q-Deformed and λ-parametrized hyperbolic tangent function relied complex valued trigonometric and hyperbolic neural network high order approximations

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis,
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we study the univariate quantitative approximation of complex valued continuous functions on a compact interval by complex valued neural network operators. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the used function's high order derivatives. The kind of our approximations are trigonometric and hyperbolic. Our operators are defined by using a density function generated by a q-deformed and λ-parametrized hyperbolic tangent function, which is a sigmoid function. The approximations are pointwise and of the uniform norm. The related complex valued feed-forward neural networks are with one hidden layer.

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1 Introduction

The author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats

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there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support.

Again the author inspired by [12], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3]-[7], by treating both the univariate and multivariate cases.

Let h be a general sigmoid activation function with $h(0) = 0$, and $y = \pm 1$ the horizontal asymptotes. Of course h is strictly increasing over $\mathbb{R}$. Let the parameter $0 < r < 1$ and $x > 0$. Then clearly $-x < x$ and $-x < -rx < rx < x$, furthermore it holds $h(-x) < h(-rx) < h(rx) < h(x)$. Consequently the sigmoid $y = h(rx)$ has a graph inside the graph of $y = h(x)$, of course with the same asymptotes $y = \pm 1$. Therefore $h(rx)$ has derivatives (gradients) at more points x than $h(x)$ has different than zero or not as close to zero, thus killing less number of neurons! And of course $h(rx)$ is more distant from $y = \pm 1$, than $h(x)$ it is. A highly desired fact in Neural Networks theory.

Different activation functions allow for different non-linearities which might work better for solving a specific function. So the need to use neural networks with various activation functions is vivid. Thus, performing neural network approximations using different activation functions is not only necessary but fully justified.

Also brain asymmetry has been observed in animals and humans in terms of structure, function and behaviour. This lateralization is thought to reflect evolutionary, hereditary, developmental, experiential and pathological factors. Therefore it is natural to consider for our study deformed neural network activation functions and operators. So this article is a specific study under this philosophy of approaching reality as close as possible.

Consequently the author here performs q-deformed and λ-parametrized hyperbolic tangent function activated high order neural network approximations to continuous functions over compact intervals of the real line with complex values. All convergences are with rates expressed via the modulus of continuity of the involved functions high order derivatives, deriving by very tight Jackson type inequalities.

The basis of our higher order approximations here are some newly discovered by the author trigonometric and hyperbolic type Taylor's formulae.

Our compact intervals are not necessarily symmetric to the origin. In preparation to prove our results we describe important properties of the basic density function defining our operators which is induced by a q-deformed and λ-parametrized hyperbolic tangent function, which is a sigmoid activation function.

Feed-forward X-valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, \ \ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. About neural networks in general read [13], [14], [15].

2 About q-deformed and $\lambda-$parametrized hyperbolic tangent function $g_{q,\lambda}$

Here all this background comes from [10, ch. 17].

We use $g_{q,\lambda}$, see (1), and exhibit that it is a sigmoid function and we will present several of its properties related to the approximation by neural network operators.

So, let us consider the function

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - q e^{-\lambda x}}{e^{\lambda x} + q e^{-\lambda x}}, \quad \lambda, q > 0, \ x \in \mathbb{R}. \tag{1}$$

We have that

$$g_{q,\lambda}(0) = \frac{1-q}{1+q}.$$

We notice also that

$$g_{q,\lambda}(-x) = \frac{e^{-\lambda x} - q e^{\lambda x}}{e^{-\lambda x} + q e^{\lambda x}} = \frac{\frac{1}{q} e^{-\lambda x} - e^{\lambda x}}{\frac{1}{q} e^{-\lambda x} + e^{\lambda x}} = -\frac{\left(e^{\lambda x} - \frac{1}{q} e^{-\lambda x}\right)}{e^{\lambda x} + \frac{1}{q} e^{-\lambda x}} = -g_{\frac{1}{q},\lambda}(x). \tag{2}$$

That is

$$g_{q,\lambda}(-x) = -g_{\frac{1}{q},\lambda}(x), \ \ \forall \ x \in \mathbb{R}, \tag{3}$$

and

$$g_{\frac{1}{q},\lambda}(x) = -g_{q,\lambda}(-x),$$

hence

$$g'_{\frac{1}{q},\lambda}(x) = g'_{q,\lambda}(-x). \tag{4}$$

It is

$$g_{q,\lambda}(x) = \frac{e^{2\lambda x} - q}{e^{2\lambda x} + q} = \frac{1 - \frac{q}{e^{2lx}}}{1 + \frac{q}{e^{2\lambda x}}} \underset{(x \to +\infty)}{\to} 1,$$

i.e.

$$g_{q,\lambda}(+\infty) = 1, \tag{5}$$

Furthermore

$$g_{q,\lambda}(x) = \frac{e^{2\lambda x} - q}{e^{2\lambda x} + q} \underset{(x \to -\infty)}{\to} \frac{-q}{q} = -1,$$

i.e.

$$g_{q,\lambda}(-\infty) = -1. \tag{6}$$

We find that

$$g'_{q,\lambda}(x) = \frac{4q\lambda e^{2\lambda x}}{\left(e^{2\lambda x} + q\right)^2} > 0, \tag{7}$$

therefore $g_{q,\lambda}$ is striclty increasing.

Next we obtain ($x \in \mathbb{R}$)

$$g''_{q,\lambda}(x) = 8q\lambda^2 e^{2\lambda x} \left(\frac{q - e^{2\lambda x}}{\left(e^{2\lambda x} + q\right)^3} \right) \in C(\mathbb{R}). \tag{8}$$

We observe that

$$q - e^{2\lambda x} \gtreqless 0 \Leftrightarrow q \gtreqless e^{2\lambda x} \Leftrightarrow \ln q \gtreqless 2\lambda x \Leftrightarrow x \lesseqgtr \frac{\ln q}{2\lambda}.$$

So, in case of $x < \frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave up, with $g''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = 0$.

And in case of $x > \frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave down.

Clearly, $g_{q,\lambda}$ is a shifted sigmoid function with $g_{q,\lambda}(0) = \frac{1-q}{1+q}$, and $g_{q,\lambda}(-x) = -g_{q^{-1},\lambda}(x)$, (a semi-odd function), see also [9].

By $1 > -1$, $x + 1 > x - 1$, we consider the activation function

$$M_{q.\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \tag{9}$$

$\forall x \in \mathbb{R};\ q, \lambda > 0$. Notice that $M_{q,\lambda}(\pm\infty) = 0$, so the x-axis is horizontal asymptote.

We have that

$$M_{q,\lambda}(-x) = \frac{1}{4}\left(g_{q,\lambda}(-x+1) - g_{q,\lambda}(-x-1)\right) =$$

$$\frac{1}{4}\left(g_{q,\lambda}(-(x-1)) - g_{q,\lambda}(-(x+1))\right) =$$

$$\frac{1}{4}\left(-g_{\frac{1}{q},\lambda}(x-1) + g_{\frac{1}{q},\lambda}(x+1)\right) = \tag{10}$$

$$\frac{1}{4}\left(g_{\frac{1}{q},\lambda}(x+1) - g_{\frac{1}{q},\lambda}(x-1)\right) = M_{\frac{1}{q},\lambda}(x), \quad \forall\, x \in \mathbb{R}.$$

Thus

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \ \ \forall\, x \in \mathbb{R};\ q, \lambda > 0, \tag{11}$$

a deformed symmetry.

Next, we have that

$$M'_{q,\lambda}(x) = \frac{1}{4}\left(g'_{q,\lambda}(x+1) - g'_{q,\lambda}(x-1)\right), \quad \forall\, x \in \mathbb{R}. \tag{12}$$

Let $x < \frac{\ln q}{2\lambda} - 1$, then $x - 1 < x + 1 < \frac{\ln q}{2\lambda}$ and $g'_{q,\lambda}(x+1) > g'_{q,\lambda}(x-1)$ (by $g_{q,\lambda}$ being strictly concave up for $x < \frac{\ln q}{2\lambda}$), that is $M'_{q,\lambda}(x) > 0$. Hence $M_{q,\lambda}$ is striclty increasing over $\left(-\infty, \frac{\ln q}{2\lambda} - 1\right)$.

Let now $x-1 > \frac{\ln q}{2\lambda}$, then $x+1 > x-1 > \frac{\ln q}{2\lambda}$, and $g'_{q,\lambda}(x+1) < g'_{q,\lambda}(x-1)$, that is $M'_{q,\lambda}(x) < 0$.

Therefore $M_{q,\lambda}$ is strictly decreasing over $\left(\frac{\ln q}{2\lambda} + 1, +\infty\right)$.

Let us next consider, $\frac{\ln q}{2\lambda} - 1 \leq x \leq \frac{\ln q}{2\lambda} + 1$. We have that

$$M''_{q,\lambda}(x) = \frac{1}{4}\left(g''_{q,\lambda}(x+1) - g''_{q,\lambda}(x-1)\right) =$$

$$2q\lambda^2 \left[e^{2\lambda(x+1)}\left(\frac{q - e^{2\lambda(x+1)}}{\left(e^{2\lambda(x+1)} + q\right)^3}\right) - e^{2\lambda(x-1)}\left(\frac{q - e^{2\lambda(x-1)}}{\left(e^{2\lambda(x-1)} + q\right)^3}\right)\right]. \tag{13}$$

By $\frac{\ln q}{2\lambda} - 1 \leq x \Leftrightarrow \frac{\ln q}{2\lambda} \leq x + 1 \Leftrightarrow \ln q \leq 2\lambda(x+1) \Leftrightarrow q \leq e^{2\lambda(x+1)} \Leftrightarrow q - e^{2\lambda(x+1)} \leq 0$.

