beta}}\right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}E\left(|g(W)|\right), \frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\left(t_2-t_1\right)^{\alpha}\gamma\hat{\beta}_{i,n}\left(\lambda,\beta\right)\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}E\left(|g(W)|\right)\right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\alpha}E\left(|g(W)|\right)\right\|_{\infty,[t,t_2]}\right)\right\}. \quad (77)$$

Proof. From Theorem 20. □
We continue with

Proposition 27. *Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $t \in [t_1, t_2]$, where $t_1 > 0$. Then for $i = 1, 2$*

i)

$$\left|{}_iL_n\left(\frac{\partial E\left(|g(W)|\right)}{\partial t}\right)(t) - \left(\frac{\partial E\left(|g(W)|\right)}{\partial t}\right)(t)\right| \leq$$

$$K_i\left(q\right)\left[\omega_1\left(\frac{\partial E\left(|g(W)|\right)}{\partial t}, \frac{1}{n^{\alpha}}\right) + 2\left\|\frac{\partial E\left(|g(W)|\right)}{\partial t}\right\|_{\infty}\gamma\hat{\beta}_{i,n}\left(\lambda,\alpha\right)\right] =: \rho_i, \quad (78)$$

and
ii)

$$\left\|{}_iL_n\left(\frac{\partial E\left(|g(W)|\right)}{\partial t}\right) - \frac{\partial E\left(|g(W)|\right)}{\partial t}\right\|_{\infty} \leq \rho_i. \quad (79)$$

We get that $\lim_{n\to\infty} {}_iL_n\left(\frac{\partial E(|g(W)|)}{\partial t}\right) = \frac{\partial E(|g(W)|)}{\partial t}$, pointwise and uniformly.

Proof. From Theorem 19. □

6 Applications

For a function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $[0, \pi]$ and Lebesgue measurable on $\mathbb{R}$ and $W(t, \phi)$ the Brownian motion on S^3. We will use the following notations

$$E\left(|g(W)|\right) := E\left(|g(W)|\right)^{(0)} \quad (80)$$

and

$$\frac{\partial E\left(|g(W)|\right)}{\partial t} := E\left(|g(W)|\right)^{(1)}. \tag{81}$$

We can apply our main results to the function $g(W) = W$. Consider the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = x$ for every $x \in \mathbb{R}$. Let also $W(t, \phi)$ be the Brownian motion on S^3. Then the expectation

$$E\left(|W|\right)(t) = \int_0^\pi \phi p(t, \phi) d\phi$$

is continuous in t.
Moreover,

Corollary 28. *Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $t \in [t_1, t_2]$, where $t_1 > 0$. Then for $i = 1, 2$ and $j = 0, 1$*

i)

$$\left| {}_iL_n \left(E\left(|W|\right)^{(j)} \right)(t) - E\left(|W|\right)^{(j)}(t) \right| \leq$$

$$K_i(q) \left[\omega_1 \left(E\left(|W|\right)^{(j)}, \frac{1}{n^\alpha} \right) + 2 \left\| E\left(|W|\right)^{(j)} \right\|_\infty \gamma \hat{\beta}_{i,n}(\lambda, \alpha) \right] =: \rho_i, \tag{82}$$

and
ii)

$$\left\| {}_iL_n \left(E\left(|W|\right)^{(j)} \right) - E\left(|W|\right)^{(j)} \right\|_\infty \leq \rho_i \tag{83}$$

We get that $\lim_{n \to \infty} {}_iL_n \left(E\left(|W|\right)^{(j)} \right) = E\left(|W|\right)^{(j)}$, pointwise and uniformly.

