GENERAL INEQUALITIES FOR THE NUMERICAL RADIUS OF WEIGHTED SUMS OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. Assume that f_i and g_i, $i \in \{1, ..., n\}$ are non-negative functions on $[0, \infty)$ which are continuous and satisfying the relation $f_i(t)_i g(t) = t$ for all $t \in [0, \infty)$ and $i \in \{1, ..., n\}$. In this paper we obtained among others the following vector inequality

$$\left|\left\langle \left(\sum_{i=1}^{n} p_i B_i T_i A_i\right) x, y \right\rangle\right|^2$$
$$\le \|x\|^{2/q} \|y\|^{2/p}$$
$$\times \left\langle \left(\sum_{i=1}^{n} p_i \left|f_i\left(|T_i|\right) A_i\right|^{2p}\right) x, x \right\rangle^{1/p} \left\langle \left(\sum_{i=1}^{n} p_i \left|g_i\left(|T_i^*|\right) B_i^*\right|^{2q}\right) y, y \right\rangle^{1/q}$$

for any A_i, B_i, $T_i \in \mathcal{B}(H)$ for $i \in \{1, ..., n\}$, for all x, $y \in H$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. We also established the numerical radius inequality

$$\omega^2 \quad \sum_{i=1}^{n} p_i B_i T_i A_i \Bigg) \le \left\|\sum_{i=1}^{n} p_i \left[\frac{1}{p} \left|f_i\left(|T_i|\right) A_i\right|^{2p} + \frac{1}{q} \left|g_i\left(|T_i^*|\right) B_i^*\right|^{2q}\right]\right\|$$

for all p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \le w(T) \|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \ge 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T + V) \le \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \le \|T\| \le 2\omega(T) \tag{1.3}$$

for any $T \in B(H)$.

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F. Kittaneh, in 2003 [8], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [9] improved the inequality (1.3) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{1.5}$$

for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [5]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{1.6}$$

and

$$\omega^{2r}(T) \leq \left\|\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|, \tag{1.7}$$

where $\alpha \in (0,1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (1.6) that

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{1.8}$$

and from (1.7) that

$$\omega^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [4] and [1].

In 1988, F. Kittaneh obtained the following generalization of Schwarz inequality [7]:

Theorem 1. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. For any $T \in \mathcal{B}(H)$*

$$|\langle Tx, y\rangle| \leq \|f(|T|)x\| \|g(|T^*|)y\| \tag{1.10}$$

for all x, $y \in H$.

If we take $f(t) = t^\lambda$, $g(t) = t^{1-\lambda}$ with $\lambda \in [0,1]$, then we obtain *Kato's inequality* [6]

$$|\langle Tx, y\rangle| \leq \left\||T|^\lambda x\right\| \left\||T^*|^{1-\lambda} y\right\| \tag{1.11}$$

for all x, $y \in H$.

We recall the following vector inequality for positive operators $A \geq 0$, obtained by C. A. McCarthy in [3]

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle, \ p \geq 1$$

for $x \in H$, $\|x\| = 1$ and Buzano's inequality [2],

$$|\langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{2}[\|x\| \|y\| + |\langle x, y\rangle|] \tag{1.12}$$

that holds for any $x,\ y,\ e\in H$ with $\|e\|=1$.

If we replace x by $\frac{y}{\|y\|}$, $y\neq 0$, we get

$$\left\langle A\frac{y}{\|y\|},\frac{y}{\|y\|}\right\rangle^{p}\leq\left\langle A^{p}\frac{y}{\|y\|},\frac{y}{\|y\|}\right\rangle,\ p\geq 1,$$

namely

$$\langle Ay,y\rangle^{p}\leq\|y\|^{2(p-1)}\langle A^{p}y,y\rangle,\ p\geq 1, \tag{1.13}$$

for all $y\in H$.

