POWER INEQUALITIES FOR THE NUMERICAL RADIUS OF WEIGHTED SUMS OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. In this paper we obtain among others the following vector inequality for $r \geq 1$ and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$,

$$\left| \left\langle \left(\sum_{i=1}^{n} p_i V_i^* S_i^* \right) x, x \right\rangle \right|^{2r} \leq \|x\|^{2r} \left[\frac{1}{p} \left\langle \left(\sum_{i=1}^{n} p_i \left| S_i^* \right|^{2p} \right) x, x \right\rangle^r + \frac{1}{q} \left\langle \left(\sum_{i=1}^{n} p_i \left| V_i \right|^{2q} \right) x, x \right\rangle^r \right]$$

for all $x \in H$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, where $S_i, V_i \in \mathcal{B}(H)$ with $i \in \{1, ..., n\}$. We also established the numerical radius inequality

$$\omega^{2r} \left(\sum_{i=1}^{n} p_i V_i^* S_i^* \right) \leq \frac{1}{2} \left[\left\| \sum_{i=1}^{n} p_i \left| S_i^* \right|^2 \right\|^r \left\| \sum_{i=1}^{n} p_i \left| V_i \right|^2 \right\|^r + \omega^r \left(\sum_{i=1}^{n} p_i \left| V_i \right|^2 \sum_{i=1}^{n} p_i \left| S_i^* \right|^2 \right) \right]$$

for $r \geq 1$, where $S_i, V_i \in \mathcal{B}(H)$ with $i \in \{1, ..., n\}$.

1. Introduction

The *numerical radius* $\omega(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x \rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x \rangle| \leq w(T) \|x\|^2. \tag{1.2}$$

It is well known that $\omega(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T + V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{1.3}$$

for any $T \in B(H)$.

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F. Kittaneh, in 2003 [13], showed that for any operator $T\in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T)\leq\frac{1}{2}\left(\|T\|+\left\|T^{2}\right\|^{1/2}\right). \tag{1.4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [14] improved the inequality (1.3) as follows:

$$\frac{1}{4}\|T^{*}T+TT^{*}\|\leq\omega^{2}(T)\leq\frac{1}{2}\|T^{*}T+TT^{*}\| \tag{1.5}$$

for any operator $T\in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [11]:

If for an operator $T\in B(H)$ we denote $|T|:=(T^{*}T)^{1/2}$, then

$$\omega^{r}(T)\leq\frac{1}{2}\left\||T|^{2\alpha r}+|T^{*}|^{2(1-\alpha)r}\right\| \tag{1.6}$$

and

$$\omega^{2r}(T)\leq\left\|\alpha|T|^{2r}+(1-\alpha)|T^{*}|^{2r}\right\|, \tag{1.7}$$

where $\alpha\in(0,1)$ and $r\geq1$.

If we take $\alpha=\frac{1}{2}$ and $r=1$ we get from (1.6) that

$$\omega(T)\leq\frac{1}{2}\||T|+|T^{*}|\| \tag{1.8}$$

and from (1.7) that

$$\omega^{2}(T)\leq\frac{1}{2}\left\||T|^{2}+|T^{*}|^{2}\right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [9] and [5].

We recall the following vector inequality for positive operators $A\geq0$, obtained by C. A. McCarthy in [8]

$$\langle Ax,x\rangle^{p}\leq\langle A^{p}x,x\rangle,\ p\geq1$$

for $x\in H$, $\|x\|=1$ and Buzano's inequality [7],

$$|\langle x,e\rangle\langle e,y\rangle|\leq\frac{1}{2}\left[\|x\|\|y\|+|\langle x,y\rangle|\right] \tag{1.10}$$

that holds for any x, y, $e\in H$ with $\|e\|=1$.

If we replace x by $\frac{y}{\|y\|}$, $y\neq0$, we get

$$\left\langle A\frac{y}{\|y\|},\frac{y}{\|y\|}\right\rangle^{p}\leq\left\langle A^{p}\frac{y}{\|y\|},\frac{y}{\|y\|}\right\rangle,\ p\geq1,$$

namely

$$\langle Ay,y\rangle^{p}\leq\|y\|^{2(p-1)}\langle A^{p}y,y\rangle,\ p\geq1, \tag{1.11}$$

for all $y\in H$.

Also recall the following result for operator matrices obtained by F. Kittaneh in [12]:

Lemma 1. *Let A, B, $C \in \mathcal{B}(H)$ with A, $B \geq 0$. Then the operator matrix*

$$\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \in \mathcal{B}(H \oplus H)$$

is positive, if and only if

$$|\langle Cx, y\rangle|^2 \leq \langle Ax, x\rangle \langle By, y\rangle$$

for all x, $y \in H$.

In the recent paper [10] we proved that, if A, B, $C \in \mathcal{B}(H)$ satisfy the conditions in Lemma 1, then then for $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$,

$$\omega^{2r}(C) \leq \left\| \frac{1}{p} A^{pr} + \frac{1}{q} B^{qr} \right\|. \tag{1.12}$$

If $r \geq 1$, then

$$\omega^{2r}(C) \leq \frac{1}{2} \left[\|A\|^r \|B\|^r + \omega^r(BA) \right]. \tag{1.13}$$

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then also

$$\omega^{2r}(C) \leq \frac{1}{2} \left(\left\| \frac{1}{p} A^{pr} + \frac{1}{q} B^{qr} \right\| + \omega^r(BA) \right). \tag{1.14}$$

Moreover, for $\alpha \in [0,1]$ and $r \geq 1$,

$$\omega^2(C) \leq \|(1-\alpha) A^r + \alpha B^r\|^{1/r} \|A\|^{\alpha} \|B\|^{1-\alpha} \tag{1.15}$$

and

$$\omega^2(C) \leq \|(1-\alpha) A^r + \alpha B^r\|^{1/r} \|\alpha A^r + (1-\alpha) B^r\|^{1/r}. \tag{1.16}$$

By the use of the inequalities (1.12)-(1.16) we also obtained in [10] that:

Let S, $V \in \mathcal{B}(H)$, then for $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$,

$$\omega^{2r}(SV) \leq \left\| \frac{1}{p} |S^*|^{2pr} + \frac{1}{q} |V|^{2qr} \right\|. \tag{1.17}$$

If $r \geq 1$, then

$$\omega^{2r}(SV) \leq \frac{1}{2} \left[\|S\|^{2r} \|V\|^{2r} + \omega^r\left(|V|^2 |S^*|^2 \right) \right]. \tag{1.18}$$