Proof. From Propositions 25 and 27. □
Next we present

Corollary 29. *Let $0 < \alpha, \beta < 1$, $t \in [t_1, t_2]$, where $t_1 > 0$ $n \in \mathbb{N} : n^{1-\beta} > 2$. Then for $i = 1, 2$*
i)

$$\left| {}_iL_n \left(E\left(|W|\right)\right)(t) - E\left(|W|\right)(t) \right| \leq$$

$$K_i(q) \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{t-}^\alpha E\left(|W|\right), \frac{1}{n^\beta} \right)_{[t_1,t]} + \omega_1 \left(D_{*t}^\alpha E\left(|W|\right), \frac{1}{n^\beta} \right)_{[t,t_2]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \gamma \hat{\beta}_{i,n}(\lambda, \beta) \left(\left\| D_{t-}^\alpha E\left(|W|\right) \right\|_{\infty,[t_1,t]} (t - t_1)^\alpha + \left\| D_{*t}^\alpha E\left(|W|\right) \right\|_{\infty,[t,t_2]} (t_2 - t)^\alpha \right) \right\}, \tag{84}$$

and
ii)

$$\left\| {}_iL_n E\left(|W|\right) - E\left(|W|\right) \right\|_\infty \leq K_i(q) \frac{1}{\Gamma(\alpha+1)} \cdot$$

$$\left\{ \frac{\left(\sup_{t \in [t_1,t_2]} \omega_1 \left(D_{t-}^\alpha E\left(|W|\right), \frac{1}{n^\beta} \right)_{[t_1,t]} + \sup_{t \in [t_1,t_2]} \omega_1 \left(D_{*t}^\alpha E\left(|W|\right), \frac{1}{n^\beta} \right)_{[t,t_2]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. (t_2 - t_1)^\alpha \gamma \hat{\beta}_{i,n}(\lambda, \beta) \left(\sup_{t \in [t_1,t_2]} \left\| D_{t-}^\alpha E\left(|W|\right) \right\|_{\infty,[t_1,t]} + \sup_{t \in [t_1,t_2]} \left\| D_{*t}^\alpha E\left(|W|\right) \right\|_{\infty,[t,t_2]} \right) \right\}. \tag{85}$$

Proof. From Proposition 26. $\square$

For the next corollaries we consider the function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=cosx$ for every $x\in\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^3. Then the expectation

$$E\left(\left|cosW\right|\right)(t)=\int_0^\pi\left|\cos\phi\right|p(t,\phi)d\phi$$

is continuous in t.

Moreover,

Corollary 30. *Let $0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $t\in[t_1,t_2]$, where $t_1>0$. Then for $i=1,2$ and $j=0,1$*

i)

$$\left|{}_iL_n\left(E\left(\left|\cos W\right|\right)^{(j)}\right)(t)-E\left(\left|\cos W\right|\right)^{(j)}(t)\right|\leq$$

$$K_i(q)\left[\omega_1\left(E\left(\left|\cos W\right|\right)^{(j)},\frac{1}{n^\alpha}\right)+2\left\|E\left(\left|\cos W\right|\right)^{(j)}\right\|_\infty\gamma\hat{\beta}_{i,n}(\lambda,\alpha)\right]=:\rho_i, \tag{86}$$

and

ii)

$$\left\|{}_iL_n\left(E\left(\left|\cos W\right|\right)^{(j)}\right)-E\left(\left|\cos W\right|\right)^{(j)}\right\|_\infty\leq\rho_i \tag{87}$$

We get that $\lim_{n\to\infty}{}_iL_n\left(E\left(\left|\cos W\right|\right)^{(j)}\right)=E\left(\left|\cos W\right|\right)^{(j)}$, pointwise and uniformly.

Proof. From Propositions 25 and 27. $\square$

Next we present

Corollary 31. *Let $0<\alpha,\beta<1$, $t\in[t_1,t_2]$, where $t_1>0$ $n\in\mathbb{N}:n^{1-\beta}>2$. Then for $i=1,2$*

i)

$$\left|{}_iL_n\left(E\left(\left|\cos W\right|\right)\right)(t)-E\left(\left|\cos W\right|\right)(t)\right|\leq$$

$$K_i(q)\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^\alpha E\left(\left|\cos W\right|\right),\frac{1}{n^\beta}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^\alpha E\left(\left|\cos W\right|\right),\frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\gamma\hat{\beta}_{i,n}(\lambda,\beta)\left(\left\|D_{t-}^\alpha E\left(\left|\cos W\right|\right)\right\|_{\infty,[t_1,t]}(t-t_1)^\alpha+\left\|D_{*t}^\alpha E\left(\left|\cos W\right|\right)\right\|_{\infty,[t,t_2]}(t_2-t)^\alpha\right)\right\}, \tag{88}$$