Assume that f_i and g_i, $i\in\{1,...,n\}$ are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f_i(t)_i g(t)=t$ for all $t\in[0,\infty)$ and $i\in\{1,...,n\}$. Motivated by the above results, in this paper we obtained among others the following vector inequality

$$\begin{aligned}&\left|\left\langle\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)x,y\right\rangle\right|^{2}\\&\leq\|x\|^{2/q}\|y\|^{2/p}\\&\times\left\langle\left(\sum_{i=1}^{n}p_i\left|f_i\left(|T_i|\right)A_i\right|^{2p}\right)x,x\right\rangle^{1/p}\left\langle\left(\sum_{i=1}^{n}p_i\left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q}\right)y,y\right\rangle^{1/q}\end{aligned}$$

for any $A_i,\ B_i,\ T_i\in\mathcal{B}(H)$ for $i\in\{1,...,n\}$, for all $x,\ y\in H$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. We also established the numerical radius inequality

$$\omega^{2}\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)\leq\left\|\sum_{i=1}^{n}p_i\left[\frac{1}{p}\left|f_i\left(|T_i|\right)A_i\right|^{2p}+\frac{1}{q}\left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q}\right]\right\|$$

for all $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

2. Vector Inequalities

In what follows we assume everywhere that the weights $p_i\geq 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}p_i=1$.

The first result is as follows:

Theorem 2. *Assume that f_i and g_i, $i\in\{1,...,n\}$ are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f_i(t)_i g(t)=t$ for all $t\in[0,\infty)$ and $i\in\{1,...,n\}$. For any $A_i,\ B_i,\ T_i\in\mathcal{B}(H)$ for $i\in\{1,...,n\}$, we have*

$$\begin{aligned}&\left|\left\langle\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)x,y\right\rangle\right|^{2}\\&\leq\|x\|^{2/q}\|y\|^{2/p}\\&\times\left\langle\left(\sum_{i=1}^{n}p_i\left|f_i\left(|T_i|\right)A_i\right|^{2p}\right)x,x\right\rangle^{1/p}\left\langle\left(\sum_{i=1}^{n}p_i\left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q}\right)y,y\right\rangle^{1/q}\end{aligned} \tag{2.1}$$

for all x, $y \in H$ *and* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *In particular,*

$$\left|\left\langle \left(\sum_{i=1}^{n} p_i B_i T_i A_i\right) x, y\right\rangle\right|^2 \le \|x\| \|y\| \times \left\langle \left(\sum_{i=1}^{n} p_i \left|f_i\left(|T_i|\right) A_i\right|^4\right) x, x\right\rangle^{1/2} \left\langle \left(\sum_{i=1}^{n} p_i \left|g_i\left(|T_i^*|\right) B_i^*\right|^4\right) y, y\right\rangle^{1/2} \tag{2.2}$$

for all x, $y \in H$.

Proof. Observe that by (1.10) we have

$$\begin{aligned} |\langle T_i x, y\rangle|^2 &\le \|f_i(|T_i|) x\|^2 \|g_i(|T_i^*|) y\|^2 \\ &= \langle f_i(|T_i|) x, f_i(|T_i|) x\rangle \langle g_i(|T_i^*|) y, g_i(|T_i^*|) y\rangle \\ &= \langle f_i^2(|T_i|) x, x\rangle \langle g_i^2(|T_i^*|) y, y\rangle \end{aligned}$$

for all x, $y \in H$ and $i \in \{1, ..., n\}$.

If we take $A_i x$ instead of x and $B_i^* y$ instead of y, then we get

$$\begin{aligned} |\langle T_i A_i x, B_i^* y\rangle|^2 &\le \langle f_i^2(|T_i|) A_i x, A_i x\rangle \langle g_i^2(|T_i^*|) B_i^* y, B_i^* y\rangle \\ &= \langle A_i^* f_i^2(|T_i|) A_i x, x\rangle \langle B_i g_i^2(|T_i^*|) B_i^* y, y\rangle \\ &= \langle (f_i(|T_i|) A_i x)^* f_i(|T_i|) A_i x, x\rangle \langle (g_i(|T_i^*|) B_i^*)^* g_i(|T_i^*|) B_i^* y, y\rangle \\ &= \left\langle |f_i(|T_i|) A_i|^2 x, x\right\rangle \left\langle |g_i(|T_i^*|) B_i^*|^2 y, y\right\rangle, \end{aligned}$$

namely

$$|\langle B_i T_i A_i x, y\rangle|^2 \le \left\langle |f_i(|T_i|) A_i|^2 x, x\right\rangle \left\langle |g_i(|T_i^*|) B_i^*|^2 y, y\right\rangle \tag{2.3}$$

for all x, $y \in H$ and $i \in \{1, ..., n\}$.

