SPECTRAL RADIUS BOUNDS FOR THE NUMERICAL RADIUS OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. In this paper we show among others that, if T and V are operators in $\mathcal{B}(H)$ such that $|T|V=V^*|T|$, then for A, $B\in\mathcal{B}(H)$

$$\|BTVA\|\le r(V)\|f(|T|)A\|\|Bg(|T^*|)\|,$$

$$\omega(BTVA)\le\frac{1}{2}r(V)\left\||f(|T|)A|^2+|g(|T^*|)B^*|^2\right\|$$

and

$$\omega^2(BTVA)$$
$$\le\frac{1}{2}r^2(V)\left[\|f(|T|)A\|^2\|g(|T^*|)B^*\|^2+\left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|\right].$$

Some applications for the *generalized Aluthge transform* of an operator are also given.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T)=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}.\tag{1.1}$$

Obviously, by (1.1), for any $x\in H$ one has

$$|\langle Tx,x\rangle|\le w(T)\|x\|^2.\tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T:H\to H$, i.e.,

(i) $\omega(T)\ge0$ for any $T\in B(H)$ and $\omega(T)=0$ if and only if $T=0$;
(ii) $\omega(\lambda T)=|\lambda|\omega(T)$ for any $\lambda\in\mathbb{C}$ and $T\in B(H)$;
(iii) $\omega(T+V)\le\omega(T)+\omega(V)$ for any $T,V\in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T)\le\|T\|\le2\omega(T)\tag{1.3}$$

for any $T\in B(H)$.

F. Kittaneh, in 2003 [10], showed that for any operator $T\in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T)\le\frac{1}{2}\left(\|T\|+\left\|T^2\right\|^{1/2}\right).\tag{1.4}$$

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Utilizing the Cartesian decomposition for operators, F. Kittaneh in [11] improved the inequality (1.3) as follows:

$$\frac{1}{4}\left\|T^{*}T+TT^{*}\right\|\leq\omega^{2}\left(T\right)\leq\frac{1}{2}\left\|T^{*}T+TT^{*}\right\| \tag{1.5}$$

for any operator $T\in B\left(H\right).$

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [7]:

If for an operator $T\in B\left(H\right)$ we denote $\left|T\right|:=\left(T^{*}T\right)^{1/2}$, then

$$\omega^{r}\left(T\right)\leq\frac{1}{2}\left\|\left|T\right|^{2\alpha r}+\left|T^{*}\right|^{2(1-\alpha)r}\right\| \tag{1.6}$$

and

$$\omega^{2r}\left(T\right)\leq\left\|\alpha\left|T\right|^{2r}+\left(1-\alpha\right)\left|T^{*}\right|^{2r}\right\|, \tag{1.7}$$

where $\alpha\in(0,1)$ and $r\geq1$.

If we take $\alpha=\frac{1}{2}$ and $r=1$ we get from (1.6) that

$$\omega\left(T\right)\leq\frac{1}{2}\left\|\left|T\right|+\left|T^{*}\right|\right\| \tag{1.8}$$

and from (1.7) that

$$\omega^{2}\left(T\right)\leq\frac{1}{2}\left\|\left|T\right|^{2}+\left|T^{*}\right|^{2}\right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [6] and [4].

Let $T=U\left|T\right|$ be the *polar decomposition* of the bounded linear operator T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T}:=\left|T\right|^{1/2}U\left|T\right|^{1/2}$, see [1].

The following properties of $\widetilde{T}$ are as follows:

(i) $\left\|\widetilde{T}\right\|\leq\left\|T\right\|,$

(ii) $w\left(\widetilde{T}\right)\leq\omega\left(T\right),$

(iii) $r\left(\widetilde{T}\right)=\omega\left(T\right),$

(iv) $\omega\left(\widetilde{T}\right)\leq\left\|T^{2}\right\|^{1/2}\left(\leq\left\|T\right\|\right)$, [12].