and

ii)

$$\left\|{}_iL_nE\left(\left|\cos W\right|\right)-E\left(\left|\cos W\right|\right)\right\|_\infty\leq K_i(q)\frac{1}{\Gamma(\alpha+1)}\cdot$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^\alpha E\left(\left|\cos W\right|\right),\frac{1}{n^\beta}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^\alpha E\left(\left|\cos W\right|\right),\frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.(t_2-t_1)^\alpha\gamma\hat{\beta}_{i,n}(\lambda,\beta)\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^\alpha E\left(\left|\cos W\right|\right)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^\alpha E\left(\left|\cos W\right|\right)\right\|_{\infty,[t,t_2]}\right)\right\}. \tag{89}$$

Proof. From Proposition 26. □

We can obtain similar results for the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = \sin x$ for every $x \in \mathbb{R}$.

Let as consider now the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = \tanh x$ for every $x \in \mathbb{R}$. Let also $W(t, \phi)$ be the Brownian motion on S^3. Then the expectation

$$E\left(|\tanh W|\right)(t) = \int_0^\pi |\tanh(\phi)|\, p(t, \phi) d\phi$$

is continuous in t.

Moreover,

Corollary 32. *Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $t \in [t_1, t_2]$, where $t_1 > 0$. Then for $i = 1, 2$ and $j = 0, 1$*

i)

$$\left|{}_iL_n\left(E\left(|\tanh W|\right)^{(j)}\right)(t) - E\left(|\tanh W|\right)^{(j)}(t)\right| \leq$$

$$K_i(q)\left[\omega_1\left(E\left(|\tanh W|\right)^{(j)}, \frac{1}{n^\alpha}\right) + 2\left\|E\left(|\tanh W|\right)^{(j)}\right\|_\infty \gamma\hat{\beta}_{i,n}(\lambda, \alpha)\right] =: \rho_i, \tag{90}$$

and

ii)

$$\left\|{}_iL_n\left(E\left(|\tanh W|\right)^{(j)}\right) - E\left(|\tanh W|\right)^{(j)}\right\|_\infty \leq \rho_i \tag{91}$$

We get that $\lim_{n\to\infty} {}_iL_n\left(E\left(|\tanh W|\right)^{(j)}\right) = E\left(|\tanh W|\right)^{(j)}$, *pointwise and uniformly.*

Proof. From Propositions 25 and 27. □

Next we present

Corollary 33. *Let $0 < \alpha, \beta < 1$, $t \in [t_1, t_2]$, where $t_1 > 0$ $n \in \mathbb{N} : n^{1-\beta} > 2$. Then for $i = 1, 2$*

i)

$$\left|{}_iL_n\left(E\left(|\tanh W|\right)\right)(t) - E\left(|\tanh W|\right)(t)\right| \leq$$

$$K_i(q)\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^\alpha E\left(|\tanh W|\right), \frac{1}{n^\beta}\right)_{[t_1,t]} + \omega_1\left(D_{*t}^\alpha E\left(|\tanh W|\right), \frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\gamma\hat{\beta}_{i,n}(\lambda, \beta)\left(\left\|D_{t-}^\alpha E\left(|\tanh W|\right)\right\|_{\infty,[t_1,t]}(t-t_1)^\alpha + \left\|D_{*t}^\alpha E\left(|\tanh W|\right)\right\|_{\infty,[t,t_2]}(t_2-t)^\alpha\right)\right\}, \tag{92}$$

and

ii)

$$\left\|{}_iL_nE\left(|\tanh W|\right) - E\left(|\tanh W|\right)\right\|_\infty \leq K_i(q)\frac{1}{\Gamma(\alpha+1)}\cdot$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^\alpha E\left(|\tanh W|\right), \frac{1}{n^\beta}\right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^\alpha E\left(|\tanh W|\right), \frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}} + \right.$$

$$
(t_2-t_1)^{\alpha}\gamma\hat{\beta}_{i,n}(\lambda,\beta)\left(\sup_{t\in[t_1,t_2]}\left\|D^{\alpha}_{t-}E(|\tanh W|)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D^{\alpha}_{*t}E(|\tanh W|)\right\|_{\infty,[t,t_2]}\right)\Bigg\}. \tag{93}
$$