If we multiply (2.3) by $p_i \ge 0$, $i \in \{1, ..., n\}$ and sum over i from 1 to n, then we get

$$\sum_{i=1}^{n} p_i |\langle B_i T_i A_i x, y\rangle|^2 \le \sum_{i=1}^{n} p_i \left\langle |f_i(|T_i|) A_i|^2 x, x\right\rangle \left\langle |g_i(|T_i^*|) B_i^*|^2 y, y\right\rangle \tag{2.4}$$

for all x, $y \in H$.

By Cauchy-Buniakowsky-Schwarz weighted inequality we have

$$\left|\left\langle \left(\sum_{i=1}^{n} p_i B_i T_i A_i\right) x, y\right\rangle\right|^2 = \left|\sum_{i=1}^{n} p_i \langle B_i T_i A_i x, y\rangle\right|^2 \le \sum_{i=1}^{n} p_i |\langle B_i T_i A_i x, y\rangle|^2 \tag{2.5}$$

for all x, $y \in H$.

By weighted Hölder's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sum_{i=1}^{n} p_i \left\langle |f_i(|T_i|) A_i|^2 x, x\right\rangle \left\langle |g_i(|T_i^*|) B_i^*|^2 y, y\right\rangle \le \left(\sum_{i=1}^{n} p_i \left\langle |f_i(|T_i|) A_i|^2 x, x\right\rangle^p\right)^{1/p} \left(\sum_{i=1}^{n} p_i \left\langle |g_i(|T_i^*|) B_i^*|^2 y, y\right\rangle^q\right)^{1/q} \tag{2.6}$$

for all x, $y \in H$.

By the McCarthy inequality (1.13) we have

$$\left\langle \left|f_i\left(|T_i|\right)A_i\right|^2 x,x\right\rangle^p \le \|x\|^{2(p-1)}\left\langle \left|f_i\left(|T_i|\right)A_i\right|^{2p} x,x\right\rangle$$

and

$$\left\langle \left|g_i\left(|T_i^*|\right)B_i^*\right|^2 y,y\right\rangle^q \le \|y\|^{2(q-1)}\left\langle \left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q} y,y\right\rangle$$

for all x, $y \in H$.

Therefore, by (2.6) we get

$$\begin{aligned}
&\sum_{i=1}^n p_i\left\langle \left|f_i\left(|T_i|\right)A_i\right|^2 x,x\right\rangle\left\langle \left|g_i\left(|T_i^*|\right)B_i^*\right|^2 y,y\right\rangle \qquad (2.7)\\
&\le\left(\|x\|^{2(p-1)}\sum_{i=1}^n p_i\left\langle \left|f_i\left(|T_i|\right)A_i\right|^{2p} x,x\right\rangle\right)^{1/p}\\
&\times\left(\|y\|^{2(p-1)}\sum_{i=1}^n p_i\left\langle \left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q} y,y\right\rangle\right)^{1/q}\\
&=\|x\|^{2(1-1/p)}\left\langle\left(\sum_{i=1}^n p_i\left|f_i\left(|T_i|\right)A_i\right|^{2p}\right)x,x\right\rangle^{1/p}\\
&\times\|y\|^{2(1-1/q)}\left\langle\left(\sum_{i=1}^n p_i\left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q}\right)y,y\right\rangle^{1/q}\\
&=\|x\|^{2/q}\|y\|^{2/p}\left\langle\left(\sum_{i=1}^n p_i\left|f_i\left(|T_i|\right)A_i\right|^{2p}\right)x,x\right\rangle^{1/p}\\
&\times\left\langle\left(\sum_{i=1}^n p_i\left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q}\right)y,y\right\rangle^{1/q}
\end{aligned}$$

for all x, $y \in H$.