Utilizing this transform T. Yamazaki, [12] obtained in 2007 the following refinement of Kittaneh's inequality (1.4):

$$\omega\left(T\right)\leq\frac{1}{2}\left(\left\|T\right\|+\omega\left(\widetilde{T}\right)\right)\leq\frac{1}{2}\left(\left\|T\right\|+\left\|T^{2}\right\|^{1/2}\right) \tag{1.10}$$

for any operator $T\in B\left(H\right).$

We remark that if $\widetilde{T}=0$, then obviously $w\left(T\right)=\frac{1}{2}\left\|T\right\|.$

Abu-Omar and Kittaneh [2] improved on inequality (1.10) using generalized Aluthge transform to prove that

$$\omega\left(T\right)\leq\frac{1}{2}\left(\left\|T\right\|+\min_{t\in[0,1]}\omega\left(\Delta_{t}\left(T\right)\right)\right).$$

For $t=1$ this also gives the following result for the *Dougal transform*

$$\omega\left(T\right)\leq\frac{1}{2}\left(\left\|T\right\|+\omega\left(\widehat{T}\right)\right). \tag{1.11}$$

In [3] Bunia et al. also proved that

$$\omega(T) \leq \min_{t\in[0,1]} \left\{ \frac{1}{2}\omega(\Delta_t(T)) + \frac{1}{4}\left(\|T\|^{2t} + \|T\|^{2(1-t)} \right) \right\},$$

which for $t = 1/2$ gives (1.10) as well.

In 1988, F. Kittaneh obtained the following generalization of Schwarz inequality [9]:

Theorem 1. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Let T, V be operators in $V(H)$ such that $|T|V = V^*|T|$, then*

$$|\langle TVx, y\rangle| \leq r(V)\|f(|T|)x\|\|g(|T^*|)y\| \tag{1.12}$$

for all x, $y \in H$, where $r(V)$ denotes the spectral radius of V.

If we take $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$ for $\alpha \in [0,1]$ and $t > 0$,

$$|\langle TVx, y\rangle| \leq r(V)\||T|^{\alpha}x\|\left\||T^*|^{1-\alpha}y\right\| \tag{1.13}$$

for all x, $y \in H$.

Motivated by the above results, in this paper we provide some upper bounds for the numerical radius $\omega(BTVA)$ in term of the functions f, g, the operators T, V from Theorem 1 and A, $B \in \mathcal{B}(H)$. Among others, we show that, if $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$, then

$$\omega^{2r}(BTVA) \leq r^{2r}(V)\left\| \frac{1}{p}|f(|T|)A|^{2pr} + \frac{1}{q}|g(|T^*|)B^*|^{2qr} \right\|. \tag{1.14}$$

If $r \geq 1$, then also

$$\begin{aligned} \omega^{2r}(BTVA) \leq \frac{1}{2}r^{2r}(V)\Big(&\|f(|T|)A\|^{2r}\|g(|T^*|)B^*\|^{2r} \\ &+ \left\| |g(|T^*|)B^*|^2 |f(|T|)A|^2 \right\|^r \Big). \end{aligned} \tag{1.15}$$

Some applications for the *generalized Aluthge transform* of an operator are also given.

2. Inequalities Via Buzano's Result

We have:

Theorem 2. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Let T, V be operators in $\mathcal{B}(H)$ such that $|T|V = V^*|T|$, then for A, $B \in \mathcal{B}(H)$*

$$\|BTVA\| \leq r(V)\|f(|T|)A\|\|Bg(|T^*|)\| \tag{2.1}$$

and

$$\omega(BTVA) \leq \frac{1}{2}r(V)\left\| |f(|T|)A|^2 + |g(|T^*|)B^*|^2 \right\|. \tag{2.2}$$

Also,

$$\begin{aligned} &\omega^2(BTVA) \\ &\leq \frac{1}{2}r^2(V)\left[\|f(|T|)A\|^2\|g(|T^*|)B^*\|^2 + \left\| |g(|T^*|)B^*|^2 |f(|T|)A|^2 \right\| \right]. \end{aligned} \tag{2.3}$$

Proof. Observe that by (1.12) we have

$$\begin{aligned}|\langle TVx,y\rangle|^2 &\leq r^2(V)\|f(|T|)x\|^2\|g(|T^*|)y\|^2\\ &= r^2(V)\langle f(|T|)x,f(|T|)x\rangle\langle g(|T^*|)y,g(|T^*|)y\rangle\\ &= r^2(V)\left\langle f^2(|T|)x,x\right\rangle\left\langle g^2(|T^*|)y,y\right\rangle\end{aligned}$$

for all x, $y\in H$.