Proof. From Proposition 26. □

In the following let as consider the function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=e^{-\ell x}$, $\ell>0$ for every $x\in\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^3. Then the expectation

$$
E\left(e^{-\ell W}\right)(t)=\int_0^{\pi}e^{-\ell\phi}p(t,\phi)d\phi
$$

is continuous in t.
Furthermore,

Corollary 34. *Let $0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $t\in[t_1,t_2]$, where $t_1>0$. Then for $i=1,2$ and $j=0,1$*

i)

$$
\left|{}_iL_n\left(E\left(e^{-\ell W}\right)^{(j)}\right)(t)-E\left(e^{-\ell W}\right)^{(j)}(t)\right|\leq
$$

$$
K_i(q)\left[\omega_1\left(E\left(e^{-\ell W}\right)^{(j)},\frac{1}{n^{\alpha}}\right)+2\left\|E\left(e^{-\ell W}\right)^{(j)}\right\|_{\infty}\gamma\hat{\beta}_{i,n}(\lambda,\alpha)\right]=:\rho_i, \tag{94}
$$

and

ii)

$$
\left\|{}_iL_n\left(E\left(e^{-\ell W}\right)^{(j)}\right)-E\left(e^{-\ell W}\right)^{(j)}\right\|_{\infty}\leq\rho_i \tag{95}
$$

We get that $\lim_{n\to\infty}{}_iL_n\left(E\left(e^{-\ell W}\right)^{(j)}\right)=E\left(e^{-\ell W}\right)^{(j)}$, pointwise and uniformly.

Proof. From Propositions 25 and 27. □
We continue with

Corollary 35. *Let $0<\alpha,\beta<1$, $t\in[t_1,t_2]$,where $t_1>0$ $n\in\mathbb{N}:n^{1-\beta}>2$. Then for $i=1,2$*
i)

$$
\left|{}_iL_n\left(E\left(e^{-\ell W}\right)\right)(t)-E\left(e^{-\ell W}\right)(t)\right|\leq
$$

$$
K_i(q)\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D^{\alpha}_{t-}E\left(e^{-\ell W}\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D^{\alpha}_{*t}E\left(e^{-\ell W}\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.
$$

$$
\left.\gamma\hat{\beta}_{i,n}(\lambda,\beta)\left(\left\|D^{\alpha}_{t-}E\left(e^{-\ell W}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D^{\alpha}_{*t}E\left(e^{-\ell W}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}, \tag{96}
$$

and

ii)

$$
\left\|{}_iL_nE\left(e^{-\ell W}\right)-E\left(e^{-\ell W}\right)\right\|_{\infty}\leq K_i(q)\frac{1}{\Gamma(\alpha+1)}.
$$

$$\left\{ \frac{\left(\sup_{t\in[t_1,t_2]} \omega_1 \left(D_{t-}^{\alpha} E\left(e^{-\ell W}\right), \frac{1}{n^{\beta}}\right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left(D_{*t}^{\alpha} E\left(e^{-\ell W}\right), \frac{1}{n^{\beta}}\right)_{[t,t_2]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. (t_2 - t_1)^{\alpha} \gamma\hat{\beta}_{i,n}(\lambda,\beta) \left(\sup_{t\in[t_1,t_2]} \left\| D_{t-}^{\alpha} E\left(e^{-\ell W}\right) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D_{*t}^{\alpha} E\left(e^{-\ell W}\right) \right\|_{\infty,[t,t_2]} \right) \right\}. \tag{97}$$