By making use of (2.5)-(2.7) we deduce the desired result (2.1). □

Corollary 1. *With the assumptions of Theorem 2,*

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^n p_iB_iT_iA_i\right)x,x\right\rangle\right|^2 \qquad (2.8)\\
&\le\|x\|^2\left\langle\left(\sum_{i=1}^n p_i\left[\frac{1}{p}\left|f_i\left(|T_i|\right)A_i\right|^{2p}+\frac{1}{q}\left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q}\right]\right)x,x\right\rangle
\end{aligned}$$

for all $x \in H$ and p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. In particular,

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^n p_iB_iT_iA_i\right)x,x\right\rangle\right|^2 \qquad (2.9)\\
&\le\frac{1}{2}\|x\|^2\left\langle\left(\sum_{i=1}^n p_i\left[\left|f_i\left(|T_i|\right)A_i\right|^{4}+\left|g_i\left(|T_i^*|\right)B_i^*\right|^{4}\right]\right)x,x\right\rangle
\end{aligned}$$

for all $x \in H$

Proof. From the inequality (2.1) we obtain for $y=x$ that

$$\begin{aligned}(2.10)\quad & \left|\left\langle\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)x,x\right\rangle\right|^2 \\ & \leq \|x\|^{2/q}\|x\|^{2/p} \\ & \times\left\langle\left(\sum_{i=1}^{n}p_i\left|f_i\left(|T_i|\right)A_i\right|^{2p}\right)x,x\right\rangle^{1/p}\left\langle\left(\sum_{i=1}^{n}p_i\left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q}\right)x,x\right\rangle^{1/q}\end{aligned}$$

for all $x\in H$.

From Young's inequality

$$ab\leq\frac{1}{p}a^p+\frac{1}{q}b^q,\ a,\ b\geq 0,\ p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,$$

we have

$$\begin{aligned}&\left\langle\left(\sum_{i=1}^{n}p_i\left|f_i\left(|T_i|\right)A_i\right|^{2p}\right)x,x\right\rangle^{1/p}\left\langle\left(\sum_{i=1}^{n}p_i\left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q}\right)x,x\right\rangle^{1/q} \\ &\leq\frac{1}{p}\left(\left\langle\left(\sum_{i=1}^{n}p_i\left|f_i\left(|T_i|\right)A_i\right|^{2p}\right)x,x\right\rangle^{1/p}\right)^p \\ &+\frac{1}{q}\left(\left\langle\left(\sum_{i=1}^{n}p_i\left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q}\right)x,x\right\rangle^{1/q}\right)^q \\ &=\frac{1}{p}\left\langle\left(\sum_{i=1}^{n}p_i\left|f_i\left(|T_i|\right)A_i\right|^{2p}\right)x,x\right\rangle+\frac{1}{q}\left\langle\left(\sum_{i=1}^{n}p_i\left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q}\right)x,x\right\rangle \\ &=\left\langle\left(\sum_{i=1}^{n}p_i\left[\frac{1}{p}\left|f_i\left(|T_i|\right)A_i\right|^{2p}+\frac{1}{q}\left|g_i\left(|T_i^*|\right)B_i^*\right|^{2q}\right]\right)x,x\right\rangle\end{aligned}$$

and by (2.10) we obtain (2.8). □

Remark 1. *For any $A_i,\ B_i,\ T_i\in\mathcal{B}(H)$ and $\alpha_i\in[0,1]$ for $i\in\{1,...,n\}$, we have by (2.1) for the functions $f_i(t)=t^{\alpha_i}$, $g_i(t)=t^{1-\alpha_i}$ for $i\in\{1,...,n\}$ that*

$$\begin{aligned}(2.11)\quad & \left|\left\langle\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)x,y\right\rangle\right|^2 \\ & \leq \|x\|^{2/q}\|y\|^{2/p} \\ & \times\left\langle\left(\sum_{i=1}^{n}p_i\left||T_i|^{\alpha_i}A_i\right|^{2p}\right)x,x\right\rangle^{1/p}\left\langle\left(\sum_{i=1}^{n}p_i\left||T_i^*|^{1-\alpha_i}B_i^*\right|^{2q}\right)y,y\right\rangle^{1/q}\end{aligned}$$