If we take Ax instead of x and B^*y instead of y, then we get

$$\begin{aligned}|\langle TVAx,B^*y\rangle|^2 &\leq r^2(V)\left\langle f^2(|T|)Ax,Ax\right\rangle\left\langle g^2(|T^*|)B^*y,B^*y\right\rangle\\ &= r^2(V)\left\langle A^*f^2(|T|)Ax,x\right\rangle\left\langle Bg^2(|T^*|)B^*y,y\right\rangle\\ &= r^2(V)\left\langle (f(|T|)Ax)^*f(|T|)Ax,x\right\rangle\\ &\quad\times\left\langle (g(|T^*|)B^*)^*g(|T^*|)B^*y,y\right\rangle\\ &= r^2(V)\left\langle |f(|T|)A|^2x,x\right\rangle\left\langle |g(|T^*|)B^*|^2y,y\right\rangle,\end{aligned}$$

namely

$$|\langle BTVAx,y\rangle|^2\leq r^2(V)\left\langle |f(|T|)A|^2x,x\right\rangle\left\langle |g(|T^*|)B^*|^2y,y\right\rangle \tag{2.4}$$

for all x, $y\in H$.

Therefore

$$\begin{aligned}\|BTVA\|^2 &= \sup_{\|x\|=\|y\|=1}|\langle BTAx,y\rangle|^2\\ &\leq r^2(V)\sup_{\|x\|=\|y\|=1}\left[\left\langle |f(|T|)A|^2x,x\right\rangle\left\langle |g(|T^*|)B^*|^2y,y\right\rangle\right]\\ &= r^2(V)\sup_{\|x\|=1}\left\langle |f(|T|)A|^2x,x\right\rangle\sup_{\|y\|=1}\left\langle |g(|T^*|)B^*|^2y,y\right\rangle\\ &= r^2(V)\left\||f(|T|)A|^2\right\|\left\||g(|T^*|)B^*|^2\right\|\\ &= r^2(V)\|f(|T|)A\|^2\|g(|T^*|)B^*\|^2\\ &= r^2(V)\|f(|T|)A\|^2\|Bg(|T^*|)\|^2,\end{aligned}$$

which is equivalent to (2.1).

From (2.4) we also have

$$|\langle BTVAx,x\rangle|^2\leq r^2(V)\left\langle |f(|T|)A|^2x,x\right\rangle\left\langle |g(|T^*|)B^*|^2x,x\right\rangle \tag{2.5}$$

for all $x\in H$.

By the *A-G inequality*, we also have

$$\begin{aligned}|\langle BTVAx,x\rangle| &\leq r(V)\left\langle |f(|T|)A|^2x,x\right\rangle^{1/2}\left\langle |g(|T^*|)B^*|^2x,x\right\rangle^{1/2}\\ &\leq \frac{1}{2}r(V)\left[\left\langle |f(|T|)A|^2x,x\right\rangle+\left\langle |g(|T^*|)B^*|^2x,x\right\rangle\right]\\ &= r(V)\left\langle\left(\frac{|f(|T|)A|^2+|g(|T^*|)B^*|^2}{2}\right)x,x\right\rangle\end{aligned}$$

for all $x\in H$.

By taking the supremum, we get

$$\begin{aligned}\omega\left(BTVA\right) &= \sup_{\|x\|=1}\left|\left\langle BTAx,x\right\rangle\right| \\ &\le r\left(V\right)\sup_{\|x\|=1}\left\langle\left(\frac{\left|f\left(|T|\right)A\right|^{2}+\left|g\left(|T^{*}|\right)B^{*}\right|^{2}}{2}\right)x,x\right\rangle \\ &= r\left(V\right)\left\|\frac{\left|f\left(|T|\right)A\right|^{2}+\left|g\left(|T^{*}|\right)B^{*}\right|^{2}}{2}\right\|,\end{aligned}$$

which proves (2.2).

Let $x\in H$, $\|x\|=1$, then by Buzano's inequality, we recall that

$$\frac{1}{2}\left[\|u\|\,\|v\|+|\langle u,v\rangle|\right]\ge\left|\langle u,e\rangle\,\langle e,v\rangle\right|$$

holds for any u, v, $e\in H$ with $\|e\|=1$, we derive

$$\begin{aligned}&\left\langle\left|f\left(|T|\right)A\right|^{2}x,x\right\rangle\left\langle x,\left|g\left(|T^{*}|\right)B^{*}\right|^{2}x\right\rangle \\ &\le\frac{1}{2}\left[\left\|\left|f\left(|T|\right)A\right|^{2}x\right\|\left\|\left|g\left(|T^{*}|\right)B^{*}\right|^{2}x\right\|+\left\langle\left|f\left(|T|\right)A\right|^{2}x,\left|g\left(|T^{*}|\right)B^{*}\right|^{2}x\right\rangle\right] \\ &=\frac{1}{2}\left[\left\|\left|f\left(|T|\right)A\right|^{2}x\right\|\left\|\left|g\left(|T^{*}|\right)B^{*}\right|^{2}x\right\|+\left\langle\left|g\left(|T^{*}|\right)B^{*}\right|^{2}\left|f\left(|T|\right)A\right|^{2}x,x\right\rangle\right].\end{aligned}$$