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then also

$$\omega^{2r}(SV) \leq \frac{1}{2} \left(\left\| \frac{1}{p} |S^*|^{2pr} + \frac{1}{q} |V|^{2qr} \right\| + \omega^r\left(|V|^2 |S^*|^2 \right) \right). \tag{1.19}$$

Moreover, for $\alpha \in [0,1]$ and $r \geq 1$, we have that

$$\omega^2(SV) \leq \left\| (1-\alpha) |S^*|^{2r} + \alpha |V|^{2r} \right\|^{1/r} \|S\|^{2\alpha} \|V\|^{2(1-\alpha)} \tag{1.20}$$

and

$$\omega^2(SV) \leq \left\| (1-\alpha) |S^*|^{2r} + \alpha |V|^{2r} \right\|^{1/r} \left\| \alpha |S^*|^{2r} + (1-\alpha) |V|^{2r} \right\|^{1/r}. \tag{1.21}$$

Motivated by the above results, in this paper we obtain some generalizations of the previous inequalities involving weighted sums of operators. Among others we establish the following vector inequality for $r \geq 1$ and $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$,

$$\left|\left\langle \left(\sum_{i=1}^n p_i V_i^* S_i^*\right) x, x\right\rangle\right|^{2r} \leq \|x\|^{2r} \left[\frac{1}{p}\left\langle \left(\sum_{i=1}^n p_i |S_i^*|^{2p}\right) x, x\right\rangle^r + \frac{1}{q}\left\langle \left(\sum_{i=1}^n p_i |V_i|^{2q}\right) x, x\right\rangle^r\right]$$

for all $x \in H$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, where $S_i, V_i \in \mathcal{B}(H)$ with $i \in \{1,...,n\}$. We also have obtained the numerical radius inequality

$$\omega^{2r}\left(\sum_{i=1}^n p_i V_i^* S_i^*\right) \leq \frac{1}{2}\left[\left\|\sum_{i=1}^n p_i |S_i^*|^2\right\|^r \left\|\sum_{i=1}^n p_i |V_i|^2\right\|^r + \omega^r\left(\sum_{i=1}^n p_i |V_i|^2 \sum_{i=1}^n p_i |S_i^*|^2\right)\right]$$

for $r \geq 1$, where $S_i, V_i \in \mathcal{B}(H)$ with $i \in \{1,...,n\}$.

2. Vector Inequalities

In what follows we assume everywhere that the weights $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$.

Theorem 1. *Let $A_i, B_i, C_i \in \mathcal{B}(H)$ with $A_i, B_i \geq 0$ for $i \in \{1,...,n\}$ and such that the operator matrices*

$$\begin{bmatrix} A_i & C_i^* \\ C_i & B_i \end{bmatrix} \in \mathcal{B}(H \oplus H) \tag{2.1}$$

are positive for $i \in \{1,...,n\}$. Then for all $x, y \in H$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left|\left\langle \left(\sum_{i=1}^n p_i C_i\right) x, y\right\rangle\right|^2 \leq \|x\|^{2/q} \|y\|^{2/p} \left\langle \left(\sum_{i=1}^n p_i A_i^p\right) x, x\right\rangle^{1/p} \left\langle \left(\sum_{i=1}^n p_i B_i^q\right) y, y\right\rangle^{1/q}. \tag{2.2}$$

In particular

$$\left|\left\langle \left(\sum_{i=1}^n p_i C_i\right) x, y\right\rangle\right|^2 \leq \|x\| \|y\| \left\langle \left(\sum_{i=1}^n p_i A_i^2\right) x, x\right\rangle^{1/2} \left\langle \left(\sum_{i=1}^n p_i B_i^2\right) y, y\right\rangle^{1/2}. \tag{2.3}$$

Moreover,

$$\left|\left\langle \left(\sum_{i=1}^n p_i C_i\right) x, y\right\rangle\right| \leq \left\langle \left(\sum_{i=1}^n p_i A_i\right) x, x\right\rangle^{1/2} \left\langle \left(\sum_{i=1}^n p_i B_i\right) y, y\right\rangle^{1/2} \tag{2.4}$$

for all $x, y \in H$.

Proof. From Lemma 1 we have

$$|\langle C_i x, y\rangle|^2 \leq \langle A_i x, x\rangle \langle B_i y, y\rangle, \; x, y \in H \tag{2.5}$$

for $i \in \{1, ..., n\}$.

If we multiply by $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$ and sum, then we get

$$\sum_{i=1}^{n} p_i |\langle C_i x, y\rangle|^2 \leq \sum_{i=1}^{n} p_i \langle A_i x, x\rangle \langle B_i y, y\rangle, \; x, \; y \in H. \tag{2.6}$$

By Cauchy-Buniakowsky-Schwarz weighted inequality we have

$$\left|\left\langle \left(\sum_{i=1}^{n} p_i C_i\right) x, y\right\rangle\right|^2 \leq \sum_{i=1}^{n} p_i |\langle C_i x, y\rangle|^2, \tag{2.7}$$

while by weighted Hölder's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} &\sum_{i=1}^{n} p_i \langle A_i x, x\rangle \langle B_i y, y\rangle \\ &\leq \left(\sum_{i=1}^{n} p_i \langle A_i x, x\rangle^p\right)^{1/p} \left(\sum_{i=1}^{n} p_i \langle B_i y, y\rangle^q\right)^{1/q} \\ &\leq \left(\sum_{i=1}^{n} p_i \langle A_i^p x, x\rangle \|x\|^{2(p-1)}\right)^{1/p} \left(\sum_{i=1}^{n} p_i \langle B_i^q y, y\rangle \|y\|^{2(q-1)}\right)^{1/q} \\ &= \|x\|^{2\left(1-\frac{1}{p}\right)} \|y\|^{2\left(1-\frac{1}{q}\right)} \left\langle \left(\sum_{i=1}^{n} p_i A_i^p\right) x, x\right\rangle^{1/p} \left\langle \left(\sum_{i=1}^{n} p_i B_i^q\right) y, y\right\rangle^{1/q} \\ &= \|x\|^{2/q} \|y\|^{2/p} \left\langle \left(\sum_{i=1}^{n} p_i A_i^p\right) x, x\right\rangle^{1/p} \left\langle \left(\sum_{i=1}^{n} p_i B_i^q\right) y, y\right\rangle^{1/q} \end{aligned} \tag{2.8}$$

for x, $y \in H$, where for the last inequality we used McCarthy's result (1.11).

By making use of (2.6), (2.7) and (2.8) we get (2.2).