Proof. From Proposition 26. □

In the following we consider the generalized logistic sigmoid function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = (1 + e^{-x})^{-\delta}$ for every $x \in \mathbb{R}$. Let also $W(t, \phi)$ be the Brownian motion on S^3. Then the expectation

$$E\left(\left(1 + e^{-W}\right)^{-\delta}\right)(t) = \int_0^{\pi} \left(1 + e^{-\phi}\right)^{-\delta} p(t,\phi) d\phi$$

is continuous in t.
Moreover,

Corollary 36. *Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $t \in [t_1, t_2]$, where $t_1 > 0$. Then for $i = 1, 2$ and $j = 0, 1$*

i)

$$\left| {}_iL_n \left(E\left(\left(1 + e^{-W}\right)^{-\delta}\right)^{(j)} \right)(t) - E\left(\left(1 + e^{-W}\right)^{-\delta}\right)^{(j)}(t) \right| \leq$$

$$K_i(q) \left[\omega_1 \left(E\left(\left(1 + e^{-W}\right)^{-\delta}\right)^{(j)}, \frac{1}{n^{\alpha}} \right) + 2 \left\| E\left(\left(1 + e^{-W}\right)^{-\delta}\right)^{(j)} \right\|_{\infty} \gamma\hat{\beta}_{i,n}(\lambda,\alpha) \right] =: \rho_i, \tag{98}$$

and
ii)

$$\left\| {}_iL_n \left(E\left(\left(1 + e^{-W}\right)^{-\delta}\right)^{(j)} \right) - E\left(\left(1 + e^{-W}\right)^{-\delta}\right)^{(j)} \right\|_{\infty} \leq \rho_i \tag{99}$$

We get that $\lim_{n\to\infty} {}_iL_n \left(E\left(\left(1 + e^{-W}\right)^{-\delta}\right)^{(j)} \right) = E\left(\left(1 + e^{-W}\right)^{-\delta}\right)^{(j)}$, *pointwise and uniformly.*

Proof. From Propositions 25 and 27. □
It follows

Corollary 37. *Let $0 < \alpha, \beta < 1$, $t \in [t_1, t_2]$, where $t_1 > 0$ $n \in \mathbb{N} : n^{1-\beta} > 2$. Then for $i = 1, 2$*
i)

$$\left| {}_iL_n \left(E\left(\left(1 + e^{-W}\right)^{-\delta}\right) \right)(t) - E\left(\left(1 + e^{-W}\right)^{-\delta}\right)(t) \right| \leq$$

$$K_i(q) \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{t-}^{\alpha} E\left(\left(1 + e^{-W}\right)^{-\delta}\right), \frac{1}{n^{\beta}} \right)_{[t_1,t]} + \omega_1 \left(D_{*t}^{\alpha} E\left(\left(1 + e^{-W}\right)^{-\delta}\right), \frac{1}{n^{\beta}} \right)_{[t,t_2]} \right)}{n^{\alpha\beta}} + \right.$$

$$\gamma\hat{\beta}_{i,n}(\lambda,\beta)\left(\left\|D_{t-}^{\alpha}E\left(\left(1+e^{-W}\right)^{-\delta}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}E\left(\left(1+e^{-W}\right)^{-\delta}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\Bigg\}, \tag{100}$$

and

ii)

$$\left\|{}_iL_nE\left(\left(1+e^{-W}\right)^{-\delta}\right)-E\left(\left(1+e^{-W}\right)^{-\delta}\right)\right\|_{\infty}\leq K_i(q)\frac{1}{\Gamma(\alpha+1)}\cdot$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}E\left(\left(1+e^{-W}\right)^{-\delta}\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}E\left(\left(1+e^{-W}\right)^{-\delta}\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.(t_2-t_1)^{\alpha}\gamma\hat{\beta}_{i,n}(\lambda,\beta)\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}E\left(\left(1+e^{-W}\right)^{-\delta}\right)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\alpha}E\left(\left(1+e^{-W}\right)^{-\delta}\right)\right\|_{\infty,[t,t_2]}\right)\right\}. \tag{101}$$

Proof. From Proposition 26. □

When $\delta=1$ we have the usual logistic sigmoid function.

The Gompertz function $g:\mathbb{R}\to\mathbb{R}$, with $g(x)=e^{\mu e^{-x}}$, $\mu<0$ is a sigmoid function which describes growth as being slowest at the start and end of a given time period. Let $W(t,\phi)$ be the Brownian motion on S^3. Then the expectation

$$E\left(e^{\mu e^{-W}}\right)(t)=\int_0^{\pi}e^{\mu e^{-\phi}}p(t,\phi)d\phi$$

is continuous in t.