By making use of (2.5) we get

$$\begin{aligned}&\left|\left\langle BTVAx,x\right\rangle\right|^{2} \\ &\le\frac{1}{2}r^{2}\left(V\right) \\ &\times\left[\left\|\left|f\left(|T|\right)A\right|^{2}x\right\|\left\|\left|g\left(|T^{*}|\right)B^{*}\right|^{2}x\right\|+\left\langle\left|g\left(|T^{*}|\right)B^{*}\right|^{2}\left|f\left(|T|\right)A\right|^{2}x,x\right\rangle\right]\end{aligned}\tag{2.6}$$

for $x\in H$, $\|x\|=1$.

If we take the supremum, then we obtain

$$\begin{aligned}&\omega^{2}\left(BTVA\right) \\ &=\sup_{\|x\|=1}\left|\left\langle BTAx,x\right\rangle\right|^{2} \\ &\le\frac{1}{2}r^{2}\left(V\right)\sup_{\|x\|=1}\left[\left\|\left|f\left(|T|\right)A\right|^{2}x\right\|\left\|\left|g\left(|T^{*}|\right)B^{*}\right|^{2}x\right\|\right. \\ &\left.+\left\langle\left|g\left(|T^{*}|\right)B^{*}\right|^{2}\left|f\left(|T|\right)A\right|^{2}x,x\right\rangle\right] \\ &\le\frac{1}{2}r^{2}\left(V\right)\left[\sup_{\|x\|=1}\left\|\left|f\left(|T|\right)A\right|^{2}x\right\|\sup_{\|x\|=1}\left\|\left|g\left(|T^{*}|\right)B^{*}\right|^{2}x\right\|\right. \\ &\left.+\sup_{\|x\|=1}\left\langle\left|g\left(|T^{*}|\right)B^{*}\right|^{2}\left|f\left(|T|\right)A\right|^{2}x,x\right\rangle\right] \\ &=\frac{1}{2}r^{2}\left(V\right) \\ &\times\left[\left\|f\left(|T|\right)A\right\|^{2}\left\|g\left(|T^{*}|\right)B^{*}\right\|^{2}+\left\|\left|g\left(|T^{*}|\right)B^{*}\right|^{2}\left|f\left(|T|\right)A\right|^{2}\right\|\right],\end{aligned}$$

which proves (2.3). □

We observe that if we take $B=A$ in Theorem 2, then we get

$$\|ATVA\| \leq r(V)\,\|f(|T|)A\|\,\|Ag(|T^*|)\|$$

and

$$\omega(ATVA) \leq \frac{1}{2}r(V)\left\||f(|T|)A|^2 + |g(|T^*|)A^*|^2\right\|. \tag{2.7}$$

Also,

$$\omega^2(ATVA) \leq \frac{1}{2}r^2(V)\left[\|f(|T|)A\|^2\,\|g(|T^*|)A^*\|^2 + \left\||g(|T^*|)A^*|^2\,|f(|T|)A|^2\right\|\right]. \tag{2.8}$$

Further, if we choose $A=I$ in (2.7) and (2.8), then we get

$$\omega(TV) \leq \frac{1}{2}r(V)\left\||f(|T|)|^2 + |g(|T^*|)|^2\right\|$$

and

$$\omega^2(TV) \leq \frac{1}{2}r^2(V)\left[\|f(|T|)\|^2\,\|g(|T^*|)\|^2 + \left\||g(|T^*|)|^2\,|f(|T|)|^2\right\|\right].$$

Remark 1. *If we take $f(t)=t^\lambda$, $g(t)=t^{1-\lambda}$ with $\lambda\in[0,1]$ in Theorem 2, then we get*

$$\|BTVA\| \leq r(V)\left\||T|^\lambda A\right\|\left\|B|T^*|^{1-\lambda}\right\|$$

and

$$\omega(BTVA) \leq \frac{1}{2}r(V)\left\|\left||T|^\lambda A\right|^2 + \left||T^*|^{1-\lambda}B^*\right|^2\right\|. \tag{2.9}$$

Also,

$$\omega^2(BTVA) \leq \frac{1}{2}r^2(V)\left[\left\||T|^\lambda A\right\|^2\left\||T^*|^{1-\lambda}B^*\right\|^2 + \left\|\left||T^*|^{1-\lambda}B^*\right|^2\left||T|^\lambda A\right|^2\right\|\right]. \tag{2.10}$$