From (2.5) we get by taking the square root that

$$|\langle C_i x, y\rangle| \leq \langle A_i x, x\rangle^{1/2} \langle B_i y, y\rangle^{1/2}, \; x, \; y \in H,$$

which implies that

$$\sum_{i=1}^{n} p_i |\langle C_i x, y\rangle| \leq \sum_{i=1}^{n} p_i \langle A_i x, x\rangle^{1/2} \langle B_i y, y\rangle^{1/2}, \; x, \; y \in H. \tag{2.9}$$

By the weighted triangle inequality we get

$$\left|\left\langle \left(\sum_{i=1}^{n} p_i C_i\right) x, y\right\rangle\right| \leq \sum_{i=1}^{n} p_i |\langle C_i x, y\rangle|, \tag{2.10}$$

while by Cauchy-Buniakowsky-Schwarz weighted inequality we have

$$\begin{aligned}(2.11)\quad & \sum_{i=1}^{n} p_i \langle A_i x, x\rangle^{1/2} \langle B_i y, y\rangle^{1/2} \\ & \leq \left[\sum_{i=1}^{n} p_i \left(\langle A_i x, x\rangle^{1/2}\right)^2\right]^{1/2} \left[\sum_{i=1}^{n} p_i \left(\langle B_i y, y\rangle^{1/2}\right)^2\right]^{1/2} \\ & = \left[\sum_{i=1}^{n} p_i \langle A_i x, x\rangle\right]^{1/2} \left[\sum_{i=1}^{n} p_i \langle B_i y, y\rangle\right]^{1/2} \\ & = \left\langle \left(\sum_{i=1}^{n} p_i A_i\right) x, x\right\rangle^{1/2} \left\langle \left(\sum_{i=1}^{n} p_i B_i\right) y, y\right\rangle^{1/2}\end{aligned}$$

for $x,\ y \in H$.

By utilizing (2.9)-(2.11) we obtain (2.4). □

Corollary 1. *Let $A_i,\ B_i,\ C_i \in \mathcal{B}(H)$ with $A_i,\ B_i \geq 0$ for $i \in \{1, ..., n\}$ and such the operator matrices*

$$\begin{bmatrix} A_i & C_i^* \\ C_i & B_i \end{bmatrix} \in \mathcal{B}(H \oplus H)$$

are positive for $i \in \{1, ..., n\}$. Then for all $x \in H$ and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$(2.12)\quad \left|\left\langle \left(\sum_{i=1}^{n} p_i C_i\right) x, x\right\rangle\right|^2 \leq \|x\|^2 \left\langle \left(\sum_{i=1}^{n} p_i \left(\frac{1}{p} A_i^p + \frac{1}{q} B_i^q\right)\right) x, x\right\rangle.$$

In particular

$$(2.13)\quad \left|\left\langle \left(\sum_{i=1}^{n} p_i C_i\right) x, x\right\rangle\right|^2 \leq \|x\|^2 \left\langle \left(\sum_{i=1}^{n} p_i \left(\frac{A_i^2 + B_i^2}{2}\right)\right) x, x\right\rangle.$$

Moreover,

$$(2.14)\quad \left|\left\langle \left(\sum_{i=1}^{n} p_i C_i\right) x, x\right\rangle\right| \leq \left\langle \left(\sum_{i=1}^{n} p_i \frac{A_i + B_i}{2}\right) x, x\right\rangle$$

for all $x \in H$.

Proof. From (2.2) we get

$$\begin{aligned}(2.15)\quad & \left|\left\langle \left(\sum_{i=1}^{n} p_i C_i\right) x, x\right\rangle\right|^2 \\ & \leq \|x\|^{2/q} \|x\|^{2/p} \left\langle \left(\sum_{i=1}^{n} p_i A_i^p\right) x, x\right\rangle^{1/p} \left\langle \left(\sum_{i=1}^{n} p_i B_i^q\right) x, x\right\rangle^{1/q} \\ & = \|x\|^2 \left\langle \left(\sum_{i=1}^{n} p_i A_i^p\right) x, x\right\rangle^{1/p} \left\langle \left(\sum_{i=1}^{n} p_i B_i^q\right) x, x\right\rangle^{1/q}.\end{aligned}$$

From Young's inequality

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q,\ a,\ b \geq 0,\ p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1,$$

we have

$$\left\langle \left(\sum_{i=1}^{n} p_i A_i^p \right) x, x \right\rangle^{1/p} \left\langle \left(\sum_{i=1}^{n} p_i B_i^q \right) x, x \right\rangle^{1/q} \tag{2.16}$$
$$\leq \frac{1}{p} \left\langle \left(\sum_{i=1}^{n} p_i A_i^p \right) x, x \right\rangle + \frac{1}{q} \left\langle \left(\sum_{i=1}^{n} p_i B_i^q \right) x, x \right\rangle$$
$$= \left\langle \left(\sum_{i=1}^{n} p_i \left(\frac{1}{p} A_i^p + \frac{1}{q} B_i^q \right) \right) x, x \right\rangle .$$

By using (2.15) and (2.16) we obtain (2.12).

From (2.4) we have

$$\left| \left\langle \left(\sum_{i=1}^{n} p_i C_i \right) x, x \right\rangle \right| \leq \left\langle \left(\sum_{i=1}^{n} p_i A_i \right) x, x \right\rangle^{1/2} \left\langle \left(\sum_{i=1}^{n} p_i B_i \right) x, x \right\rangle^{1/2} . \tag{2.17}$$

By the elementary inequality $ab \leq \frac{1}{2} \left(a^2 + b^2 \right)$, a, $b \geq 0$, we get

$$\left\langle \left(\sum_{i=1}^{n} p_i A_i \right) x, x \right\rangle^{1/2} \left\langle \left(\sum_{i=1}^{n} p_i B_i \right) x, x \right\rangle^{1/2} \tag{2.18}$$
$$\leq \frac{1}{2} \left[\left\langle \left(\sum_{i=1}^{n} p_i A_i \right) x, x \right\rangle + \left\langle \left(\sum_{i=1}^{n} p_i B_i \right) x, x \right\rangle \right]$$
$$= \left\langle \left(\sum_{i=1}^{n} p_i \frac{A_i + B_i}{2} \right) x, x \right\rangle$$

and by (2.17) and (2.18) we get (2.14). □

Corollary 2. *With the assumptions of Corollary 1 for* A_i, B_i, $C_i \in \mathcal{B}(H)$ *with* $i \in \{1, ..., n\}$, *we have*