By $x \leq \frac{\ln q}{2\lambda} + 1 \Leftrightarrow x - 1 \leq \frac{\ln q}{2\lambda} \Leftrightarrow 2\lambda(x-1) \leq \ln q \Leftrightarrow e^{2\lambda(x-1)} \leq q \Leftrightarrow q - e^{2\lambda\beta(x-1)} \geq 0$.

Clearly by (13) we get that $M''_{q,\lambda}(x) \leq 0$, for $x \in \left[\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1\right]$.

More precisely $M_{q,\lambda}$ is concave down over $\left[\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1\right]$, and strictly concave down over $\left(\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1\right)$.

Consequently $M_{q,\lambda}$ has a bell-type shape over $\mathbb{R}$.

Of course it holds $M''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) < 0$.

At $x = \frac{\ln q}{2\lambda}$, we have

$$M'_{q,\lambda}(x) = \frac{1}{4}\left(g'_{q,\lambda}(x+1) - g'_{q,\lambda}(x-1)\right) =$$

$$q\lambda\left(\frac{e^{2\lambda(x+1)}}{\left(e^{2\lambda(x+1)} + q\right)^2} - \frac{e^{2\lambda(x-1)}}{\left(e^{2\lambda(x-1)} + q\right)^2}\right). \tag{14}$$

Thus

$$M'_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = q\lambda\left(\frac{e^{2\lambda\left(\frac{\ln q}{2\lambda}+1\right)}}{\left(e^{2\lambda\left(\frac{\ln q}{2\lambda}+1\right)} + q\right)^2} - \frac{e^{2\lambda\left(\frac{\ln q}{2\lambda}-1\right)}}{\left(e^{2\lambda\left(\frac{\ln q}{2\lambda}-1\right)} + q\right)^2}\right) =$$

$$q\lambda\left(\frac{qe^{2\lambda}}{\left(qe^{2\lambda}+q\right)^2}-\frac{qe^{-2\lambda}}{\left(qe^{-2\lambda}+q\right)^2}\right)=$$

$$\lambda\left(\frac{e^{2\lambda}}{\left(e^{2\lambda}+1\right)^2}-\frac{e^{-2\lambda}}{\left(e^{-2\lambda}+1\right)^2}\right)= \tag{15}$$

$$\lambda\left(\frac{e^{2\lambda}\left(e^{-2\lambda}+1\right)^2-e^{-2\lambda}\left(e^{2\lambda}+1\right)^2}{\left(e^{2\lambda}+1\right)^2\left(e^{-2\lambda}+1\right)^2}\right)=0.$$

That is, $\frac{\ln q}{2\lambda}$ is the only critical number of $M_{q,\lambda}$ over $\mathbb{R}$. Hence at $x=\frac{\ln q}{2\lambda}$, $M_{q,\lambda}$ achieves its global maximum, which is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)=\frac{1}{4}\left[g_{q,\lambda}\left(\frac{\ln q}{2\lambda}+1\right)-g_{q,\lambda}\left(\frac{\ln q}{2\lambda}-1\right)\right]=$$

$$\frac{1}{4}\left[\left(\frac{e^{\lambda\left(\frac{\ln q}{2\lambda}+1\right)}-qe^{-\lambda\left(\frac{\ln q}{2\lambda}+1\right)}}{e^{\lambda\left(\frac{\ln q}{2\lambda}+1\right)}+qe^{-\lambda\left(\frac{\ln q}{2\lambda}+1\right)}}\right)-\left(\frac{e^{\lambda\left(\frac{\ln q}{2\lambda}-1\right)}-qe^{-\lambda\left(\frac{\ln q}{2\lambda}-1\right)}}{e^{\lambda\left(\frac{\ln q}{2\lambda}-1\right)}+qe^{-\lambda\left(\frac{\ln q}{2\lambda}-1\right)}}\right)\right]=$$

$$\frac{1}{4}\left[\left(\frac{\sqrt{q}e^{\lambda}-qq^{-\frac{1}{2}}e^{-\lambda}}{\sqrt{q}e^{\lambda}+qq^{-\frac{1}{2}}e^{-\lambda}}\right)-\left(\frac{\sqrt{q}e^{-\lambda}-qq^{-\frac{1}{2}}e^{\lambda}}{\sqrt{q}e^{-\lambda}+qq^{-\frac{1}{2}}e^{\lambda}}\right)\right]= \tag{17}$$

$$\frac{1}{4}\left[\left(\frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+e^{-\lambda}}\right)-\left(\frac{e^{-\lambda}-e^{\lambda}}{e^{-\lambda}+e^{\lambda}}\right)\right]=$$

$$\frac{1}{4}\left[\frac{2\left(e^{\lambda}-e^{-\lambda}\right)}{e^{\lambda}+e^{-\lambda}}\right]=\frac{1}{2}\left(\frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+e^{-\lambda}}\right)=\frac{\tanh(\lambda)}{2}.$$

Conclusion: The maximum value of $M_{q,\lambda}$ is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right)=\frac{\tanh(\lambda)}{2},\quad \lambda>0. \tag{18}$$

We give

Theorem 1 *([10, ch. 17]) We have that*

$$\sum_{i=-\infty}^{\infty}M_{q,\lambda}(x-i)=1,\ \ \forall\ x\in\mathbb{R},\ \forall\ \lambda,q>0. \tag{19}$$

Thus

$$\sum_{i=-\infty}^{\infty}M_{q,\lambda}(nx-i)=1,\ \forall\ n\in\mathbb{N},\ \ \forall\ x\in\mathbb{R}. \tag{20}$$

Similarly, it holds

$$\sum_{i=-\infty}^{\infty}M_{\frac{1}{q},\lambda}(x-i)=1,\ \forall\ x\in\mathbb{R}. \tag{21}$$

But $M_{\frac{1}{q},\lambda}(x-i) \overset{(11)}{=} M_{q,\lambda}(i-x)$, $\forall\, x \in \mathbb{R}$.

Hence

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(i-x) = 1,\ \forall\, x \in \mathbb{R}, \tag{22}$$

and

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(i+x) = 1,\ \forall\, x \in \mathbb{R}. \tag{23}$$

It follows

Theorem 2 *([10, ch. 17]) It holds*

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)\,dx = 1, \quad \lambda, q > 0. \tag{24}$$

So that $M_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda, q > 0$.

We need the following result

Theorem 3 *([10, ch. 17]) Let* $0 < \alpha < 1$, *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$; $q, \lambda > 0$. *Then*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} M_{q,\lambda}(nx-k) < \max\left\{q, \frac{1}{q}\right\} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} = T e^{-2\lambda n^{(1-\alpha)}}, \tag{25}$$

where $T := \max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Let $\lceil \cdot \rceil$ the ceiling of the number, and $\lfloor \cdot \rfloor$ the integral part of the number.

Theorem 4 *([10, ch. 17]) Let* $x \in [a,b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$. *For* $q > 0$, $\lambda > 0$, *we consider the number* $\lambda_q > z_0 > 0$ *with* $M_{q,\lambda}(z_0) = M_{q,\lambda}(0)$ *and* $\lambda_q > 1$. *Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)} < max\left\{\frac{1}{M_{q,\lambda}(\lambda_q)}, \frac{1}{M_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \Delta(q). \tag{26}$$

We make

Remark 5 *([10, ch. 17]) (i) We have that*

$$\lim_{n\to+\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \neq 1, \quad \textit{for at least some } x \in [a,b], \tag{27}$$

where $\lambda, q > 0$.

(ii) Let $[a,b] \subset \mathbb{R}$. *For large* n *we always have* $\lceil na \rceil \leq \lfloor nb \rfloor$. *Also* $a \leq \frac{k}{n} \leq b$, *iff* $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. *In general it holds*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \leq 1. \tag{28}$$

Let $(\mathbb{C}, |\cdot|)$ be the Banach space of the complex numbers over the reals.

Definition 6 *Let* $f \in C([a,b], \mathbb{C})$ *and* $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. *We introduce and define the* $\mathbb{C}$*-valued linear neural network operators*

$$H_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)}, \quad x \in [a,b]; \ q > 0, q \neq 1. \tag{29}$$

For large enough n *we always obtain* $\lceil na \rceil \leq \lfloor nb \rfloor$. *Also* $a \leq \frac{k}{n} \leq b$, *iff* $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. *The same* H_n *is used for real valued functions. We study here the pointwise and uniform convergence of* $H_n(f,x)$ *to* $f(x)$ *with rates.*

Clearly here $H_n(f) \in C([a,b], \mathbb{C})$.

For convenience, also we call

$$H_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx-k), \tag{30}$$

(the same H_n^* can be defined for real valued functions) that is

$$H_n(f,x) := \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)}. \tag{31}$$

So that

$$H_n(f,x) - f(x) = \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)} - f(x) = \tag{32}$$

$$\frac{H_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)}.$$

Consequently, we derive that

$$|H_n(f,x) - f(x)| \leq \Delta(q) \left| H_n^*(f,x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \right) \right| =$$

$$\Delta(q) \left| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) M_{q,\lambda}(nx-k) \right|, \quad (33)$$

where $\Delta(q)$ as in (26).