Moreover, if we take $\lambda=1/2$, then we get

$$\|BTVA\| \leq r(V)\left\||T|^{1/2}A\right\|\left\|B|T^*|^{1/2}\right\|$$

and

$$\omega(BTVA) \leq \frac{1}{2}r(V)\left\|\left||T|^{1/2}A\right|^2 + \left||T^*|^{1/2}B^*\right|^2\right\|.$$

Also,

$$\omega^2(BTVA) \leq \frac{1}{2}r^2(V)\left[\left\||T|^{1/2}A\right\|^2\left\||T^*|^{1/2}B^*\right\|^2 + \left\|\left||T^*|^{1/2}B^*\right|^2\left||T|^{1/2}A\right|^2\right\|\right].$$

If we put $B=A$ in Remark 1, then we get

$$\|ATVA\| \leq r(V)\left\||T|^\lambda A\right\|\left\|A|T^*|^{1-\lambda}\right\|$$

and

$$\omega(ATVA) \leq \frac{1}{2}r(V)\left\|\left||T|^\lambda A\right|^2 + \left||T^*|^{1-\lambda}A^*\right|^2\right\|. \tag{2.11}$$

Also,

$$\omega^2(BTVA) \leq \frac{1}{2}r^2(V)\left[\left\||T|^\lambda A\right\|^2\left\||T^*|^{1-\lambda}A^*\right\|^2+\left\|\left||T^*|^{1-\lambda}A^*\right|^2\left||T|^\lambda A\right|^2\right\|\right]. \tag{2.12}$$

Moreover, the choice $A = I$ in (2.11) and (2.12) gives

$$\omega(TV) \leq \frac{1}{2}r(V)\left\||T|^{2\lambda}+|T^*|^{2(1-\lambda)}\right\|$$

and

$$\omega^2(TV) \leq \frac{1}{2}r^2(V)\left[\|T\|^2+\left\||T^*|^{2(1-\lambda)}|T|^{2\lambda}\right\|\right]$$

for $\lambda \in [0,1]$.

Corollary 1. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Let A, B, $X \in \mathcal{B}(H)$, then for all $\alpha \in [0,1]$,*

$$\|BXA\| \leq \|X\|^\alpha\left\|f\left(|X|^{1-\alpha}\right)A\right\|\left\|Bg\left(|X^*|^{1-\alpha}\right)\right\| \tag{2.13}$$

and

$$\omega(BXA) \leq \frac{1}{2}\|X\|^\alpha\left\|\left|f\left(|X|^{1-\alpha}\right)A\right|^2+\left|g\left(|X^*|^{1-\alpha}\right)B^*\right|^2\right\|. \tag{2.14}$$

Also,

$$\omega^2(BXA) \leq \frac{1}{2}\|X\|^{2\alpha}\left[\left\|f\left(|X|^{1-\alpha}\right)A\right\|^2\left\|g\left(|X^*|^{1-\alpha}\right)B^*\right\|^2 + \left\|\left|g\left(|X^*|^{1-\alpha}\right)B^*\right|^2\left|f\left(|X|^{1-\alpha}\right)A\right|^2\right\|\right]. \tag{2.15}$$

Proof. Let $X = U|X|$ be the *polar decomposition* of the bounded linear operator X, with U a partial isometry. If we take $T = U|X|^{1-\alpha}$ and $V = |X|^\alpha$, then we have

$$TV = U|X| = X,\quad |T| = |X|^{1-\alpha} \text{ and } |T^*| = |X^*|^{1-\alpha}$$

and since

$$|T|V = |X| = V^*|T|$$

and

$$r(|X|^\alpha) = \||X|^\alpha\| = \|X\|^\alpha,$$

hence by Theorem 2 we derive the desired inequalities (2.13)-(2.15). □

If we take $B = A$ in Corollary 1, then we get

$$\|AXA\| \leq \|X\|^\alpha\left\|f\left(|X|^{1-\alpha}\right)A\right\|\left\|Ag\left(|X^*|^{1-\alpha}\right)\right\|$$

and

$$\omega(AXA) \leq \frac{1}{2}\|X\|^\alpha\left\|\left|f\left(|X|^{1-\alpha}\right)A\right|^2+\left|g\left(|X^*|^{1-\alpha}\right)A^*\right|^2\right\|. \tag{2.16}$$

Also,

$$\omega^2(AXA) \leq \frac{1}{2}\|X\|^{2\alpha}\left[\left\|f\left(|X|^{1-\alpha}\right)A\right\|^2\left\|g\left(|X^*|^{1-\alpha}\right)A^*\right\|^2 + \left\|\left|g\left(|X^*|^{1-\alpha}\right)A^*\right|^2\left|f\left(|X|^{1-\alpha}\right)A\right|^2\right\|\right]. \tag{2.17}$$