It follows,

Corollary 38. *Let $0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $t\in[t_1,t_2]$, where $t_1>0$. Then for $i=1,2$ and $j=0,1$*

i)

$$\left|{}_iL_n\left(E\left(e^{\mu e^{-W}}\right)^{(j)}\right)(t)-E\left(e^{\mu e^{-W}}\right)^{(j)}(t)\right|\leq$$

$$K_i(q)\left[\omega_1\left(E\left(e^{\mu e^{-W}}\right)^{(j)},\frac{1}{n^{\alpha}}\right)+2\left\|E\left(e^{\mu e^{-W}}\right)^{(j)}\right\|_{\infty}\gamma\hat{\beta}_{i,n}(\lambda,\alpha)\right]=:\rho_i, \tag{102}$$

and

ii)

$$\left\|{}_iL_n\left(E\left(e^{\mu e^{-W}}\right)^{(j)}\right)-E\left(e^{\mu e^{-W}}\right)^{(j)}\right\|_{\infty}\leq\rho_i \tag{103}$$

We get that $\lim_{n\to\infty}{}_iL_n\left(E\left(e^{\mu e^{-W}}\right)^{(j)}\right)=E\left(e^{\mu e^{-W}}\right)^{(j)}$, *pointwise and uniformly.*

Proof. From Propositions 25 and 27. □

We finish with

Corollary 39. *Let* $0 < \alpha, \beta < 1$, $t \in [t_1, t_2]$,*where* $t_1 > 0$ $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then for* $i = 1, 2$

i)

$$\left| {}_iL_n \left(e^{\mu e^{-W}} \right)(t) - E\left(e^{\mu e^{-W}} \right)(t) \right| \leq$$

$$K_i(q) \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{t-}^{\alpha} E\left(e^{\mu e^{-W}} \right), \frac{1}{n^\beta} \right)_{[t_1,t]} + \omega_1 \left(D_{*t}^{\alpha} E\left(e^{\mu e^{-W}} \right), \frac{1}{n^\beta} \right)_{[t,t_2]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \gamma \hat{\beta}_{i,n}(\lambda, \beta) \left(\left\| D_{t-}^{\alpha} E\left(e^{\mu e^{-W}} \right) \right\|_{\infty,[t_1,t]} (t-t_1)^\alpha + \left\| D_{*t}^{\alpha} E\left(e^{\mu e^{-W}} \right) \right\|_{\infty,[t,t_2]} (t_2 - t)^\alpha \right) \right\}, \quad (104)$$

and

ii)

$$\left\| {}_iL_n E\left(e^{\mu e^{-W}} \right) - E\left(e^{\mu e^{-W}} \right) \right\|_\infty \leq K_i(q) \frac{1}{\Gamma(\alpha+1)} \cdot$$

$$\left\{ \frac{\left(\sup_{t\in[t_1,t_2]} \omega_1 \left(D_{t-}^{\alpha} E\left(e^{\mu e^{-W}} \right), \frac{1}{n^\beta} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left(D_{*t}^{\alpha} E\left(e^{\mu e^{-W}} \right), \frac{1}{n^\beta} \right)_{[t,t_2]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. (t_2 - t_1)^\alpha \gamma \hat{\beta}_{i,n}(\lambda, \beta) \left(\sup_{t\in[t_1,t_2]} \left\| D_{t-}^{\alpha} E\left(e^{\mu e^{-W}} \right) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D_{*t}^{\alpha} E\left(e^{\mu e^{-W}} \right) \right\|_{\infty,[t,t_2]} \right) \right\}. \quad (105)$$

Proof. From Proposition 26. □

References

[1] G.A. Anastassiou, Rate of convergence of some neural network operators to the unit-univariate case, J. Math Anal. Appl, 212 (1997), 237-262.

[2] G.A. Anastassiou, Quantitative Approximations, Chapman and Hall / CRC Boca Raton, New York, 2001.

[3] G.A. Anastassiou, On Right Fractional Calculus, Chaos, solitons and fractals, 42 (2009), 365-376.

[4] G.A. Anastassiou, Inteligent Systems: Approximation by Artificial Neural Networks, Intelligent Systems Reference Library, Vol 19, Springer, Heidelberg, 2011.