$$\left| \left\langle \left(\sum_{i=1}^{n} p_i C_i \right) x, x \right\rangle \right|^4 \leq \frac{1}{2} \left[\left\| \left(\sum_{i=1}^{n} p_i A_i^2 \right) x \right\| \left\| \left(\sum_{i=1}^{n} p_i B_i^2 \right) x \right\| + \left| \left\langle \left(\sum_{i=1}^{n} p_i B_i^2 \right) \left(\sum_{i=1}^{n} p_i A_i^2 \right) x, x \right\rangle \right| \right] \tag{2.19}$$

and

$$\left| \left\langle \left(\sum_{i=1}^{n} p_i C_i \right) x, x \right\rangle \right|^2 \leq \frac{1}{2} \left[\left\| \left(\sum_{i=1}^{n} p_i A_i \right) x \right\| \left\| \left(\sum_{i=1}^{n} p_i B_i \right) x \right\| + \left| \left\langle \left(\sum_{i=1}^{n} p_i B_i \right) \left(\sum_{i=1}^{n} p_i A_i \right) x, x \right\rangle \right| \right] \tag{2.20}$$

for $x \in H$, $\|x\| = 1$.

Proof. From (2.3) we get for $x \in H$, $\|x\| = 1$ that

$$\left| \left\langle \left(\sum_{i=1}^{n} p_i C_i \right) x, x \right\rangle \right|^4 \leq \left\langle \left(\sum_{i=1}^{n} p_i A_i^2 \right) x, x \right\rangle \left\langle x, \left(\sum_{i=1}^{n} p_i B_i^2 \right) x \right\rangle . \tag{2.21}$$

By utilizing Buzano's inequality we have for $x \in H$, $\|x\| = 1$ that

$$\begin{aligned}
&\left| \left\langle \left(\sum_{i=1}^{n} p_i A_i^2 \right) x, x \right\rangle \left\langle x, \left(\sum_{i=1}^{n} p_i B_i^2 \right) x \right\rangle \right| \\
&\leq \frac{1}{2} \left[\left\| \left(\sum_{i=1}^{n} p_i A_i^2 \right) x \right\| \left\| \left(\sum_{i=1}^{n} p_i B_i^2 \right) x \right\| \right. \\
&\left. + \left| \left\langle \left(\sum_{i=1}^{n} p_i A_i^2 \right) x, \left(\sum_{i=1}^{n} p_i B_i^2 \right) x \right\rangle \right| \right] \\
&= \frac{1}{2} \left[\left\| \left(\sum_{i=1}^{n} p_i A_i^2 \right) x \right\| \left\| \left(\sum_{i=1}^{n} p_i B_i^2 \right) x \right\| \right. \\
&\left. + \left| \left\langle \left(\sum_{i=1}^{n} p_i B_i^2 \right) \left(\sum_{i=1}^{n} p_i A_i^2 \right) x, x \right\rangle \right| \right]
\end{aligned}$$

and by (2.21) we get (2.19).

From (2.4) we get for $x \in H$, $\|x\| = 1$

$$\left| \left\langle \left(\sum_{i=1}^{n} p_i C_i \right) x, x \right\rangle \right|^2 \leq \left\langle \left(\sum_{i=1}^{n} p_i A_i \right) x, x \right\rangle \left\langle x, \left(\sum_{i=1}^{n} p_i B_i \right) x \right\rangle . \tag{2.22}$$

By utilizing Buzano's inequality we have for $x \in H$, $\|x\| = 1$ that

$$\begin{aligned}
&\left\langle \left(\sum_{i=1}^{n} p_i A_i \right) x, x \right\rangle \left\langle x, \left(\sum_{i=1}^{n} p_i B_i \right) x \right\rangle \\
&\leq \frac{1}{2} \left[\left\| \left(\sum_{i=1}^{n} p_i A_i \right) x \right\| \left\| \left(\sum_{i=1}^{n} p_i B_i \right) x \right\| \right. \\
&\left. + \left| \left\langle \left(\sum_{i=1}^{n} p_i B_i \right) \left(\sum_{i=1}^{n} p_i A_i \right) x, x \right\rangle \right| \right],
\end{aligned}$$

which proves (2.20). □

We also have the power inequalities:

Corollary 3. *With the assumptions of Corollary 1 for A_i, B_i, $C_i \in \mathcal{B}(H)$ with $i \in \{1, ..., n\}$, we have for $r \geq 1$ that*

$$\begin{aligned}
&\left| \left\langle \left(\sum_{i=1}^{n} p_i C_i \right) x, x \right\rangle \right|^{2r} \\
&\leq \|x\|^{2r} \left[\frac{1}{p} \left\langle \left(\sum_{i=1}^{n} p_i A_i^p \right) x, x \right\rangle^r + \frac{1}{q} \left\langle \left(\sum_{i=1}^{n} p_i B_i^q \right) x, x \right\rangle^r \right].
\end{aligned} \tag{2.23}$$

for all $x \in H$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular

$$\left|\left\langle\left(\sum_{i=1}^{n}p_iC_i\right)x,x\right\rangle\right|^{2r} \tag{2.24}$$
$$\leq \frac{1}{2}\left\|x\right\|^{2r}\left[\left\langle\left(\sum_{i=1}^{n}p_iA_i^2\right)x,x\right\rangle^r+\left\langle\left(\sum_{i=1}^{n}p_iB_i^2\right)x,x\right\rangle^r\right].$$

Moreover,

$$\left|\left\langle\left(\sum_{i=1}^{n}p_iC_i\right)x,x\right\rangle\right|^{r} \tag{2.25}$$
$$\leq \frac{1}{2}\left[\left\langle\left(\sum_{i=1}^{n}p_iA_i\right)x,x\right\rangle^r+\left\langle\left(\sum_{i=1}^{n}p_iB_i\right)x,x\right\rangle^r\right]$$

for all $x\in H$.

Proof. If we take the power $r\geq 1$ in (2.12), then we get, by the convexity of power function, that

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n}p_iC_i\right)x,x\right\rangle\right|^{2r}\\
&\leq \left\|x\right\|^{2r}\left\langle\left(\sum_{i=1}^{n}p_i\left(\frac{1}{p}A_i^p+\frac{1}{q}B_i^q\right)\right)x,x\right\rangle^r\\
&= \left\|x\right\|^{2r}\left(\frac{1}{p}\sum_{i=1}^{n}p_i\left\langle A_i^px,x\right\rangle+\frac{1}{q}\sum_{i=1}^{n}p_i\left\langle B_i^qx,x\right\rangle\right)^r\\
&\leq \left\|x\right\|^{2r}\left[\frac{1}{p}\left(\sum_{i=1}^{n}p_i\left\langle A_i^px,x\right\rangle\right)^r+\frac{1}{q}\left(\sum_{i=1}^{n}p_i\left\langle B_i^qx,x\right\rangle\right)^r\right]\\
&= \left\|x\right\|^{2r}\left[\frac{1}{p}\left\langle\left(\sum_{i=1}^{n}p_iA_i^p\right)x,x\right\rangle^r+\frac{1}{q}\left\langle\left(\sum_{i=1}^{n}p_iB_i^q\right)x,x\right\rangle^r\right],
\end{aligned}$$

for all $x\in H$. This proves (2.23).