Moreover, by taking $A = I$ in (2.16) and (2.17) we obtain

$$\omega(X) \leq \frac{1}{2}\|X\|^{\alpha}\left\|\left|f\left(|X|^{1-\alpha}\right)\right|^2 + \left|g\left(|X^*|^{1-\alpha}\right)\right|^2\right\|$$

and

$$\omega^2(X) \leq \frac{1}{2}\|X\|^{2\alpha}\left[\left\|f\left(|X|^{1-\alpha}\right)\right\|^2\left\|g\left(|X^*|^{1-\alpha}\right)\right\|^2 + \left\|\left|g\left(|X^*|^{1-\alpha}\right)\right|^2\left|f\left(|X|^{1-\alpha}\right)\right|^2\right\|\right]$$

for $\alpha \in [0,1]$.

Remark 2. *If we take $f(t) = t^{\lambda}$, $g(t) = t^{1-\lambda}$ with $\lambda \in [0,1]$ in Corollary 1, then we get*

$$\|BXA\| \leq \|X\|^{\alpha}\left\||X|^{\lambda(1-\alpha)}A\right\|\left\|B|X^*|^{(1-\lambda)(1-\alpha)}\right\| \tag{2.18}$$

and

$$\omega(BXA) \leq \frac{1}{2}\|X\|^{\alpha}\left\|\left||X|^{\lambda(1-\alpha)}A\right|^2 + \left||X^*|^{(1-\lambda)(1-\alpha)}B^*\right|^2\right\|. \tag{2.19}$$

Also,

$$\omega^2(BXA) \leq \frac{1}{2}\|X\|^{2\alpha}\left[\left\||X|^{\lambda(1-\alpha)}A\right\|^2\left\||X^*|^{(1-\lambda)(1-\alpha)}B^*\right\|^2 + \left\|\left||X^*|^{(1-\lambda)(1-\alpha)}B^*\right|^2\left||X|^{\lambda(1-\alpha)}A\right|^2\right\|\right]. \tag{2.20}$$

For $\lambda = 1/2$ we get

$$\|BXA\| \leq \|X\|^{\alpha}\left\||X|^{(1-\alpha)/2}A\right\|\left\|B|X^*|^{(1-\alpha)/2}\right\| \tag{2.21}$$

and

$$\omega(BXA) \leq \frac{1}{2}\|X\|^{\alpha}\left\|\left||X|^{(1-\alpha)/2}A\right|^2 + \left||X^*|^{(1-\alpha)/2}B^*\right|^2\right\|. \tag{2.22}$$

Also,

$$\omega^2(BXA) \leq \frac{1}{2}\|X\|^{2\alpha}\left[\left\||X|^{(1-\alpha)/2}A\right\|^2\left\||X^*|^{(1-\alpha)/2}B^*\right\|^2 + \left\|\left||X^*|^{(1-\alpha)/2}B^*\right|^2\left||X|^{(1-\alpha)/2}A\right|^2\right\|\right]. \tag{2.23}$$

Moreover, if we take $\alpha = 1/2$, then we get

$$\|BXA\| \leq \|X\|^{1/2}\left\||X|^{1/4}A\right\|\left\|B|X^*|^{1/4}\right\| \tag{2.24}$$

and

$$\omega(BXA) \le \frac{1}{2}\|X\|^{1/2}\left\|\left||X|^{1/4}A\right|^2+\left||X^*|^{1/4}B^*\right|^2\right\|. \tag{2.25}$$

Also,

$$\omega^2(BXA) \le \frac{1}{2}\|X\|\left[\left\||X|^{1/4}A\right\|^2\left\||X^*|^{1/4}B^*\right\|^2+\left\|\left||X^*|^{1/4}B^*\right|^2\left||X|^{1/4}A\right|^2\right\|\right]. \tag{2.26}$$

If we take $B=A$ in Remark 2 then we get

$$\|AXA\| \le \|X\|^{\alpha}\left\||X|^{\lambda(1-\alpha)}A\right\|\left\|A|X^*|^{(1-\lambda)(1-\alpha)}\right\|$$

and

$$\omega(AXA) \le \frac{1}{2}\|X\|^{\alpha}\left\|\left||X|^{\lambda(1-\alpha)}A\right|^2+\left||X^*|^{(1-\lambda)(1-\alpha)}A^*\right|^2\right\|. \tag{2.27}$$