[5] G.A. Anastassiou, Multivariate hyperbolic tangent neural network approximation, Computer and Mathematics, 61 (2011), 809-821.

[6] G.A. Anastassiou, Multivariate sigmoidal neural network approximation, Neural Networks, 24 (2011), 378-386.

[7] G.A. Anastassiou, Univariate hyperbolic tangent neural network approximation, Mathematics and Computer Modeling, 53 (2011), 1111-1132.

[8] G.A. Anastassiou, Fractional neural network approximation, Computer and Mathematics with Applications, 64 (2012), 1655-1676.

[9] G.A. Anastassiou, Univariate sigmoidal neural network approximation, J. of Computational Analysis and Applications, Vol. 14, No. 4, 2012, 659-690.

[10] G.A. Anastassiou, Inteligent Systems II: Complete Approximation by Neural Network Operators, Springer, Heidelberg, New York, 2016.

[11] G.A. Anastassiou, *A strong Fractional Calculus Theory for Banach space valued functions*, Nonlinear Functional Analysis and Applications (Korea), 22(3) (2017), 495-524.

[12] G.A. Anastassiou, *Strong Right Fractional Calculus for Banach space valued functions*, Revista Proyecciones, Vol. 36, No. 1(2017), 149-186.

[13] G.A. Anastassiou, *Vector fractional Korovkin type Approximations*, Dynamic Systems and Applications, 26(2017), 81-104.

[14] G.A. Anastassiou, *Nonlinearity: Ordinary and Fractional Approximations by Sublinear and Max-Product Operators*, Springer, Heidelberg, New York, 2018.

[15] G.A. Anastassiou, Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations, Springer, Heidelberg, New York, 2018.

[16] G.A. Anastasiou, *Abstract univariate neural network approximation using a q-deformed and λ-parametrized hyperbolic tangent activation function*, FRACT.FRACT./MDPI/ 2023,7,208.

[17] G.A. Anastassiou, *Banach Space Valued Neural Network*, Springer, Heidelberg, New York, 2023.

[18] G.A. Anastassiou, *General sigmoid based Banach space valued neural network approximation*, J. of Computational Analysis and Applications, 31(4) (2023), 520-534.

[19] G.A. Anastassiou, *Parametrized, deformed and general neural networks*, accepted, Springer, Heidelberg, New York, 2023.

[20] G.A. Anastassiou, *q-Deformed and λ-parametrized hyperbolic tangent function based Banach space valued multivariate multi layer neural network approximations*, Annales Univ. Sci.Budapest, Sect. Comp., accepted, 2023.

[21] G.A. Anastassiou, *q-Deformed and λ-parametrized A-generalized logistic function based Banach space valued ordinary and fractional neural network approximations*, submitted 2023.

[22] G.A. Anastassiou, *q-Deformed and λ-parametrized A-generalized logistic function indused Banach space valued multivariate multi layer neural network approximations*, submitted 2023.

[23] Z. Chen and F.Cao, The approximation operators with sigmoidal functions, Computers and Mathematics with Applications, 58 (2009), 758-765.

[24] K. Diethelm, The Analysis of Fractional Diferential Equations, Lecture Notes in Mathematics 2004, Springer-Verlag, Berlin, Heidelberg, 2010.

[25] A.M.A El-sayed and M. Geber, On the finite Caputo and finite Riesz derivatives, Electronic journal of Theoritical Physics, Vol. 3, No. 12 (2006), 81-95.

[26] G.S. Frederico and D.F.M. Torres, Fractional Optimal Control in the sense of Caputo and the fractional Noether's theorem, International Mathematical Forum, Vol. 3, No. 10 (2008), 479-493.

[27] D. Kouloumpou and V.G. Papanicolaou, Certain Calculation Regarding the Brownian Motion on the Sphere. Journal of Concrete and Applicable Mathematics. 2013 Jan 1;11(1).

[28] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, (Gordon and Breach, Amsterdam, 1993)[English translation from the Russian, Integrals and Derivatives of Fractional Order and Some of Their Applications (Nauka i Tekhnika, Minsk,1987)].