We will estimate the right hand side of the last quantity.

For that we need, for $f \in C([a,b], \mathbb{C})$ the first modulus of continuity

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0. \quad (34)$$

The fact $f \in C([a,b], \mathbb{C})$ is equivalent to $\lim_{\delta \to 0} \omega_1(f,\delta) = 0$, see [8].

3 Main Results

We present $\mathbb{C}$-valued neural network high order approximations to a function given with rates. We start with a trigonometric approximation.

Theorem 7 *Let $f \in C^2([a,b], \mathbb{C})$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

1)

$$|H_n(f,x) - f(x)| \leq \Delta(q) \left[|f'(x)| \left(\frac{1}{n^\alpha} + (b-a) T e^{-2\lambda n^{(1-\alpha)}} \right) \right. \quad (35)$$

$$+ \frac{|f''(x)|}{2} \left(\frac{1}{n^{2\alpha}} + (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}} \right) +$$

$$\left. \left(\frac{\omega_1\left(f'' + f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f'' + f\|_\infty (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}} \right) \right],$$

2) if $f'(x) = f''(x) = 0$, we obtain

$$|H_n(f,x) - f(x)| \leq \Delta(q) \left[\frac{\omega_1\left(f'' + f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f'' + f\|_\infty (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}} \right], \quad (36)$$

notice here the high rate of convergence at $n^{-3\alpha}$, *3) furthermore we get*

$$\|H_n f - f\|_\infty \leq \Delta(q) \left[\|f'\|_\infty \left(\frac{1}{n^\alpha} + (b-a) T e^{-2\lambda n^{(1-\alpha)}} \right) \right.$$

$$+ \frac{\|f''\|_\infty}{2} \left(\frac{1}{n^{2\alpha}} + (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}} \right) +$$

$$\left. \left(\frac{\omega_1 \left(f'' + f, \frac{1}{n^\alpha} \right)}{2n^{2\alpha}} + \|f'' + f\|_\infty (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}} \right) \right], \tag{37}$$

i.e. $\lim_{n \to +\infty} H_n(f) = f$, *pointwise and uniformly,*

4) and finally, it holds

$$\left| H_n(f, x) - f'(x) H_n(\sin(\cdot - x), x) - 2f''(x) H_n\left(\sin^2\left(\frac{\cdot - x}{2} \right), x \right) - f(x) \right| \leq$$

$$\Delta(q) \left[\frac{\omega_1 \left(f'' + f, \frac{1}{n^\alpha} \right)}{2n^{2\alpha}} + \|f'' + f\|_\infty (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}} \right], \tag{38}$$

again here we achieve high speed of convergence at $n^{-3\alpha}$.

Proof. Here $f \in C^2([a,b], \mathbb{C})$, and we apply the trigonometric Taylor's formula for $f \in C^2([a,b], \mathbb{C})$, see Theorem 6 of [11].

Let $\frac{k}{n}, x \in [a,b]$, then

$$f\left(\frac{k}{n}\right) = f(x) + f'(x) \sin\left(\frac{k}{n} - x\right) + 2f''(x) \sin^2\left(\frac{\frac{k}{n} - x}{2}\right) +$$

$$\int_x^{\frac{k}{n}} \left[(f''(t) + f(t)) - (f''(x) + f(x)) \right] \sin\left(\frac{k}{n} - t\right) dt. \tag{39}$$

Hence it holds

$$f\left(\frac{k}{n}\right) M_{q,\lambda}(nx - k) = f(x) M_{q,\lambda}(nx - k) +$$

$$f'(x) \sin\left(\frac{k}{n} - x\right) M_{q,\lambda}(nx - k) + 2f''(x) \sin^2\left(\frac{\frac{k}{n} - x}{2}\right) M_{q,\lambda}(nx - k) +$$

$$M_{q,\lambda}(nx - k) \left(\int_x^{\frac{k}{n}} \left[(f''(t) + f(t)) - (f''(x) + f(x)) \right] \sin\left(\frac{k}{n} - t\right) dt \right). \tag{40}$$

So that we have

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx - k) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx - k) =$$

$$f'(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\sin\left(\frac{k}{n}-x\right)+2f''(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\sin^2\left(\frac{\frac{k}{n}-x}{2}\right)+$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\left(\int_x^{\frac{k}{n}}\left[\left(f''(t)+f(t)\right)-\left(f''(x)+f(x)\right)\right]\sin\left(\frac{k}{n}-t\right)dt\right). \tag{41}$$

Thus, we obtain

$$H_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)=$$

$$f'(x)\,H_n^*\left(\sin(\cdot-x),x\right)+2f''(x)\,H_n^*\left(\sin^2\left(\frac{\cdot-x}{2}\right),x\right)+\Lambda_n(x), \tag{42}$$

where

$$\Lambda_n(x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\left(\int_x^{\frac{k}{n}}\left[\left(f''(t)+f(t)\right)-\left(f''(x)+f(x)\right)\right]\sin\left(\frac{k}{n}-t\right)dt\right).$$

We call

$$R_2(n):=\int_x^{\frac{k}{n}}\left[\left(f''(t)+f(t)\right)-\left(f''(x)+f(x)\right)\right]\sin\left(\frac{k}{n}-t\right)dt. \tag{43}$$

We assume that $b-a>\frac{1}{n^\alpha}$, which is always the case for large enough $n\in\mathbb{N}$, that is when $n>\left\lceil (b-a)^{-\frac{1}{\alpha}}\right\rceil$.

Thus $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^\alpha}$ or $\left|\frac{k}{n}-x\right|>\frac{1}{n^\alpha}$.

In case of $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^\alpha}$, we have the following cases:

i) if $\frac{k}{n}\geq x$, then

$$|R_2(n)|=\left|\int_x^{\frac{k}{n}}\left[\left(f''(t)+f(t)\right)-\left(f''(x)+f(x)\right)\right]\sin\left(\frac{k}{n}-t\right)dt\right|\leq$$

$$\int_x^{\frac{k}{n}}\omega_1\left(f''+f,t-x\right)\left|\sin\left(\frac{k}{n}-t\right)\right|dt\leq \tag{44}$$

(by $|\sin x|\leq|x|$, $\forall\ x\in\mathbb{R}$)

$$\int_x^{\frac{k}{n}}\omega_1\left(f''+f,t-x\right)\left(\frac{k}{n}-t\right)dt\leq\omega_1\left(f''+f,\frac{k}{n}-x\right)\frac{\left(\frac{k}{n}-x\right)^2}{2}$$

$$\leq\frac{\omega_1\left(f''+f,\frac{1}{n^\alpha}\right)}{2n^{2\alpha}},$$

that is

$$|R_2(n)| \leq \frac{\omega_1\left(f''+f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}}. \tag{45}$$

ii) if $\frac{k}{n} < x$, then

$$|R_2(n)| = \left|\int_x^{\frac{k}{n}} \left[(f''(t)+f(t)) - (f''(x)+f(x))\right] \sin\left(\frac{k}{n}-t\right) dt\right| =$$

$$\left|\int_{\frac{k}{n}}^x \left[(f''(t)+f(t)) - (f''(x)+f(x))\right] \sin\left(\frac{k}{n}-t\right) dt\right| \leq$$

$$\int_{\frac{k}{n}}^x \left|(f''(t)+f(t)) - (f''(x)+f(x))\right| \left|\sin\left(\frac{k}{n}-t\right)\right| dt \leq \tag{46}$$

$$\int_{\frac{k}{n}}^x \omega_1\left(f''+f, x-\frac{k}{n}\right)\left(t-\frac{k}{n}\right) dt \leq \omega_1\left(f''+f, x-\frac{k}{n}\right) \frac{\left(x-\frac{k}{n}\right)^2}{2}$$

$$\leq \frac{\omega_1\left(f''+f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}}.$$

That is

$$|R_2(n)| \leq \frac{\omega_1\left(f''+f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}}. \tag{47}$$

So, we have proved when $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}$, always it holds

$$|R_2(n)| \leq \frac{\omega_1\left(f''+f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}}. \tag{48}$$

Next assume again $\frac{k}{n} \geq x$, then

$$|R_2(n)| = \left|\int_x^{\frac{k}{n}} \left[(f''(t)+f(t)) - (f''(x)+f(x))\right] \sin\left(\frac{k}{n}-t\right) dt\right| \leq$$

$$\int_x^{\frac{k}{n}} \left|(f''(t)+f(t)) - (f''(x)+f(x))\right| \left|\sin\left(\frac{k}{n}-t\right)\right| dt \leq$$

(by $|\sin x| \leq |x|$, $\forall\ x \in \mathbb{R}$)

$$2\|f''+f\|_\infty \left(\int_x^{\frac{k}{n}} \left|\sin\left(\frac{k}{n}-t\right)\right| dt\right) \leq$$

$$2\|f''+f\|_\infty \left(\int_x^{\frac{k}{n}} \left(\frac{k}{n}-t\right) dt\right) =$$

$$2\|f''+f\|_\infty \frac{\left(\frac{k}{n}-x\right)^2}{2} \leq \|f''+f\|_\infty (b-a)^2. \tag{49}$$