From (2.14) we obtain, by taking the power $r\geq 1$, that

$$\begin{aligned}
\left|\left\langle\left(\sum_{i=1}^{n}p_iC_i\right)x,x\right\rangle\right|^{r} &\leq \left\langle\left(\sum_{i=1}^{n}p_i\frac{A_i+B_i}{2}\right)x,x\right\rangle^r\\
&= \frac{1}{2}\left(\sum_{i=1}^{n}p_i\left\langle A_ix,x\right\rangle+\sum_{i=1}^{n}p_i\left\langle B_ix,x\right\rangle\right)^r\\
&\leq \frac{1}{2}\left[\left(\sum_{i=1}^{n}p_i\left\langle A_ix,x\right\rangle\right)^r+\left(\sum_{i=1}^{n}p_i\left\langle B_ix,x\right\rangle\right)^r\right]\\
&= \frac{1}{2}\left[\left\langle\left(\sum_{i=1}^{n}p_iA_i\right)x,x\right\rangle^r+\left\langle\left(\sum_{i=1}^{n}p_iB_i\right)x,x\right\rangle^r\right],
\end{aligned}$$

which proves (2.25). □

Corollary 4. *With the assumptions of Corollary 1 for A_i, B_i, $C_i \in \mathcal{B}(H)$ with $i \in \{1,...,n\}$, we have for $r \geq 1$ that*

$$\left|\left\langle\left(\sum_{i=1}^{n} p_i C_i\right)x,x\right\rangle\right|^{4r} \leq \frac{1}{2}\left[\left\|\left(\sum_{i=1}^{n} p_i A_i^2\right)x\right\|^r \left\|\left(\sum_{i=1}^{n} p_i B_i^2\right)x\right\|^r + \left|\left\langle\left(\sum_{i=1}^{n} p_i B_i^2\right)\left(\sum_{i=1}^{n} p_i A_i^2\right)x,x\right\rangle\right|^r\right] \tag{2.26}$$

and

$$\left|\left\langle\left(\sum_{i=1}^{n} p_i C_i\right)x,x\right\rangle\right|^{2r} \leq \frac{1}{2}\left[\left\|\left(\sum_{i=1}^{n} p_i A_i\right)x\right\|^r \left\|\left(\sum_{i=1}^{n} p_i B_i\right)x\right\|^r + \left|\left\langle\left(\sum_{i=1}^{n} p_i B_i\right)\left(\sum_{i=1}^{n} p_i A_i\right)x,x\right\rangle\right|^r\right] \tag{2.27}$$

for $x \in H$, $\|x\| = 1$.

We also have:

Theorem 2. *Let S_i, $V_i \in \mathcal{B}(H)$ with $i \in \{1,...,n\}$. Then for all x, $y \in H$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$*

$$\left|\left\langle\left(\sum_{i=1}^{n} p_i V_i^* S_i^*\right)x,y\right\rangle\right|^2 \leq \|x\|^{2/q}\|y\|^{2/p}\left\langle\left(\sum_{i=1}^{n} p_i |S_i^*|^{2p}\right)x,x\right\rangle^{1/p}\left\langle\left(\sum_{i=1}^{n} p_i |V_i|^{2q}\right)y,y\right\rangle^{1/q}. \tag{2.28}$$

In particular

$$\left|\left\langle\left(\sum_{i=1}^{n} p_i V_i^* S_i^*\right)x,y\right\rangle\right|^2 \leq \|x\|\|y\|\left\langle\left(\sum_{i=1}^{n} p_i |S_i^*|^{4}\right)x,x\right\rangle^{1/2}\left\langle\left(\sum_{i=1}^{n} p_i |V_i|^{4}\right)y,y\right\rangle^{1/2}. \tag{2.29}$$

Moreover,

$$\left|\left\langle\left(\sum_{i=1}^{n} p_i V_i^* S_i^*\right)x,y\right\rangle\right| \leq \left\langle\left(\sum_{i=1}^{n} p_i |S_i^*|^{2}\right)x,x\right\rangle^{1/2}\left\langle\left(\sum_{i=1}^{n} p_i |V_i|^{2}\right)y,y\right\rangle^{1/2} \tag{2.30}$$

for all x, $y \in H$.

Proof. Let S_i, $V_i \in \mathcal{B}(H)$ with $i \in \{1,...,n\}$. Observe that the operator matrices

$$\begin{bmatrix} S_i S_i^* & S_i V_i \\ V_i^* S_i^* & V_i^* V_i \end{bmatrix} \in \mathcal{B}(H \oplus H)$$

are positive for $i \in \{1,...,n\}$.

Then by Theorem 1 for $A_i = |S_i^*|^2$, $B_i = |V_i|^2$ and $C_i = V_i^* S_i^*$ we get the desired results. □

Corollary 5. *With the assumptions of Theorem 2, we have*

$$\left|\left\langle\left(\sum_{i=1}^n p_i V_i^* S_i^*\right)x, x\right\rangle\right|^2 \le \|x\|^2 \left\langle\left(\sum_{i=1}^n p_i \left(\frac{1}{p}|S_i^*|^{2p} + \frac{1}{q}|V_i|^{2q}\right)\right)x, x\right\rangle \tag{2.31}$$

for all $x \in H$ and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular

$$\left|\left\langle\left(\sum_{i=1}^n p_i V_i^* S_i^*\right)x, x\right\rangle\right|^2 \le \|x\|^2 \left\langle\left(\sum_{i=1}^n p_i \left(\frac{|S_i^*|^4 + |V_i|^4}{2}\right)\right)x, x\right\rangle. \tag{2.32}$$

Moreover,

$$\left|\left\langle\left(\sum_{i=1}^n p_i V_i^* S_i^*\right)x, x\right\rangle\right| \le \left\langle\left(\sum_{i=1}^n p_i \frac{|S_i^*|^2 + |V_i|^2}{2}\right)x, x\right\rangle \tag{2.33}$$

for all $x \in H$.