for all $x,\ y\in H$ *and* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *In particular,*

$$\begin{aligned}\left|\left\langle\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)x,y\right\rangle\right|^2 &\\ \le \|x\|\,\|y\| &\\ \times\left\langle\left(\sum_{i=1}^{n}p_i\left||T_i|^{\alpha_i}A_i\right|^4\right)x,x\right\rangle^{1/2}\left\langle\left(\sum_{i=1}^{n}p_i\left||T_i^*|^{1-\alpha_i}B_i^*\right|^4\right)y,y\right\rangle^{1/2} &\end{aligned}\tag{2.12}$$

for all $x,\ y\in H$.

We also have

$$\begin{aligned}&\left|\left\langle\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)x,x\right\rangle\right|^2\\ &\le\|x\|^2\left\langle\left(\sum_{i=1}^{n}p_i\left[\frac{1}{p}\left||T_i|^{\alpha_i}A_i\right|^{2p}+\frac{1}{q}\left||T_i^*|^{1-\alpha_i}B_i^*\right|^{2q}\right]\right)x,x\right\rangle\end{aligned}\tag{2.13}$$

for all $x\in H$ *and* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *In particular,*

$$\begin{aligned}&\left|\left\langle\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)x,x\right\rangle\right|^2\\ &\le\frac{1}{2}\|x\|^2\left\langle\left(\sum_{i=1}^{n}p_i\left[\left||T_i|^{\alpha_i}A_i\right|^4+\left||T_i^*|^{1-\alpha_i}B_i^*\right|^4\right]\right)x,x\right\rangle\end{aligned}\tag{2.14}$$

for all $x\in H$.

We also have:

Theorem 3. *Assume that* f_i *and* g_i, $i\in\{1,...,n\}$ *are non-negative functions on* $[0,\infty)$ *which are continuous and satisfying the relation* $f_i(t)_ig(t)=t$ *for all* $t\in[0,\infty)$ *and* $i\in\{1,...,n\}$. *For any* $A_i,\ B_i,\ T_i\in\mathcal{B}(H)$ *for* $i\in\{1,...,n\}$, *we have*

$$\begin{aligned}&\left|\left\langle\sum_{i=1}^{n}p_iB_iT_iA_ix,y\right\rangle\right|^2\\ &\le\left\langle\sum_{i=1}^{n}p_i\left|f_i(|T_i|)A_i\right|^2x,x\right\rangle\left\langle\sum_{i=1}^{n}p_i\left|g_i(|T_i^*|)B_i^*\right|^2y,y\right\rangle\end{aligned}\tag{2.15}$$

for all $x,\ y\in H$.

Proof. By taking the square root in (2.3), we get

$$|\langle B_iT_iA_ix,y\rangle|\le\left\langle\left|f_i(|T_i|)A_i\right|^2x,x\right\rangle^{1/2}\left\langle\left|g_i(|T_i^*|)B_i^*\right|^2y,y\right\rangle^{1/2}$$

for all $x,\ y\in H$ and $i\in\{1,...,n\}$.

If we multiply by $p_i\ge0$, $i\in\{1,...,n\}$ and sum over i from 1 to n, then we get

$$\sum_{i=1}^{n}p_i\left|\langle B_iT_iA_ix,y\rangle\right|\le\sum_{i=1}^{n}p_i\left\langle\left|f_i(|T_i|)A_i\right|^2x,x\right\rangle^{1/2}\left\langle\left|g_i(|T_i^*|)B_i^*\right|^2y,y\right\rangle^{1/2}.$$