Hence it is

$$|R_2(n)| \leq \|f'' + f\|_\infty (b-a)^2 . \tag{50}$$

When $\frac{k}{n} < x$, we have

$$|R_2(n)| = \left| \int_x^{\frac{k}{n}} \left[(f''(t) + f(t)) - (f''(x) + f(x)) \right] \sin\left(\frac{k}{n} - t\right) dt \right| =$$

$$\left| \int_{\frac{k}{n}}^x \left[(f''(x) + f(x)) - (f''(t) + f(t)) \right] \sin\left(\frac{k}{n} - t\right) dt \right| \leq$$

$$\int_{\frac{k}{n}}^x \left| (f''(x) + f(x)) - (f''(t) + f(t)) \right| \left| \sin\left(\frac{k}{n} - t\right) \right| dt \leq$$

$$2 \|f'' + f\|_\infty \int_{\frac{k}{n}}^x \left| \sin\left(\frac{k}{n} - t\right) \right| dt \leq$$

$$2 \|f'' + f\|_\infty \int_{\frac{k}{n}}^x \left(t - \frac{k}{n}\right) dt = \|f'' + f\|_\infty \left(x - \frac{k}{n}\right)^2 \leq$$

$$\|f'' + f\|_\infty (b-a)^2 . \tag{51}$$

Therefore, always it holds

$$|R_2(n)| \leq \|f'' + f\|_\infty (b-a)^2 . \tag{52}$$

And we have

$$\Lambda_n(x) = \sum_{\substack{k = \lceil na \rceil \\ : \left|\frac{k}{n} - x\right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} M_{q,\lambda}(nx - k) R_2(n) +$$

$$\sum_{\substack{k = \lceil na \rceil \\ : \left|\frac{k}{n} - x\right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} M_{q,\lambda}(nx - k) R_2(n) . \tag{53}$$

Hence it holds

$$|\Lambda_n(x)| \leq \sum_{\substack{k = \lceil na \rceil \\ : \left|\frac{k}{n} - x\right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} M_{q,\lambda}(nx - k) |R_2(n)| +$$

$$\sum_{\substack{k = \lceil na \rceil \\ : \left|\frac{k}{n} - x\right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} M_{q,\lambda}(nx - k) |R_2(n)| \leq \tag{54}$$

$$\left(\sum_{\left\{ \begin{array}{c} k=\lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha} \end{array} \right.}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \right) \frac{\omega_1\left(f''+f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} +$$

$$\sum_{\left\{ \begin{array}{c} k=\lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha} \end{array} \right.}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \|f''+f\|_\infty (b-a)^2 \overset{\text{(by (28))}}{\leq}$$

$$\frac{\omega_1\left(f''+f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f''+f\|_\infty (b-a)^2$$

$$\left(\sum_{\left\{ \begin{array}{c} k=\lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha} \end{array} \right.}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \right) \overset{\text{(by Theorem 3)}}{\leq}$$

$$\frac{\omega_1\left(f''+f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f''+f\|_\infty (b-a)^2 \, Te^{-2\lambda n^{(1-\alpha)}}. \tag{55}$$

Consequently, we have derived that

$$|\Lambda_n(x)| \leq \frac{\omega_1\left(f''+f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f''+f\|_\infty (b-a)^2 \, Te^{-2\lambda n^{(1-\alpha)}}. \tag{56}$$

Next we use again $|\sin x| \leq |x|$, $\forall \ x \in \mathbb{R}$.

We have that

$$H_n^*(\sin(\cdot - x), x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \sin\left(\frac{k}{n} - x\right), \tag{57}$$

and

$$|H_n^*(\sin(\cdot - x), x)| \leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \left| \sin\left(\frac{k}{n} - x\right) \right| =$$

$$\sum_{\left\{ \begin{array}{c} k=\lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha} \end{array} \right.}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \left| \sin\left(\frac{k}{n} - x\right) \right| +$$

$$\sum_{\left\{ \begin{array}{c} k=\lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha} \end{array} \right.}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \left| \sin\left(\frac{k}{n} - x\right) \right| \leq \tag{58}$$

$$\sum_{\substack{k=\lceil na\rceil \\ : \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\left|\frac{k}{n}-x\right| +$$

$$\sum_{\substack{k=\lceil na\rceil \\ : \left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\left|\frac{k}{n}-x\right| \leq$$

$$\frac{1}{n^{\alpha}}+(b-a)\left(\sum_{\substack{k=\lceil na\rceil \\ : \left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\right) \overset{\text{(by (25))}}{\leq} \tag{59}$$

$$\frac{1}{n^{\alpha}}+(b-a)\,Te^{-2\lambda n^{(1-\alpha)}}.$$

We found that

$$\left|H_n^*(\sin(\cdot-x),x)\right| \leq \frac{1}{n^{\alpha}}+(b-a)\,Te^{-2\lambda n^{(1-\alpha)}}. \tag{60}$$

Next we estimate

$$H_n^*\left(\sin^2\left(\frac{\cdot-x}{2}\right),x\right)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\sin^2\left(\frac{\frac{k}{n}-x}{2}\right), \tag{61}$$

We have that (by $|\sin x|\leq|x|$, $\forall\ x\in\mathbb{R}$)

$$H_n^*\left(\sin^2\left(\frac{\cdot-x}{2}\right),x\right)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\left|\sin\left(\frac{\frac{k}{n}-x}{2}\right)\right|^2 \leq$$

$$\frac{1}{4}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\left|\frac{k}{n}-x\right|^2 =$$

$$\frac{1}{4}\left[\sum_{\substack{k=\lceil na\rceil \\ : \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\left|\frac{k}{n}-x\right|^2 +\right.$$

$$\sum_{\substack{k=\lceil na\rceil \\ : \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\left|\frac{k}{n}-x\right|^{2}\Bigg] \leq \tag{62}$$

$$\frac{1}{4}\left[\frac{1}{n^{2\alpha}}+(b-a)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right].$$

That is

$$H_{n}^{*}\left(\sin^{2}\left(\frac{\cdot-x}{2}\right),x\right) \leq \frac{1}{4}\left[\frac{1}{n^{2\alpha}}+(b-a)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right]. \tag{63}$$

Consequently we have derived:

1)

$$|H_{n}(f,x)-f(x)| \leq \Delta(q)\Bigg[|f'(x)|\left(\frac{1}{n^{\alpha}}+(b-a)Te^{-2\lambda n^{(1-\alpha)}}\right) \tag{64}$$

$$+\frac{|f''(x)|}{2}\left(\frac{1}{n^{2\alpha}}+(b-a)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right)+$$

$$\left(\frac{\omega_{1}\left(f''+f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''+f\|_{\infty}(b-a)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right)\Bigg].$$

2) if $f'(x)=f''(x)=0$, by (64), we obtain

$$|H_{n}(f,x)-f(x)| \leq \Delta(q)\left[\frac{\omega_{1}\left(f''+f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''+f\|_{\infty}(b-a)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right], \tag{65}$$

notice here the high rate of convergence at $n^{-3\alpha}$.

3) Furthermore, by (64), we get

$$\|H_{n}f-f\|_{\infty} \leq \Delta(q)\Bigg[\|f'\|_{\infty}\left(\frac{1}{n^{\alpha}}+(b-a)Te^{-2\lambda n^{(1-\alpha)}}\right)+$$

$$\frac{\|f''\|_{\infty}}{2}\left(\frac{1}{n^{2\alpha}}+(b-a)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right)+$$

$$\left(\frac{\omega_{1}\left(f''+f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''+f\|_{\infty}(b-a)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right)\Bigg]. \tag{66}$$

We derive that $\lim_{n\to+\infty} H_{n}(f)=f$, pointwise and uniformly.

We observe that

$$H_{n}(f,x)-f'(x)H_{n}(\sin(\cdot-x),x)-2f''(x)H_{n}\left(\sin^{2}\left(\frac{\cdot-x}{2}\right),x\right)-f(x)=$$

$$\frac{H_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)} - f'(x)\frac{H_n^*(\sin(\cdot - x),x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)} -$$

$$2f''(x)\frac{H_n^*\left(\sin^2\left(\frac{\cdot - x}{2}\right),x\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)} - f(x)\left(\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)}\right) = \tag{67}$$

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)}\left[H_n^*(f,x) - f'(x)H_n^*(\sin(\cdot - x),x) -\right.$$

$$\left.2f''(x)H_n^*\left(\sin^2\left(\frac{\cdot - x}{2}\right),x\right) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\right] =$$

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)}\left(\Lambda_n(x)\right). \tag{68}$$

Finally, we obtain ($\forall\ x \in [a,b]$, $n \in \mathbb{N}$):

4)

$$\left|H_n(f,x) - f'(x)H_n(\sin(\cdot - x),x) - 2f''(x)H_n\left(\sin^2\left(\frac{\cdot - x}{2}\right),x\right) - f(x)\right| \overset{(26)}{\leq}$$

$$\Delta(q)\left|\Lambda_n(x)\right| \overset{(56)}{\leq}$$

$$\Delta(q)\left[\frac{\omega_1\left(f''+f,\frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f''+f\|_\infty (b-a)^2 Te^{-2\lambda n^{(1-\alpha)}}\right]. \tag{69}$$

The theorem is proved. ■

We continue with a hyperbolic high order neural network approximation.