Corollary 6. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}\left|\left\langle\left(\sum_{i=1}^n p_i V_i^* S_i^*\right)x, x\right\rangle\right|^4 \le \frac{1}{2}\Bigg[&\left\|\left(\sum_{i=1}^n p_i |S_i^*|^4\right)x\right\|\left\|\left(\sum_{i=1}^n p_i |V_i|^4\right)x\right\| \\ &+\left|\left\langle\left(\sum_{i=1}^n p_i |V_i|^2\right)\left(\sum_{i=1}^n p_i |S_i^*|^4\right)x, x\right\rangle\right|\Bigg]\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}\left|\left\langle\left(\sum_{i=1}^n p_i V_i^* S_i^*\right)x, x\right\rangle\right|^2 \le \frac{1}{2}\Bigg[&\left\|\left(\sum_{i=1}^n p_i |S_i^*|^2\right)x\right\|\left\|\left(\sum_{i=1}^n p_i |V_i|^2\right)x\right\| \\ &+\left|\left\langle\left(\sum_{i=1}^n p_i |V_i|^2\right)\left(\sum_{i=1}^n p_i |S_i^*|^2\right)x, x\right\rangle\right|\Bigg]\end{aligned} \tag{2.35}$$

for $x \in H$, $\|x\| = 1$.

Remark 1. *Let $S_i,\ V_i \in \mathcal{B}(H)$ with $i \in \{1, ..., n\}$ and $r \ge 1$. Then we get from Corollary 3 that*

$$\begin{aligned}&\left|\left\langle\left(\sum_{i=1}^n p_i V_i^* S_i^*\right)x, x\right\rangle\right|^{2r} \\ &\le \|x\|^{2r}\left[\frac{1}{p}\left\langle\left(\sum_{i=1}^n p_i |S_i^*|^{2p}\right)x, x\right\rangle^r + \frac{1}{q}\left\langle\left(\sum_{i=1}^n p_i |V_i|^{2q}\right)x, x\right\rangle^r\right]\end{aligned} \tag{2.36}$$

for all $x \in H$ *and* $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *In particular,*

$$\left|\left\langle \left(\sum_{i=1}^{n} p_i V_i^* S_i^*\right) x, x\right\rangle\right|^{2r} \leq \frac{1}{2}\|x\|^{2r}\left[\left\langle \left(\sum_{i=1}^{n} p_i |S_i^*|^4\right) x, x\right\rangle^{r} + \left\langle \left(\sum_{i=1}^{n} p_i |V_i|^4\right) x, x\right\rangle^{r}\right]. \tag{2.37}$$

Moreover,

$$\left|\left\langle \left(\sum_{i=1}^{n} p_i V_i^* S_i^*\right) x, x\right\rangle\right|^{r} \leq \frac{1}{2}\left[\left\langle \left(\sum_{i=1}^{n} p_i |S_i^*|^2\right) x, x\right\rangle^{r} + \left\langle \left(\sum_{i=1}^{n} p_i |V_i|^2\right) x, x\right\rangle^{r}\right] \tag{2.38}$$

for all $x \in H$.

Also, from Corollary 4 we get

$$\left|\left\langle \left(\sum_{i=1}^{n} p_i V_i^* S_i^*\right) x, x\right\rangle\right|^{4r} \leq \frac{1}{2}\left[\left\|\left(\sum_{i=1}^{n} p_i |S_i^*|^4\right) x\right\|^{r}\left\|\left(\sum_{i=1}^{n} p_i |V_i|^4\right) x\right\|^{r} + \left|\left\langle \left(\sum_{i=1}^{n} p_i |V_i|^4\right)\left(\sum_{i=1}^{n} p_i |S_i^*|^4\right) x, x\right\rangle\right|^{r}\right] \tag{2.39}$$

and

$$\left|\left\langle \left(\sum_{i=1}^{n} p_i V_i^* S_i^*\right) x, x\right\rangle\right|^{2r} \leq \frac{1}{2}\left[\left\|\left(\sum_{i=1}^{n} p_i |S_i^*|^2\right) x\right\|^{r}\left\|\left(\sum_{i=1}^{n} p_i |V_i|^2\right) x\right\|^{r} + \left|\left\langle \left(\sum_{i=1}^{n} p_i |V_i|^2\right)\left(\sum_{i=1}^{n} p_i |S_i^*|^2\right) x, x\right\rangle\right|^{r}\right] \tag{2.40}$$

for $x \in H$, $\|x\| = 1$.

3. Norm and Numerical Radius Inequalities

We also have:

Theorem 3. *Let* $A_i,\ B_i,\ C_i \in \mathcal{B}(H)$ *with* $A_i,\ B_i \geq 0$ *for* $i \in \{1,...,n\}$ *and such that the operator matrices*

$$\begin{bmatrix} A_i & C_i^* \\ C_i & B_i \end{bmatrix} \in \mathcal{B}(H \oplus H)$$

are positive for $i \in \{1,...,n\}$. *Then for all and* $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$,

$$\left\|\sum_{i=1}^{n} p_i C_i\right\|^{2} \leq \left\|\sum_{i=1}^{n} p_i A_i^p\right\|^{1/p}\left\|\sum_{i=1}^{n} p_i B_i^q\right\|^{1/q}. \tag{3.1}$$

In particular

$$\left\|\sum_{i=1}^{n} p_i C_i\right\|^{2} \leq \left\|\sum_{i=1}^{n} p_i A_i^2\right\|^{1/2}\left\|\sum_{i=1}^{n} p_i B_i^2\right\|^{1/2}. \tag{3.2}$$

Moreover,

$$\left\| \sum_{i=1}^{n} p_i C_i \right\|^2 \leq \left\| \sum_{i=1}^{n} p_i A_i \right\| \left\| \sum_{i=1}^{n} p_i B_i \right\|. \tag{3.3}$$

Proof. We have, by (2.2) that

$$\begin{aligned}
\left\| \sum_{i=1}^{n} p_i C_i \right\|^2 &= \sup_{\|x\|=\|y\|=1} \left| \left\langle \left(\sum_{i=1}^{n} p_i C_i \right) x, y \right\rangle \right|^2 \\
&\leq \sup_{\|x\|=\|y\|=1} \left[\left\langle \left(\sum_{i=1}^{n} p_i A_i^p \right) x, x \right\rangle^{1/p} \left\langle \left(\sum_{i=1}^{n} p_i B_i^q \right) y, y \right\rangle^{1/q} \right] \\
&= \sup_{\|x\|=1} \left\langle \left(\sum_{i=1}^{n} p_i A_i^p \right) x, x \right\rangle^{1/p} \sup_{\|y\|=1} \left\langle \left(\sum_{i=1}^{n} p_i B_i^q \right) y, y \right\rangle^{1/q} \\
&= \left\| \sum_{i=1}^{n} p_i A_i^p \right\|^{1/p} \left\| \sum_{i=1}^{n} p_i B_i^q \right\|^{1/q},
\end{aligned}$$

which proves (3.1).