Also,

$$\omega^2(AXA) \le \frac{1}{2}\|X\|^{2\alpha}\left[\left\||X|^{\lambda(1-\alpha)}A\right\|^2\left\||X^*|^{(1-\lambda)(1-\alpha)}A^*\right\|^2 + \left\|\left||X^*|^{(1-\lambda)(1-\alpha)}A^*\right|^2\left||X|^{\lambda(1-\alpha)}A\right|^2\right\|\right]. \tag{2.28}$$

Moreover, if we take $A=I$, then we obtain for $\alpha,\ \lambda\in[0,1]$

$$\omega(X) \le \frac{1}{2}\|X\|^{\alpha}\left\||X|^{2\lambda(1-\alpha)}+|X^*|^{2(1-\lambda)(1-\alpha)}\right\|$$

and

$$\omega^2(X) \le \frac{1}{2}\|X\|^{2\alpha}\left[\|X\|^{2(1-\alpha)}+\left\||X^*|^{2(1-\lambda)(1-\alpha)}|X|^{2\lambda(1-\alpha)}\right\|\right].$$

3. Some Inequalities Via Young's Result

We also have:

Theorem 3. *Assume that the conditions of Theorem 2 are satisfied. If $r>0$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\ge 1$, then*

$$\omega^{2r}(BTVA) \le r^{2r}(V)\left\|\frac{1}{p}|f(|T|)A|^{2pr}+\frac{1}{q}|g(|T^*|)B^*|^{2qr}\right\|. \tag{3.1}$$

If $r\ge 1$, then

$$\omega^{2r}(BTVA) \le \frac{1}{2}r^{2r}(V)\left(\|f(|T|)A\|^{2r}\|g(|T^*|)B^*\|^{2r}+\left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|^r\right). \tag{3.2}$$

If $r\ge 1$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\ge 2$, then also

$$\omega^{2r}(BTVA) \le \frac{1}{2}r^{2r}(V)\left(\left\|\frac{1}{p}|f(|T|)A|^{2pr}+\frac{1}{q}|g(|T^*|)B^*|^{2qr}\right\| + \left\||g(|T^*|)B^*|^2|f(|T|)A|^2\right\|^r\right). \tag{3.3}$$

Proof. If we take the power $r>0$ in (2.5), then we get, by Young and McCarthy inequalities that

$$\begin{aligned}
&\left|\langle BTVAx,x\rangle\right|^{2r}\\
&\leq r^{2r}\left(V\right)\left\langle \left|f\left(|T|\right)A\right|^{2}x,x\right\rangle^{r}\left\langle \left|g\left(|T^{*}|\right)B^{*}\right|^{2}x,x\right\rangle^{r}\\
&\leq r^{2r}\left(V\right)\left[\frac{1}{p}\left\langle \left|f\left(|T|\right)A\right|^{2}x,x\right\rangle^{pr}+\frac{1}{q}\left\langle \left|g\left(|T^{*}|\right)B^{*}\right|^{2}x,x\right\rangle^{qr}\right]\\
&\leq r^{2r}\left(V\right)\left[\frac{1}{p}\left\langle \left|f\left(|T|\right)A\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle \left|g\left(|T^{*}|\right)B^{*}\right|^{2qr}x,x\right\rangle\right]\\
&=r^{2r}\left(V\right)\left\langle \left[\frac{1}{p}\left|f\left(|T|\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(|T^{*}|\right)B^{*}\right|^{2qr}\right]x,x\right\rangle
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, then we get that

$$\begin{aligned}
\omega^{2r}\left(BTVA\right)&=\sup_{\|x\|=1}\left|\langle BTVAx,x\rangle\right|^{2r}\\
&\leq r^{2r}\left(V\right)\sup_{\|x\|=1}\left\langle \left[\frac{1}{p}\left|f\left(|T|\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(|T^{*}|\right)B^{*}\right|^{2qr}\right]x,x\right\rangle\\
&=r^{2r}\left(V\right)\left\|\frac{1}{p}\left|f\left(|T|\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(|T^{*}|\right)B^{*}\right|^{2qr}\right\|,
\end{aligned}$$

which proves (3.1).