Theorem 8 *Let $f \in C^2([a,b],\mathbb{C})$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

1)

$$|H_n(f,x) - f(x)| \leq \Delta(q)\cosh(b-a)\left[|f'(x)|\left(\frac{1}{n^\alpha} + (b-a)Te^{-2\lambda n^{(1-\alpha)}}\right)\right.$$

$$\frac{|f''(x)|}{2}\left(\frac{1}{n^{2\alpha}} + (b-a)^2 Te^{-2\lambda n^{(1-\alpha)}}\right) +$$

$$\left.\left(\frac{\omega_1\left(f''-f,\frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f''-f\|_\infty (b-a)^2 Te^{-2\lambda n^{(1-\alpha)}}\right)\right], \tag{70}$$

2) if $f'(x) = f''(x) = 0$, we obtain

$$|H_n(f,x) - f(x)| \leq \Delta(q)\cosh(b-a)$$

$$\left[\frac{\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''-f\|_{\infty}(b-a)^2\,Te^{-2\lambda n^{(1-\alpha)}}\right], \tag{71}$$

notice here the high rate of convergence at $n^{-3\alpha}$,

3) furthermore, we get

$$\|H_nf-f\|_{\infty}\leq\Delta(q)\cosh(b-a)\left[\|f'\|_{\infty}\left(\frac{1}{n^{\alpha}}+(b-a)\,Te^{-2\lambda n^{(1-\alpha)}}\right)\right.$$

$$\frac{\|f''\|_{\infty}}{2}\left(\frac{1}{n^{2\alpha}}+(b-a)^2\,Te^{-2\lambda n^{(1-\alpha)}}\right)+$$

$$\left.\left(\frac{\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''-f\|_{\infty}(b-a)^2\,Te^{-2\lambda n^{(1-\alpha)}}\right)\right], \tag{72}$$

it follows that $\lim_{n\to+\infty}H_n(f)=f$, *pointwise and uniformly,*

and

4)

$$\left|H_n(f,x)-f'(x)\,H_n(\sinh(\cdot-x),x)-2f''(x)\,H_n\left(\sinh^2\left(\frac{\cdot-x}{2}\right),x\right)-f(x)\right|\leq$$

$$\Delta(q)\cosh(b-a)\left[\frac{\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''-f\|_{\infty}(b-a)^2\,Te^{-2\lambda n^{(1-\alpha)}}\right], \tag{73}$$

again here we achieve high speed of convergence at $n^{-3\alpha}$.

Proof. By the mean value theorem we have that

$$\sinh x=\sinh x-\sinh 0=(\cosh\xi)(x-0),$$

for some ξ between $\{0,x\}$, for any $x\in\mathbb{R}$.

Hence

$$|\sinh x|\leq\|\cosh\|_{\infty,[-(b-a),b-a]}|x|,\ \ \forall\ x\in[-(b-a),b-a]. \tag{74}$$

That is, there exists $M\geq 1$ such that

$$|\sinh x|\leq M|x|,\ \ \forall\ x\in[-(b-a),b-a], \tag{75}$$

where $M:=\|\cosh\|_{\infty,[-(b-a),b-a]}=\cosh(b-a)$.

Here $f\in C^2([a,b],\mathbb{C})$, and we apply the hyperbolic Taylor's formula for $f\in C^2([a,b],\mathbb{C})$, see Theorem 7 of [11].

Let $\frac{k}{n},x\in[a,b]$, then

$$f\left(\frac{k}{n}\right)=f(x)+f'(x)\sinh\left(\frac{k}{n}-x\right)+2f''(x)\sinh^2\left(\frac{\frac{k}{n}-x}{2}\right)+$$

$$\int_{x}^{\frac{k}{n}} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh \left(\frac{k}{n} - t \right) dt. \tag{76}$$

Hence it holds

$$f\left(\frac{k}{n}\right) M_{q,\lambda}(nx-k) = f(x) M_{q,\lambda}(nx-k) +$$

$$f'(x) \sinh\left(\frac{k}{n} - x\right) M_{q,\lambda}(nx-k) + 2f''(x) \sinh^2\left(\frac{\frac{k}{n} - x}{2}\right) M_{q,\lambda}(nx-k) +$$

$$M_{q,\lambda}(nx-k) \left(\int_{x}^{\frac{k}{n}} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh \left(\frac{k}{n} - t \right) dt \right). \tag{77}$$

So that we have

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx-k) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) =$$

$$f'(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \sinh\left(\frac{k}{n} - x\right) + 2f''(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \sinh^2\left(\frac{\frac{k}{n} - x}{2}\right) +$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \left(\int_{x}^{\frac{k}{n}} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh \left(\frac{k}{n} - t \right) dt \right). \tag{78}$$

Thus, we obtain

$$H_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) =$$

$$f'(x) H_n^*(\sinh(\cdot - x), x) + 2f''(x) H_n^*\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) + \Lambda_n(x), \tag{79}$$

where

$$\Lambda_n(x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \left(\int_{x}^{\frac{k}{n}} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh \left(\frac{k}{n} - t \right) dt \right). \tag{80}$$

We call

$$R_2(n) := \int_{x}^{\frac{k}{n}} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh \left(\frac{k}{n} - t \right) dt. \tag{81}$$

We assume that $b - a > \frac{1}{n^{\alpha}}$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\alpha}} \right\rceil$.

Thus $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}$ or $\left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}}$.

In case of $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}$, we have the following cases:

i) if $\frac{k}{n} \geq x$, then

$$|R_2(n)| = \left|\int_x^{\frac{k}{n}} \left[\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right]\sinh\left(\frac{k}{n}-t\right)dt\right| \leq$$

$$\int_x^{\frac{k}{n}} \omega_1\left(f''-f, t-x\right)\left|\sinh\left(\frac{k}{n}-t\right)\right| dt \overset{(75)}{\leq}$$

$$\int_x^{\frac{k}{n}} \omega_1\left(f''-f, t-x\right) M\left(\frac{k}{n}-t\right)dt \leq M\omega_1\left(f''-f, \frac{k}{n}-x\right)\int_x^{\frac{k}{n}}\left(\frac{k}{n}-t\right)dt = \tag{82}$$

$$M\omega_1\left(f''-f, \frac{k}{n}-x\right)\frac{\left(\frac{k}{n}-x\right)^2}{2} \leq \frac{M\omega_1\left(f''-f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}},$$

that is

$$|R_2(n)| \leq \frac{M\omega_1\left(f''-f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}. \tag{83}$$

ii) if $\frac{k}{n} < x$, then

$$|R_2(n)| = \left|\int_x^{\frac{k}{n}} \left[\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right]\sinh\left(\frac{k}{n}-t\right)dt\right| =$$

$$\left|\int_{\frac{k}{n}}^{x} \left[\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right]\sinh\left(\frac{k}{n}-t\right)dt\right| \leq$$

$$\int_{\frac{k}{n}}^{x} \left|\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right|\left|\sinh\left(\frac{k}{n}-t\right)\right| dt \leq \tag{84}$$

$$M\omega_1\left(f''-f, x-\frac{k}{n}\right)\int_{\frac{k}{n}}^{x}\left(t-\frac{k}{n}\right)dt = M\omega_1\left(f''-f, x-\frac{k}{n}\right)\frac{\left(x-\frac{k}{n}\right)^2}{2}$$

$$\leq \frac{M\omega_1\left(f''-f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}},$$

that is

$$|R_2(n)| \leq \frac{M\omega_1\left(f''-f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}. \tag{85}$$

So, we have proved when $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}$, always it holds

$$|R_2(n)| \leq \frac{M\omega_1\left(f''-f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}. \tag{86}$$

Next assume again $\frac{k}{n} \geq x$, then

$$|R_2(n)| = \left|\int_x^{\frac{k}{n}} \left[\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right]\sinh\left(\frac{k}{n}-t\right)dt\right| \leq$$

$$\int_x^{\frac{k}{n}} \left|\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right| \left|\sinh\left(\frac{k}{n}-t\right)\right| dt \leq$$

$$2M\left\|f''-f\right\|_\infty \int_x^{\frac{k}{n}} \left(\frac{k}{n}-t\right) dt = \tag{87}$$

$$2M\left\|f''-f\right\|_\infty \frac{\left(\frac{k}{n}-x\right)^2}{2} \leq M\left\|f''-f\right\|_\infty (b-a)^2.$$

Hence

$$|R_2(n)| \leq M\left\|f''-f\right\|_\infty (b-a)^2. \tag{88}$$

When $\frac{k}{n} < x$, we have

$$|R_2(n)| = \left|\int_{\frac{k}{n}}^x \left[\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right] \sinh\left(\frac{k}{n}-t\right) dt\right| \leq$$

$$\int_{\frac{k}{n}}^x \left|\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right| \left|\sinh\left(\frac{k}{n}-t\right)\right| dt \leq$$

$$2M\left\|f''-f\right\|_\infty \int_{\frac{k}{n}}^x \left(t-\frac{k}{n}\right) dt = \tag{89}$$

$$2M\left\|f''-f\right\|_\infty \frac{\left(x-\frac{k}{n}\right)^2}{2} \leq M\left\|f''-f\right\|_\infty (b-a)^2.$$