From (2.4) we get

$$\begin{aligned}
\left\| \sum_{i=1}^{n} p_i C_i \right\| &= \sup_{\|x\|=\|y\|=1} \left| \left\langle \left(\sum_{i=1}^{n} p_i C_i \right) x, y \right\rangle \right| \\
&\leq \sup_{\|x\|=\|y\|=1} \left\langle \left(\sum_{i=1}^{n} p_i A_i \right) x, x \right\rangle^{1/2} \left\langle \left(\sum_{i=1}^{n} p_i B_i \right) y, y \right\rangle^{1/2} \\
&= \sup_{\|x\|=1} \left\langle \left(\sum_{i=1}^{n} p_i A_i \right) x, x \right\rangle^{1/2} \sup_{\|y\|=1} \left\langle \left(\sum_{i=1}^{n} p_i B_i \right) y, y \right\rangle^{1/2} \\
&= \left\| \sum_{i=1}^{n} p_i A_i \right\|^{1/2} \left\| \sum_{i=1}^{n} p_i B_i \right\|^{1/2},
\end{aligned}$$

which proves (3.3). □

We also have:

Theorem 4. *With the assumptions of Theorem 3 we have*

$$\omega^2 \left(\sum_{i=1}^{n} p_i C_i \right) \leq \left\| \frac{1}{p} \sum_{i=1}^{n} p_i A_i^p + \frac{1}{q} \sum_{i=1}^{n} p_i B_i^q \right\| \tag{3.4}$$

for all and p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

In particular

$$\omega^2 \left(\sum_{i=1}^{n} p_i C_i \right) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} p_i \left(A_i^2 + B_i^2 \right) \right\|. \tag{3.5}$$

Moreover,

$$\omega \left(\sum_{i=1}^{n} p_i C_i \right) \leq \left\| \sum_{i=1}^{n} p_i \left(\frac{A_i + B_i}{2} \right) \right\|. \tag{3.6}$$

We also have

$$\omega^4\left(\sum_{i=1}^n p_i C_i\right) \le \frac{1}{2}\left[\left\|\sum_{i=1}^n p_i A_i^2\right\|\left\|\sum_{i=1}^n p_i B_i^2\right\| + \omega\left(\sum_{i=1}^n p_i B_i^2 \sum_{i=1}^n p_i A_i^2\right)\right] \tag{3.7}$$

and

$$\omega^2\left(\sum_{i=1}^n p_i C_i\right) \le \frac{1}{2}\left[\left\|\sum_{i=1}^n p_i A_i\right\|\left\|\sum_{i=1}^n p_i B_i\right\| + \omega\left(\sum_{i=1}^n p_i B_i \sum_{i=1}^n p_i A_i\right)\right]. \tag{3.8}$$

Proof. From (2.12) we have

$$\begin{aligned}\omega^2\left(\sum_{i=1}^n p_i C_i\right) &= \sup_{\|x\|=1}\left|\left\langle\left(\sum_{i=1}^n p_i C_i\right)x, x\right\rangle\right|^2 \\ &\le \sup_{\|x\|=1}\left\langle\left(\sum_{i=1}^n p_i\left(\frac{1}{p}A_i^p + \frac{1}{q}B_i^q\right)\right)x, x\right\rangle \\ &= \left\|\sum_{i=1}^n p_i\left(\frac{1}{p}A_i^p + \frac{1}{q}B_i^q\right)\right\| = \left\|\frac{1}{p}\sum_{i=1}^n p_i A_i^p + \frac{1}{q}\sum_{i=1}^n p_i B_i^q\right\|,\end{aligned}$$

which proves (3.4).

The inequality (3.6) follows by (2.14) on taking $\sup_{\|x\|=1}$.

From (2.19) we have

$$\begin{aligned}\omega^4\left(\sum_{i=1}^n p_i C_i\right) &= \sup_{\|x\|=1}\left|\left\langle\left(\sum_{i=1}^n p_i C_i\right)x, x\right\rangle\right|^4 \\ &\le \frac{1}{2}\sup_{\|x\|=1}\left[\left\|\left(\sum_{i=1}^n p_i A_i^2\right)x\right\|\left\|\left(\sum_{i=1}^n p_i B_i^2\right)x\right\|\right. \\ &\left.+\left|\left\langle\left(\sum_{i=1}^n p_i B_i^2\right)\left(\sum_{i=1}^n p_i A_i^2\right)x, x\right\rangle\right|\right] \\ &\le \frac{1}{2}\left[\sup_{\|x\|=1}\left\|\left(\sum_{i=1}^n p_i A_i^2\right)x\right\|\left\|\left(\sum_{i=1}^n p_i B_i^2\right)x\right\|\right. \\ &\left.+\sup_{\|x\|=1}\left|\left\langle\left(\sum_{i=1}^n p_i B_i^2\right)\left(\sum_{i=1}^n p_i A_i^2\right)x, x\right\rangle\right|\right] \\ &= \frac{1}{2}\left[\left\|\sum_{i=1}^n p_i A_i^2\right\|\left\|\sum_{i=1}^n p_i B_i^2\right\| + \omega\left(\sum_{i=1}^n p_i B_i^2 \sum_{i=1}^n p_i A_i^2\right)\right],\end{aligned}$$

which proves (3.7).