By the weighted triangle inequality we get

$$\left|\left\langle \sum_{i=1}^{n} p_i B_i T_i A_i x, y \right\rangle\right| \le \sum_{i=1}^{n} p_i \left|\left\langle B_i T_i A_i x, y \right\rangle\right|, \tag{2.16}$$

while by Cauchy-Buniakowsky-Schwarz weighted inequality we have

$$\begin{aligned} &\sum_{i=1}^{n} p_i \left\langle \left|f_i\left(|T_i|\right) A_i\right|^2 x, x \right\rangle^{1/2} \left\langle \left|g_i\left(|T_i^*|\right) B_i^*\right|^2 y, y \right\rangle^{1/2} \\ &\le \left[\sum_{i=1}^{n} p_i \left(\left\langle \left|f_i\left(|T_i|\right) A_i\right|^2 x, x \right\rangle^{1/2}\right)^2\right]^{1/2} \\ &\times \left[\sum_{i=1}^{n} p_i \left(\left\langle \left|g_i\left(|T_i^*|\right) B_i^*\right|^2 y, y \right\rangle^{1/2}\right)^2\right]^{1/2} \\ &= \left[\sum_{i=1}^{n} p_i \left\langle \left|f_i\left(|T_i|\right) A_i\right|^2 x, x \right\rangle\right]^{1/2} \left[\sum_{i=1}^{n} p_i \left\langle \left|g_i\left(|T_i^*|\right) B_i^*\right|^2 y, y \right\rangle\right]^{1/2} \\ &= \left\langle \sum_{i=1}^{n} p_i \left|f_i\left(|T_i|\right) A_i\right|^2 x, x \right\rangle^{1/2} \left\langle \sum_{i=1}^{n} p_i \left|g_i\left(|T_i^*|\right) B_i^*\right|^2 y, y \right\rangle^{1/2} \end{aligned} \tag{2.17}$$

for all $x,\ y \in H$.

By making use of (2.16) and (2.17) we get (2.15). □

Corollary 2. *With the assumptions of Theorem 3, we have*

$$\left|\left\langle \sum_{i=1}^{n} p_i B_i T_i A_i x, x \right\rangle\right|^2 \le \frac{1}{2}\left\langle \sum_{i=1}^{n} p_i \left(\left|f_i\left(|T_i|\right) A_i\right|^2 + \left|g_i\left(|T_i^*|\right) B_i^*\right|^2\right) x, x \right\rangle \tag{2.18}$$

for all $x \in H$.

Remark 2. *For any $A_i,\ B_i,\ T_i \in \mathcal{B}(H)$ and $\alpha_i \in [0,1]$ for $i \in \{1, ..., n\}$, we have by (2.15) for the functions $f_i(t) = t^{\alpha_i},\ g_i(t) = t^{1-\alpha_i}$ for $i \in \{1, ..., n\}$ that*

$$\begin{aligned} &\left|\left\langle \sum_{i=1}^{n} p_i B_i T_i A_i x, y \right\rangle\right|^2 \\ &\le \left\langle \sum_{i=1}^{n} p_i \left|\left|T_i\right|^{\alpha_i} A_i\right|^2 x, x \right\rangle \left\langle \sum_{i=1}^{n} p_i \left|\left|T_i^*\right|^{1-\alpha_i} B_i^*\right|^2 y, y \right\rangle \end{aligned} \tag{2.19}$$

for all $x,\ y \in H$.

From (2.19) we also have

$$\left|\left\langle \sum_{i=1}^{n} p_i B_i T_i A_i x, x \right\rangle\right|^2 \le \frac{1}{2}\left\langle \sum_{i=1}^{n} p_i \left(\left|\left|T_i\right|^{\alpha_i} A_i\right|^2 + \left|\left|T_i^*\right|^{1-\alpha_i} B_i^*\right|^2\right) x, x \right\rangle \tag{2.20}$$

for all $x,\ y \in H$.