Therefore, always it holds

$$|R_2(n)| \leq M\left\|f''-f\right\|_\infty (b-a)^2. \tag{90}$$

And we have

$$\Lambda_n(x) = \sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k) R_2(n) + \tag{91}$$

$$\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k) R_2(n).$$

Hence it holds

$$|\Lambda_n(x)| \leq \sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k) |R_2(n)| + \tag{92}$$

$$\sum_{\begin{cases} k=\lceil na \rceil \\ : \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)\left|R_2(n)\right| \leq$$

$$\left(\sum_{\begin{cases} k=\lceil na \rceil \\ : \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \right) \frac{\omega_1\left(f''-f, \frac{1}{n^{\alpha}}\right) M}{2n^{2\alpha}} +$$

$$\left(\sum_{\begin{cases} k=\lceil na \rceil \\ : \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \right) M \left\|f''-f\right\|_{\infty} (b-a)^2 \overset{((28))}{\leq}$$

$$\frac{M\omega_1\left(f''-f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}} + M\left\|f''-f\right\|_{\infty}(b-a)^2$$

$$\left(\sum_{\begin{cases} k=\lceil na \rceil \\ : \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \right) \overset{\text{(by Theorem 3)}}{\leq} \tag{93}$$

$$M\frac{\omega_1\left(f''-f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}} + M\left\|f''-f\right\|_{\infty}(b-a)^2 Te^{-2\lambda n^{(1-\alpha)}}.$$

Consequently, we have derived that

$$\left|\Lambda_n(x)\right| \leq M\left[\frac{\omega_1\left(f''-f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}} + \left\|f''-f\right\|_{\infty}(b-a)^2 Te^{-2\lambda n^{(1-\alpha)}}\right]. \tag{94}$$

We have that

$$H_n^*\left(\sinh(\cdot - x), x\right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)\sinh\left(\frac{k}{n}-x\right), \tag{95}$$

and

$$\left|H_n^*\left(\sinh(\cdot - x), x\right)\right| \leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)\left|\sinh\left(\frac{k}{n}-x\right)\right| =$$

$$\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}} \end{array}\right.}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right)\left|\sinh\left(\frac{k}{n}-x\right)\right|+$$

$$\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}} \end{array}\right.}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right)\left|\sinh\left(\frac{k}{n}-x\right)\right|\leq \tag{96}$$

$$M\left[\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}} \end{array}\right.}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right)\left|\frac{k}{n}-x\right|+\right.$$

$$\left.\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}} \end{array}\right.}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right)\left|\frac{k}{n}-x\right|\right]\leq \tag{97}$$

$$M\left[\frac{1}{n^{\alpha}}+(b-a)\left(\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}} \end{array}\right.}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right)\right)\right]\overset{\text{(by (25))}}{\leq}$$

$$M\left[\frac{1}{n^{\alpha}}+(b-a)\,Te^{-2\lambda n^{(1-\alpha)}}\right].$$

We found that

$$\left|H_{n}^{*}\left(\sinh\left(\cdot-x\right),x\right)\right|\leq M\left[\frac{1}{n^{\alpha}}+(b-a)\,Te^{-2\lambda n^{(1-\alpha)}}\right]. \tag{98}$$

Next we estimate

$$H_{n}^{*}\left(\sinh^{2}\left(\frac{\cdot-x}{2}\right),x\right)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right)\sinh^{2}\left(\frac{\frac{k}{n}-x}{2}\right), \tag{99}$$

We have that

$$H_{n}^{*}\left(\sinh^{2}\left(\frac{\cdot-x}{2}\right),x\right)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}\left(nx-k\right)\left(\sinh\left(\frac{\frac{k}{n}-x}{2}\right)\right)^{2}\leq$$

$$\frac{M}{4}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}\left(nx-k\right)\left(\frac{k}{n}-x\right)^{2}= \tag{100}$$

$$\frac{M}{4}\left[\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}\end{cases}}^{\lfloor nb\rfloor}M_{q,\lambda}\left(nx-k\right)\left(\frac{k}{n}-x\right)^{2}+\right.$$

$$\left.\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}\end{cases}}^{\lfloor nb\rfloor}M_{q,\lambda}\left(nx-k\right)\left(\frac{k}{n}-x\right)^{2}\right]\leq$$

$$\frac{M}{4}\left[\frac{1}{n^{2\alpha}}+\left(b-a\right)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right].$$

That is

$$H_{n}^{*}\left(\sinh^{2}\left(\frac{\cdot-x}{2}\right),x\right)\leq\frac{M}{4}\left[\frac{1}{n^{2\alpha}}+\left(b-a\right)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right]. \tag{101}$$

By (33) and putting together (79), (94), (98) and (101) we derive
1)

$$\left|H_{n}\left(f,x\right)-f\left(x\right)\right|\leq\Delta\left(q\right)M\left[\left|f'\left(x\right)\right|\left(\frac{1}{n^{\alpha}}+\left(b-a\right)Te^{-2\lambda n^{(1-\alpha)}}\right)+\right.$$

$$\frac{\left|f''\left(x\right)\right|}{2}\left(\frac{1}{n^{2\alpha}}+\left(b-a\right)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right)+$$

$$\left.\left(\frac{\omega_{1}\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f''-f\right\|_{\infty}\left(b-a\right)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right)\right]. \tag{102}$$

2) If $f'\left(x\right)=f''\left(x\right)=0$, by (102), we obtain

$$\left|H_{n}\left(f,x\right)-f\left(x\right)\right|\leq$$

$$M\Delta\left(q\right)\left[\frac{\omega_{1}\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f''-f\right\|_{\infty}\left(b-a\right)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right], \tag{103}$$

notice here the high rate of convergence at $n^{-3\alpha}$.

3) Furthermore, by (102), we get

$$\left\|H_{n}f-f\right\|_{\infty}\leq\Delta\left(q\right)M\left[\left\|f'\right\|_{\infty}\left(\frac{1}{n^{\alpha}}+\left(b-a\right)Te^{-2\lambda n^{(1-\alpha)}}\right)+\right.$$

$$\frac{\|f''\|_{\infty}}{2}\left(\frac{1}{n^{2\alpha}}+(b-a)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right)+$$

$$\left(\frac{\omega_{1}\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''-f\|_{\infty}(b-a)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right)\Bigg]. \tag{104}$$

It follows that $\lim_{n\to+\infty}H_{n}(f)=f$, pointwise and uniformly.

We observe that

$$H_{n}(f,x)-f'(x)H_{n}(\sinh(\cdot-x),x)-2f''(x)H_{n}\left(\sinh^{2}\left(\frac{\cdot-x}{2}\right),x\right)-f(x)=$$

$$\frac{H_{n}^{*}(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)}-f'(x)\frac{H_{n}^{*}(\sinh(\cdot-x),x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)}-$$

$$2f''(x)\frac{H_{n}^{*}\left(\sinh^{2}\left(\frac{\cdot-x}{2}\right),x\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)}-f(x)\left(\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)}\right)= \tag{105}$$

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)}\left[H_{n}^{*}(f,x)-f'(x)H_{n}^{*}(\sinh(\cdot-x),x)-\right.$$

$$\left.2f''(x)H_{n}^{*}\left(\sinh^{2}\left(\frac{\cdot-x}{2}\right),x\right)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)\right]= \tag{106}$$

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)}(\Lambda_{n}(x)).$$

Finally, we obtain ($\forall\ x\in[a,b]$, $n\in\mathbb{N}$):

4)

$$\left|H_{n}(f,x)-f'(x)H_{n}(\sinh(\cdot-x),x)-2f''(x)H_{n}\left(\sinh^{2}\left(\frac{\cdot-x}{2}\right),x\right)-f(x)\right|\overset{(26)}{\leq}$$

$$\Delta(q)|\Lambda_{n}(x)|\overset{(94)}{\leq}$$

$$\Delta(q)M\left[\frac{\omega_{1}\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''-f\|_{\infty}(b-a)^{2}Te^{-2\lambda n^{(1-\alpha)}}\right]. \tag{107}$$

The theorem is established. ■

Next follows a mixed hyperbolic-trigonometric high order neural network approximation.

Theorem 9 *Let* $f\in C^{4}([a,b],\mathbb{C})$, $0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in[a,b]$. *Then*

1)

$$\Bigg| H_n(f,x) - f(x) - \frac{f'(x)}{2} H_n((\sinh(\cdot - x) + \sin(\cdot - x)), x)$$

$$-\frac{f''(x)}{2} H_n((\cosh(\cdot - x) - \cos(\cdot - x)), x) \tag{108}$$

$$-\frac{f^{(3)}(x)}{2} H_n((\sinh(\cdot - x) - \sin(\cdot - x)), x)$$

$$-f^{(4)}(x) H_n\left(\left(\sinh^2\left(\frac{\cdot - x}{2}\right) - \sin^2\left(\frac{\cdot - x}{2}\right)\right), x\right)\Bigg| \leq$$

$$\frac{\Delta(q)(\cosh(b-a)+1)}{2}$$

$$\left[\frac{\omega_1\left(f^{(4)} - f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \left\|f^{(4)} - f\right\|_\infty (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}}\right], \tag{109}$$

2) if $f^{(i)}(x) = 0$, $i = 1, 2, 3, 4$, *we get*

$$|H_n(f,x) - f(x)| \leq \frac{\Delta(q)(\cosh(b-a)+1)}{2}$$

$$\left[\frac{\omega_1\left(f^{(4)} - f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \left\|f^{(4)} - f\right\|_\infty (b-a)^2 T e^{-2\lambda n^{(1-\alpha)}}\right], \tag{110}$$

in the last (110) observe the high speed of convergence at $n^{-3\alpha}$.

Proof. Here $f \in C^4([a,b], \mathbb{C})$, and we apply the hyperbolic-trigonometric Taylor's formula for $f \in C^4([a,b], \mathbb{C})$, see Theorem 8 of [11].