The inequality (3.8) follows in a similar way from (2.20). □

Remark 2. *If we use Corollary 3 we have for* $r \geq 1$ *that*

$$\omega^{2r}\left(\sum_{i=1}^{n} p_i C_i\right) \leq \frac{1}{p}\left\|\sum_{i=1}^{n} p_i A_i^p\right\|^r + \frac{1}{q}\left\|\sum_{i=1}^{n} p_i B_i^q\right\|^r. \tag{3.9}$$

for all $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *In particular*

$$\omega^{2r}\left(\sum_{i=1}^{n} p_i C_i\right) \leq \frac{1}{2}\left(\left\|\sum_{i=1}^{n} p_i A_i^2\right\|^r + \left\|\sum_{i=1}^{n} p_i B_i^2\right\|^r\right). \tag{3.10}$$

Moreover,

$$\omega^{r}\left(\sum_{i=1}^{n} p_i C_i\right) \leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i A_i\right\|^r + \left\|\sum_{i=1}^{n} p_i B_i\right\|^r\right]. \tag{3.11}$$

If we use Corollary 4 we also have

$$\omega^{4r}\left(\sum_{i=1}^{n} p_i C_i\right) \leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i A_i^2\right\|^r \left\|\sum_{i=1}^{n} p_i B_i^2\right\|^r + \omega^r\left(\sum_{i=1}^{n} p_i B_i^2 \sum_{i=1}^{n} p_i A_i^2\right)\right] \tag{3.12}$$

and

$$\omega^{2r}\left(\sum_{i=1}^{n} p_i C_i\right) \leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i A_i\right\|^r \left\|\sum_{i=1}^{n} p_i B_i\right\|^r + \omega^r\left(\sum_{i=1}^{n} p_i B_i \sum_{i=1}^{n} p_i A_i\right)\right]. \tag{3.13}$$

We also have the following results for two sequences of operators:

Theorem 5. *Let* $S_i,\ V_i \in \mathcal{B}(H)$ *with* $i \in \{1, ..., n\}$. *Then for all* $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$

$$\left\|\sum_{i=1}^{n} p_i V_i^* S_i^*\right\|^2 \leq \left\|\sum_{i=1}^{n} p_i \left|S_i^*\right|^{2p}\right\|^{1/p} \left\|\sum_{i=1}^{n} p_i \left|V_i\right|^{2q}\right\|^{1/q}.$$

In particular

$$\left\|\sum_{i=1}^{n} p_i V_i^* S_i^*\right\|^2 \leq \left\|\sum_{i=1}^{n} p_i \left|S_i^*\right|^{4}\right\|^{1/2} \left\|\sum_{i=1}^{n} p_i \left|V_i\right|^{4}\right\|^{1/2}.$$

Moreover,

$$\left\|\sum_{i=1}^{n} p_i V_i^* S_i^*\right\|^2 \leq \left\|\sum_{i=1}^{n} p_i \left|S_i^*\right|^{2}\right\| \left\|\sum_{i=1}^{n} p_i \left|V_i\right|^{2}\right\|.$$

The proof follows by Theorem 2 by taking the supremum over $\|x\| = \|y\| = 1$.
We also have the following numerical radius inequalities:

Theorem 6. *Let* $S_i,\ V_i \in \mathcal{B}(H)$ *with* $i \in \{1, ..., n\}$. *Then*

$$\omega^2\left(\sum_{i=1}^{n} p_i V_i^* S_i^*\right) \leq \left\|\sum_{i=1}^{n} p_i \left(\frac{1}{p}\left|S_i^*\right|^{2p} + \frac{1}{q}\left|V_i\right|^{2q}\right)\right\|,$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *In particular*

$$\omega^{2}\left(\sum_{i=1}^{n}p_{i}V_{i}^{*}S_{i}^{*}\right)\leq\left\Vert\sum_{i=1}^{n}p_{i}\left(\frac{\left\vert S_{i}^{*}\right\vert^{4}+\left\vert V_{i}\right\vert^{4}}{2}\right)\right\Vert .$$

Moreover,

$$\omega\left(\sum_{i=1}^{n}p_{i}V_{i}^{*}S_{i}^{*}\right)\leq\left\Vert\sum_{i=1}^{n}p_{i}\frac{\left\vert S_{i}^{*}\right\vert^{2}+\left\vert V_{i}\right\vert^{2}}{2}\right\Vert .$$

Also, for $r\geq1$, *we have*

$$\omega^{2r}\left(\sum_{i=1}^{n}p_{i}V_{i}^{*}S_{i}^{*}\right)\leq\frac{1}{p}\left\Vert\sum_{i=1}^{n}p_{i}\left\vert S_{i}^{*}\right\vert^{2p}\right\Vert^{r}+\frac{1}{q}\left\Vert\sum_{i=1}^{n}p_{i}\left\vert V_{i}\right\vert^{2q}\right\Vert^{r}$$

for all $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *In particular*

$$\omega^{2r}\left(\sum_{i=1}^{n}p_{i}V_{i}^{*}S_{i}^{*}\right)\leq\frac{1}{2}\left[\left\Vert\sum_{i=1}^{n}p_{i}\left\vert S_{i}^{*}\right\vert^{4}\right\Vert^{r}+\left\Vert\sum_{i=1}^{n}p_{i}\left\vert V_{i}\right\vert^{4}\right\Vert^{r}\right].$$

Moreover,

$$\omega^{r}\left(\sum_{i=1}^{n}p_{i}V_{i}^{*}S_{i}^{*}\right)\leq\frac{1}{2}\left[\left\Vert\sum_{i=1}^{n}p_{i}\left\vert S_{i}^{*}\right\vert^{2}\right\Vert^{r}+\left\Vert\sum_{i=1}^{n}p_{i}\left\vert V_{i}\right\vert^{2}\right\Vert^{r}\right].$$

Finally, we have for $r\geq1$ *that*

$$\begin{aligned}\omega^{4r}\left(\sum_{i=1}^{n}p_{i}V_{i}^{*}S_{i}^{*}\right)&\leq\frac{1}{2}\left[\left\Vert\sum_{i=1}^{n}p_{i}\left\vert S_{i}^{*}\right\vert^{4}\right\Vert^{r}\left\Vert\sum_{i=1}^{n}p_{i}\left\vert V_{i}\right\vert^{4}\right\Vert^{r}\right.\\&\left.+\omega^{r}\left(\sum_{i=1}^{n}p_{i}\left\vert V_{i}\right\vert^{4}\sum_{i=1}^{n}p_{i}\left\vert S_{i}^{*}\right\vert^{4}\right)\right]\end{aligned}$$

and

$$\begin{aligned}\omega^{2r}\left(\sum_{i=1}^{n}p_{i}V_{i}^{*}S_{i}^{*}\right)&\leq\frac{1}{2}\left[\left\Vert\sum_{i=1}^{n}p_{i}\left\vert S_{i}^{*}\right\vert^{2}\right\Vert^{r}\left\Vert\sum_{i=1}^{n}p_{i}\left\vert V_{i}\right\vert^{2}\right\Vert^{r}\right.\\&\left.+\omega^{r}\left(\sum_{i=1}^{n}p_{i}\left\vert V_{i}\right\vert^{2}\sum_{i=1}^{n}p_{i}\left\vert S_{i}^{*}\right\vert^{2}\right)\right].\end{aligned}$$

The inequalities follow by Corollary 5 and Remark 1 by taking the supremum over $\|x\|=1$.

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