3. Norm and Numerical Radius Inequalities

We have the following operator norm inequalities:

Theorem 4. *Assume that f_i and g_i, $i \in \{1,...,n\}$ are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f_i(t)_i g(t) = t$ for all $t \in [0,\infty)$ and $i \in \{1,...,n\}$. For any A_i, B_i, $T_i \in \mathcal{B}(H)$ for $i \in \{1,...,n\}$, we have*

$$\left\| \sum_{i=1}^{n} p_i B_i T_i A_i \right\|^2 \leq \left\| \sum_{i=1}^{n} p_i \left| f_i\left(|T_i|\right) A_i \right|^{2p} \right\|^{1/p} \left\| \sum_{i=1}^{n} p_i \left| g_i\left(|T_i^*|\right) B_i^* \right|^{2q} \right\|^{1/q} \tag{3.1}$$

for all p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. In particular,

$$\left\| \sum_{i=1}^{n} p_i B_i T_i A_i \right\|^4 \leq \left\| \sum_{i=1}^{n} p_i \left| f_i\left(|T_i|\right) A_i \right|^{4} \right\| \left\| \sum_{i=1}^{n} p_i \left| g_i\left(|T_i^*|\right) B_i^* \right|^{4} \right\|. \tag{3.2}$$

Also, we have

$$\left\| \sum_{i=1}^{n} p_i B_i T_i A_i \right\|^2 \leq \left\| \sum_{i=1}^{n} p_i \left| f_i\left(|T_i|\right) A_i \right|^{2} \right\| \left\| \sum_{i=1}^{n} p_i \left| g_i\left(|T_i^*|\right) B_i^* \right|^{2} \right\|. \tag{3.3}$$

Proof. We have by (2.1) we have

$$\begin{aligned}
\left\| \sum_{i=1}^{n} p_i B_i T_i A_i \right\|^2 &= \sup_{\|x\|=\|y\|=1} \left| \left\langle \left(\sum_{i=1}^{n} p_i B_i T_i A_i \right) x, y \right\rangle \right|^2 \\
&\leq \sup_{\|x\|=\|y\|=1} \left[\left\langle \left(\sum_{i=1}^{n} p_i \left| f_i\left(|T_i|\right) A_i \right|^{2p} \right) x, x \right\rangle^{1/p} \right. \\
&\quad \left. \times \left\langle \left(\sum_{i=1}^{n} p_i \left| g_i\left(|T_i^*|\right) B_i^* \right|^{2q} \right) y, y \right\rangle^{1/q} \right] \\
&= \sup_{\|x\|=1} \left\langle \left(\sum_{i=1}^{n} p_i \left| f_i\left(|T_i|\right) A_i \right|^{2p} \right) x, x \right\rangle^{1/p} \\
&\quad \times \sup_{\|y\|=1} \left\langle \left(\sum_{i=1}^{n} p_i \left| g_i\left(|T_i^*|\right) B_i^* \right|^{2q} \right) y, y \right\rangle^{1/q} \\
&= \left\| \sum_{i=1}^{n} p_i \left| f_i\left(|T_i|\right) A_i \right|^{2p} \right\|^{1/p} \left\| \sum_{i=1}^{n} p_i \left| g_i\left(|T_i^*|\right) B_i^* \right|^{2q} \right\|^{1/q}
\end{aligned}$$

which proves (3.1).

By using (2.15) we also get (3.3). □

Corollary 3. *For any A_i, B_i, $T_i \in \mathcal{B}(H)$ and $\alpha_i \in [0,1]$ for $i \in \{1,...,n\}$, we have*

$$\left\| \sum_{i=1}^{n} p_i B_i T_i A_i \right\|^2 \leq \left\| \sum_{i=1}^{n} p_i \left| |T_i|^{\alpha_i} A_i \right|^{2p} \right\|^{1/p} \left\| \sum_{i=1}^{n} p_i \left| |T_i^*|^{1-\alpha_i} B_i^* \right|^{2q} \right\|^{1/q} \tag{3.4}$$

for all $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *In particular,*

$$\left\|\sum_{i=1}^{n}p_iB_iT_iA_i\right\|^4\leq\left\|\sum_{i=1}^{n}p_i\left|\left|T_i\right|^{\alpha_i}A_i\right|^4\right\|\left\|\sum_{i=1}^{n}p_i\left|\left|T_i^*\right|^{1-\alpha_i}B_i^*\right|^4\right\|. \tag{3.5}$$