Let $\frac{k}{n}, x \in [a,b]$, then

$$f\left(\frac{k}{n}\right) - f(x) - f'(x)\left(\frac{\sinh\left(\frac{k}{n} - x\right) + \sin\left(\frac{k}{n} - x\right)}{2}\right)$$

$$-f''(x)\left(\frac{\cosh\left(\frac{k}{n} - x\right) - \cos\left(\frac{k}{n} - x\right)}{2}\right)$$

$$-f^{(3)}(x)\left(\frac{\sinh\left(\frac{k}{n} - x\right) - \sin\left(\frac{k}{n} - x\right)}{2}\right)$$

$$-f^{(4)}(x)\left(\sinh^2\left(\frac{\frac{k}{n} - x}{2}\right) - \sin^2\left(\frac{\frac{k}{n} - x}{2}\right)\right) = \tag{111}$$

$$\int_x^{\frac{k}{n}} \left[\left(f^{(4)}(t) - f(t)\right) - \left(f^{(4)}(x) - f(x)\right)\right]\left(\frac{\sinh\left(\frac{k}{n} - t\right) - \sin\left(\frac{k}{n} - t\right)}{2}\right) dt$$

$$=: R_4\left(\frac{k}{n}, x\right).$$

As in Theorems 7, 8 we derive

$$H_n^*(f,x) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k) -$$

$$\frac{f'(x)}{2}H_n^*\left(\left(\sinh(\cdot - x) + \sin(\cdot - x)\right), x\right) -$$

$$\frac{f''(x)}{2}H_n^*\left(\left(\cosh(\cdot - x) - \cos(\cdot - x)\right), x\right) -$$

$$\frac{f^{(3)}(x)}{2}H_n^*\left(\left(\sinh(\cdot - x) - \sin(\cdot - x)\right), x\right) -$$

$$\frac{f^{(4)}(x)}{2}H_n^*\left(\left(\sinh^2\left(\frac{\cdot - x}{2}\right) - \sin^2\left(\frac{\cdot - x}{2}\right)\right), x\right) = \Phi_n(x), \tag{112}$$

where

$$\Phi_n(x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k) R_4\left(\frac{k}{n}, x\right). \tag{113}$$

Without loss of generality we can assume that $n > \left\lceil (b-a)^{-\frac{1}{\alpha}}\right\rceil$.

Thus $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^\alpha}$ or $\left|\frac{k}{n} - x\right| > \frac{1}{n^\alpha}$.

In case of $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^\alpha}$, we have the following cases:

i) if $\frac{k}{n} \geq x$, then

$$\left|R_4\left(\frac{k}{n}, x\right)\right| =$$

$$\left|\frac{1}{2}\int_x^{\frac{k}{n}} \left[\left(f^{(4)}(t) - f(t)\right) - \left(f^{(4)}(x) - f(x)\right)\right]\left(\sinh\left(\frac{k}{n} - t\right) - \sin\left(\frac{k}{n} - t\right)\right) dt\right| \leq$$

$$\frac{1}{2}\int_x^{\frac{k}{n}} \left|\left(f^{(4)}(t) - f(t)\right) - \left(f^{(4)}(x) - f(x)\right)\right| \left|\sinh\left(\frac{k}{n} - t\right) - \sin\left(\frac{k}{n} - t\right)\right| dt \leq \tag{114}$$

$$\frac{1}{2}\int_x^{\frac{k}{n}} \omega_1\left(f^{(4)} - f, t - x\right)\left(\left|\sinh\left(\frac{k}{n} - t\right)\right| + \left|\sin\left(\frac{k}{n} - t\right)\right|\right) dt \leq$$

$$\frac{\omega_1\left(f^{(4)} - f, \frac{k}{n} - x\right)}{2}\int_x^{\frac{k}{n}} \left(\cosh(b-a)\left(\frac{k}{n} - t\right) + \left(\frac{k}{n} - t\right)\right) dt =$$

$$\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)} - f, \frac{k}{n} - x\right)}{2}\int_x^{\frac{k}{n}} \left(\frac{k}{n} - t\right) dt =$$

$$\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)} - f, \frac{k}{n} - x\right)}{4}\left(\frac{k}{n} - x\right)^2 \leq$$

$$\frac{\left(\cosh\left(b-a\right)+1\right)\omega_{1}\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}. \tag{115}$$

That is , when $\frac{k}{n} \geq x$, then

$$\left|R_{4}\left(\frac{k}{n},x\right)\right| \leq \frac{\left(\cosh\left(b-a\right)+1\right)\omega_{1}\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}. \tag{116}$$

ii) if $\frac{k}{n} < x$, then

$$\left|R_{4}\left(\frac{k}{n},x\right)\right| =$$

$$\left|\frac{1}{2}\int_{\frac{k}{n}}^{x}\left[\left(f^{(4)}\left(t\right)-f\left(t\right)\right)-\left(f^{(4)}\left(x\right)-f\left(x\right)\right)\right]\left(\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)\right)dt\right| \leq$$

$$\frac{\omega_{1}\left(f^{(4)}-f,x-\frac{k}{n}\right)}{2}\int_{\frac{k}{n}}^{x}\left(\cosh\left(b-a\right)\left(t-\frac{k}{n}\right)+\left(t-\frac{k}{n}\right)\right)dt =$$

$$\frac{\left(\cosh\left(b-a\right)+1\right)\omega_{1}\left(f^{(4)}-f,x-\frac{k}{n}\right)}{2}\int_{\frac{k}{n}}^{x}\left(t-\frac{k}{n}\right)dt =$$

$$\frac{\left(\cosh\left(b-a\right)+1\right)\omega_{1}\left(f^{(4)}-f,x-\frac{k}{n}\right)}{4}\left(x-\frac{k}{n}\right)^{2} \leq \tag{117}$$

$$\leq \frac{\left(\cosh\left(b-a\right)+1\right)\omega_{1}\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}.$$

Consequently, when $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}$, we always obtain that

$$\left|R_{4}\left(\frac{k}{n},x\right)\right| \leq \frac{\left(\cosh\left(b-a\right)+1\right)\omega_{1}\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}. \tag{118}$$

Next assume again $\frac{k}{n} \geq x$, then

$$\left|R_{4}\left(\frac{k}{n},x\right)\right| \leq$$

$$\frac{1}{2}\int_{x}^{\frac{k}{n}}\left|\left(f^{(4)}\left(t\right)-f\left(t\right)\right)-\left(f^{(4)}\left(x\right)-f\left(x\right)\right)\right|\left|\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)\right|dt \leq$$

$$\left\|f^{(4)}-f\right\|_{\infty}\int_{x}^{\frac{k}{n}}\left[\cosh\left(b-a\right)\left(\frac{k}{n}-t\right)+\left(\frac{k}{n}-t\right)\right]dt =$$

$$\left\|f^{(4)}-f\right\|_{\infty}\left(\cosh\left(b-a\right)+1\right)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-t\right)dt =$$

$$\frac{\left\|f^{(4)}-f\right\|_{\infty}\left(\cosh\left(b-a\right)+1\right)}{2}\left(\frac{k}{n}-x\right)^{2} \leq \tag{119}$$

$$\leq \frac{\left\| f^{(4)} - f \right\|_{\infty} \left(\cosh\left(b-a\right) + 1 \right) \left(b-a\right)^2}{2}.$$

Hence

$$\left| R_4 \left(\frac{k}{n}, x \right) \right| \leq \frac{\left\| f^{(4)} - f \right\|_{\infty} \left(\cosh\left(b-a\right) + 1 \right) \left(b-a\right)^2}{2}. \tag{120}$$

When $\frac{k}{n} < x$, we have

$$\left| R_4 \left(\frac{k}{n}, x \right) \right| \leq$$

$$\frac{1}{2} \int_{\frac{k}{n}}^{x} \left| \left(f^{(4)}(t) - f(t) \right) - \left(f^{(4)}(x) - f(x) \right) \right| \left| \sinh\left(\frac{k}{n} - t \right) - \sin\left(\frac{k}{n} - t \right) \right| dt \leq$$

$$\left\| f^{(4)} - f \right\|_{\infty} \int_{\frac{k}{n}}^{x} \left[\cosh\left(b-a\right) \left(t - \frac{k}{n} \right) + \left(t - \frac{k}{n} \right) \right] dt =$$

$$\left\| f^{(4)} - f \right\|_{\infty} \left(\cosh\left(b-a\right) + 1 \right) \int_{\frac{k}{n}}^{x} \left(t - \frac{k}{n} \right) dt =$$

$$\frac{\left\| f^{(4)} - f \right\|_{\infty} \left(\cosh\left(b-a\right) + 1 \right)}{2} \left(x - \frac{k}{n} \right)^2 \leq \tag{121}$$

$$\leq \frac{\left\| f^{(4)} - f \right\|_{\infty} \left(\cosh\left(b-a\right) + 1 \right) \left(b-a\right)^2}{2}.$$

So, it is always true that

$$\left| R_4 \left(\frac{k}{n}, x \right) \right| \leq \frac{\left\| f^{(4)} - f \right\|_{\infty} \left(\cosh\left(b-a\right) + 1 \right) \left(b-a\right)^2}{2}. \tag{122}$$

Thus

$$\left| \Phi_n(x) \right| \leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \left| R_4 \left(\frac{k}{n}, x \right) \right| =$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \left| R_4 \left(\frac{k}{n}, x \right) \right| + \tag{123}$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \left| R_4 \left(\frac{k}{n}, x \right) \right| \leq$$

$$\left(\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\right)\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}+$$

$$\left(\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor} M_{q,\lambda}(nx-k)\right) \tag{124}$$

$$\left(\frac{\left\|f^{(4)}-f\right\|_{\infty}(\cosh(b-a)+1)(b-a)^2}{2}\right)\overset{\text{(by Theorem 3)}}{\leq}$$

$$\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}+ \tag{125}$$

$$\frac{\left\|f^{(4)}-f\right\|_{\infty}(\cosh(b-a)+1)(b-a)^2}{2}Te^{-2\lambda n^{(1-\alpha)}}.$$

We have proved that

$$|\Phi_n(x)|\leq\frac{(\cosh(b-a)+1)}{2}$$

$$\left[\frac{\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f^{(4)}-f\right\|_{\infty}(b-a)^2\,Te^{-2\lambda n^{(1-\alpha)}}\right]. \tag{126}$$

We observe that

$$H_n(f,x)-f(x)-\frac{f'(x)}{2}H_n((\sinh(\cdot-x)+\sin(\cdot-x)),x)$$

$$-\frac{f''(x)}{2}H_n((\cosh(\cdot-x)-\cos(\cdot-x)),x) \tag{127}$$

$$-\frac{f^{(3)}(x)}{2}H_n((\sinh(\cdot-x)-\sin(\cdot-x)),x)$$

$$-f^{(4)}(x)\,H_n\left(\left(\sinh^2\left(\frac{\cdot-x}{2}\right)-\sin^2\left(\frac{\cdot-x}{2}\right)\right),x\right)=$$

$$\left[H_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)-\frac{f'(x)}{2}H_n^*((\sinh(\cdot-x)+\sin(\cdot-x)),x)\right.$$

$$-\frac{f''(x)}{2}H_n^*((\cosh(\cdot-x)-\cos(\cdot-x)),x) \tag{128}$$

$$-\frac{f^{(3)}(x)}{2}H_n^*\left(\left(\sinh(\cdot-x)-\sin(\cdot-x)\right),x\right)$$

$$\left.-f^{(4)}(x)\,H_n^*\left(\left(\sinh^2\left(\frac{\cdot-x}{2}\right)-\sin^2\left(\frac{\cdot-x}{2}\right)\right),x\right)\right]\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)}$$

$$=\frac{\Phi_n(x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)}.$$

Finally, we obtain ($\forall\ x\in[a,b]$, $n\in\mathbb{N}$):

$$\left|H_n(f,x)-f(x)-\frac{f'(x)}{2}H_n\left(\left(\sinh(\cdot-x)+\sin(\cdot-x)\right),x\right)\right.$$

$$-\frac{f''(x)}{2}H_n\left(\left(\cosh(\cdot-x)-\cos(\cdot-x)\right),x\right) \tag{129}$$

$$-\frac{f^{(3)}(x)}{2}H_n\left(\left(\sinh(\cdot-x)-\sin(\cdot-x)\right),x\right)$$

$$\left.-f^{(4)}(x)\,H_n\left(\left(\sinh^2\left(\frac{\cdot-x}{2}\right)-\sin^2\left(\frac{\cdot-x}{2}\right)\right),x\right)\right|=$$

$$\frac{|\Phi_n(x)|}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}M_{q,\lambda}(nx-k)}\leq\Delta(q)\,|\Phi_n(x)|\overset{(\text{by }(126))}{\leq}\frac{\Delta(q)\left(\cosh(b-a)+1\right)}{2}$$

$$\left[\frac{\omega_1\left(f^{(4)}-f,\frac{1}{n^\alpha}\right)}{2n^{2\alpha}}+\left\|f^{(4)}-f\right\|_\infty(b-a)^2\,Te^{-2\lambda n^{(1-\alpha)}}\right]. \tag{130}$$

The theorem is proved. ■

We continue with a general trigonometric result.

Theorem 10 *Let $f\in C^4([a,b],\mathbb{C})$, $0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in[a,b]$. Let also $\overline{\alpha},\overline{\beta}\in\mathbb{R}$ with $\overline{\alpha}\overline{\beta}\left(\overline{\alpha}^2-\overline{\beta}^2\right)\neq0$. Then*

1)

$$\left|H_n(f,x)-f(x)-\frac{f'(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)}H_n\left(\left(\overline{\beta}^3\sin\left(\overline{\alpha}(\cdot-x)\right)-\overline{\alpha}^3\sin\left(\overline{\beta}(\cdot-x)\right)\right),x\right)\right.$$

$$-\frac{f''(x)}{\left(\overline{\beta}^2-\overline{\alpha}^2\right)}H_n\left(\left(\cos\left(\overline{\alpha}(\cdot-x)\right)-\cos\left(\overline{\beta}(\cdot-x)\right)\right),x\right)$$

$$-\frac{f'''(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)}H_n\left(\left(\overline{\beta}\sin\left(\overline{\alpha}(\cdot-x)\right)-\overline{\alpha}\sin\left(\overline{\beta}(\cdot-x)\right)\right),x\right)$$

$$-\left(\frac{2f^{(4)}(x)+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''(x)}{\left(\overline{\alpha}\overline{\beta}\right)^2\left(\overline{\beta}^2-\overline{\alpha}^2\right)}\right)$$

$$H_n\left(\left(\overline{\beta}^2\sin^2\left(\frac{\overline{\alpha}(\cdot-x)}{2}\right)-\overline{\alpha}^2\sin^2\left(\frac{\overline{\beta}(\cdot-x)}{2}\right)\right),x\right)\Bigg|\leq \tag{131}$$

$$\frac{\Delta(q)}{\left|\overline{\beta}^2-\overline{\alpha}^2\right|}\left[\frac{\omega_1\left(\left(f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2f\right),\frac{1}{n^{\alpha}}\right)}{n^{2\alpha}}+\right.$$

$$\left.2\left\|f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2f\right\|_{\infty}(b-a)^2Te^{-2\lambda n^{(1-\alpha)}}\right],$$

2) if $f^{(i)}(x)=0$, $i=1,2,3,4$, *we get*

$$|H_n(f,x)-f(x)|\leq\frac{\Delta(q)}{\left|\overline{\beta}^2-\overline{\alpha}^2\right|}$$

$$\left[\frac{\omega_1\left(\left(f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2f\right),\frac{1}{n^{\alpha}}\right)}{n^{2\alpha}}+\right. \tag{132}$$

$$\left.2\left\|f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2f\right\|_{\infty}(b-a)^2Te^{-2\lambda n^{(1-\alpha)}}\right].$$

The high speed of convergence in (1) and (2) is $n^{-3\alpha}$.

Proof. As similar to Theorem 9 is omitted. It is based on Theorem 9 of [11]. ■

We finish with a general hyperbolic result.

Theorem 11 *Let* $f\in C^4([a,b],\mathbb{C})$, $0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in[a,b]$. *Let also* $\overline{\alpha},\overline{\beta}\in\mathbb{R}$ *with* $\overline{\alpha}\overline{\beta}\left(\overline{\alpha}^2-\overline{\beta}^2\right)\neq0$. *Then*

1)

$$\Bigg|H_n(f,x)-f(x)-\frac{f'(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)}H_n\left(\left(\overline{\beta}^3\sinh(\overline{\alpha}(\cdot-x))-\overline{\alpha}^3\sinh\left(\overline{\beta}(\cdot-x)\right)\right),x\right)$$

$$+\frac{f''(x)}{\overline{\beta}^2-\overline{\alpha}^2}H_n\left(\left(\cosh\left(\overline{\beta}(\cdot-x)\right)-\cosh(\overline{\alpha}(\cdot-x))\right),x\right)$$

$$+\frac{f'''(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)}H_n\left(\left(\overline{\alpha}\sinh\left(\overline{\beta}(\cdot-x)\right)-\overline{\beta}\sinh(\overline{\alpha}(\cdot-x))\right),x\right)$$

$$+\left(\frac{2(f^{(4)}(x)-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''(x))}{\left(\overline{\alpha}\overline{\beta}\right)^2\left(\overline{\beta}^2-\overline{\alpha}^2\right)}\right)$$

$$H_n\left(\left(\overline{\alpha}^2\sinh^2\left(\frac{\overline{\beta}(\cdot-x)}{2}\right)-\overline{\beta}^2\sinh^2\left(\frac{\overline{\alpha}(\cdot-x)}{2}\right)\right),x\right)\Bigg|\leq \tag{133}$$

$$\frac{\Delta(q)\cosh(b-a)}{\left|\overline{\beta}^2-\overline{\alpha}^2\right|}\left[\frac{\omega_1\left(\left(f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right),\frac{1}{n^{\alpha}}\right)}{n^{2\alpha}}+\right.$$

$$\left.2\left\|f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right\|_{\infty}(b-a)^2\,Te^{-2\lambda n^{(1-\alpha)}}\right],$$

2) if $f^{(i)}(x)=0,\ i=1,2,3,4,$ *we get*

$$|H_n(f,x)-f(x)|\leq\frac{\Delta(q)\cosh(b-a)}{\left|\overline{\beta}^2-\overline{\alpha}^2\right|} \tag{134}$$

$$\left[\frac{\omega_1\left(\left(f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right),\frac{1}{n^{\alpha}}\right)}{n^{2\alpha}}+\right.$$

$$\left.2\left\|f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right\|_{\infty}(b-a)^2\,Te^{-2\lambda n^{(1-\alpha)}}\right].$$

The high speed of convergence in (1) and (2) is $n^{-3\alpha}$.

Proof. As similar to Theorem 9 is omitted. It is based on Theorem 10 of [11]. ■

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