Also, we have

$$\left\|\sum_{i=1}^{n}p_iB_iT_iA_i\right\|^2\leq\left\|\sum_{i=1}^{n}p_i\left|\left|T_i\right|^{\alpha_i}A_i\right|^2\right\|\left\|\sum_{i=1}^{n}p_i\left|\left|T_i^*\right|^{1-\alpha_i}B_i^*\right|^2\right\|. \tag{3.6}$$

We also have the following inequalities for numerical radius:

Theorem 5. *Assume that* f_i *and* g_i, $i\in\{1,...,n\}$ *are non-negative functions on* $[0,\infty)$ *which are continuous and satisfying the relation* $f_i(t)_ig(t)=t$ *for all* $t\in[0,\infty)$ *and* $i\in\{1,...,n\}$. *For any* $A_i,\ B_i,\ T_i\in\mathcal{B}(H)$ *for* $i\in\{1,...,n\}$, *we have*

$$\omega^2\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)\leq\left\|\sum_{i=1}^{n}p_i\left[\frac{1}{p}\left|f_i\left(\left|T_i\right|\right)A_i\right|^{2p}+\frac{1}{q}\left|g_i\left(\left|T_i^*\right|\right)B_i^*\right|^{2q}\right]\right\| \tag{3.7}$$

for all $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *In particular,*

$$\omega^2\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)\leq\frac{1}{2}\left\|\sum_{i=1}^{n}p_i\left[\left|f_i\left(\left|T_i\right|\right)A_i\right|^4+\left|g_i\left(\left|T_i^*\right|\right)B_i^*\right|^4\right]\right\|. \tag{3.8}$$

Moreover, we have

$$\omega^2\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)\leq\frac{1}{2}\left\|\sum_{i=1}^{n}p_i\left(\left|f_i\left(\left|T_i\right|\right)A_i\right|^2+\left|g_i\left(\left|T_i^*\right|\right)B_i^*\right|^2\right)\right\|. \tag{3.9}$$

Proof. From (2.8) we get

$$\begin{aligned}\omega^2\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)&=\sup_{\|x\|=1}\left|\left\langle\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)x,x\right\rangle\right|^2\\&\leq\sup_{\|x\|=1}\left\langle\left(\sum_{i=1}^{n}p_i\left[\frac{1}{p}\left|f_i\left(\left|T_i\right|\right)A_i\right|^{2p}+\frac{1}{q}\left|g_i\left(\left|T_i^*\right|\right)B_i^*\right|^{2q}\right]\right)x,x\right\rangle\\&=\left\|\sum_{i=1}^{n}p_i\left[\frac{1}{p}\left|f_i\left(\left|T_i\right|\right)A_i\right|^{2p}+\frac{1}{q}\left|g_i\left(\left|T_i^*\right|\right)B_i^*\right|^{2q}\right]\right\|\end{aligned}$$

which proves (3.7).

The inequality (3.9) follows by (2.18) on taking the supremum over $\|x\|=1$. □

In particular, we can state:

Corollary 4. *For any* $A_i,\ B_i,\ T_i\in\mathcal{B}(H)$ *and* $\alpha_i\in[0,1]$ *for* $i\in\{1,...,n\}$,

$$\omega^2\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)\leq\left\|\sum_{i=1}^{n}p_i\left[\frac{1}{p}\left|\left|T_i\right|^{\alpha_i}A_i\right|^{2p}+\frac{1}{q}\left|\left|T_i^*\right|^{1-\alpha_i}B_i^*\right|^{2q}\right]\right\| \tag{3.10}$$

for all $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *In particular,*

$$\omega^2\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)\leq\frac{1}{2}\left\|\sum_{i=1}^{n}p_i\left[\left|\left|T_i\right|^{\alpha_i}A_i\right|^4+\left|\left|T_i^*\right|^{1-\alpha_i}B_i^*\right|^4\right]\right\|. \tag{3.11}$$

Moreover, we have

$$\omega^2\left(\sum_{i=1}^{n}p_iB_iT_iA_i\right)\leq\frac{1}{2}\left\|\sum_{i=1}^{n}p_i\left(\left||T_i|^{\alpha_i}A_i\right|^2+\left||T_i^*|^{1-\alpha_i}B_i^*\right|^2\right)\right\|. \tag{3.12}$$

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