By taking the power $r\geq 1$ in (2.6) and using the convexity of the power function, we get

$$\begin{aligned}
(3.4)\quad &\left|\langle BTVAx,x\rangle\right|^{2r}\\
&\leq r^{2r}\left(V\right)\\
&\times\left[\frac{\left\|\left|f\left(|T|\right)A\right|^{2}x\right\|\left\|\left|g\left(|T^{*}|\right)B^{*}\right|^{2}x\right\|+\left\langle \left|g\left(|T^{*}|\right)B^{*}\right|^{2}\left|f\left(|T|\right)A\right|^{2}x,x\right\rangle}{2}\right]^{r}\\
&\leq r^{2r}\left(V\right)\\
&\times\frac{\left\|\left|f\left(|T|\right)A\right|^{2}x\right\|^{r}\left\|\left|g\left(|T^{*}|\right)B^{*}\right|^{2}x\right\|^{r}+\left\langle \left|g\left(|T^{*}|\right)B^{*}\right|^{2}\left|f\left(|T|\right)A\right|^{2}x,x\right\rangle^{r}}{2}
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$
\begin{aligned}
&\omega^{2r}\left(BTVA\right) \\
&\le r^{2r}\left(V\right)\sup_{\|x\|=1}\left(\frac{1}{2}\left\|\left|f\left(|T|\right)A\right|^2 x\right\|^r\left\|\left|g\left(|T^*|\right)B^*\right|^2 x\right\|^r\right. \\
&\left.+\frac{1}{2}\left\langle\left|g\left(|T^*|\right)B^*\right|^2\left|f\left(|T|\right)A\right|^2 x, x\right\rangle^r\right) \\
&\le \frac{1}{2}r^{2r}\left(V\right)\left[\sup_{\|x\|=1}\left\|\left|f\left(|T|\right)A\right|^2 x\right\|^r\sup_{\|x\|=1}\left\|\left|g\left(|T^*|\right)B^*\right|^2 x\right\|^r\right. \\
&\left.+\sup_{\|x\|=1}\left\langle\left|g\left(|T^*|\right)B^*\right|^2\left|f\left(|T|\right)A\right|^2 x, x\right\rangle^r\right] \\
&= \frac{1}{2}r^{2r}\left(V\right) \\
&\times\left(\left\|\left|f\left(|T|\right)A\right|^2\right\|^r\left\|\left|g\left(|T^*|\right)B^*\right|^2\right\|^r+\left\|\left|g\left(|T^*|\right)B^*\right|^2\left|f\left(|T|\right)A\right|^2\right\|^r\right) \\
&= \frac{1}{2}r^{2r}\left(V\right) \\
&\times\left(\left\|f\left(|T|\right)A\right\|^{2r}\left\|g\left(|T^*|\right)B^*\right\|^{2r}+\left\|\left|g\left(|T^*|\right)B^*\right|^2\left|f\left(|T|\right)A\right|^2\right\|^r\right),
\end{aligned}
$$

which proves (3.2).

Also, observe that

$$
\begin{aligned}
&\left\|\left|f\left(|T|\right)A\right|^2 x\right\|^r\left\|\left|g\left(|T^*|\right)B^*\right|^2 x\right\|^r \\
&\le \frac{1}{p}\left\|\left|f\left(|T|\right)A\right|^2 x\right\|^{pr}+\frac{1}{q}\left\|\left|g\left(|T^*|\right)B^*\right|^2 x\right\|^{qr} \\
&= \frac{1}{p}\left\|\left|f\left(|T|\right)A\right|^2 x\right\|^{2\frac{pr}{2}}+\frac{1}{q}\left\|\left|g\left(|T^*|\right)B^*\right|^2 x\right\|^{2\frac{qr}{2}} \\
&= \frac{1}{p}\left\langle\left|f\left(|T|\right)A\right|^4 x, x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle\left|g\left(|T^*|\right)B^*\right|^4 x, x\right\rangle^{\frac{qr}{2}} \\
&\le \frac{1}{p}\left\langle\left|f\left(|T|\right)A\right|^{2pr} x, x\right\rangle+\frac{1}{q}\left\langle\left|g\left(|T^*|\right)B^*\right|^{2qr} x, x\right\rangle \\
&= \left\langle\left(\frac{1}{p}\left|f\left(|T|\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(|T^*|\right)B^*\right|^{2qr}\right)x, x\right\rangle,
\end{aligned}
$$

then

$$
\begin{aligned}
&\frac{\left\|\left|f\left(|T|\right)A\right|^2 x\right\|^r\left\|\left|g\left(|T^*|\right)B^*\right|^2 x\right\|^r+\left\langle\left|g\left(|T^*|\right)B^*\right|^2\left|f\left(|T|\right)A\right|^2 x, x\right\rangle^r}{2} \\
&\le \frac{1}{2}\left\langle\left(\frac{1}{p}\left|f\left(|T|\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(|T^*|\right)B^*\right|^{2qr}\right)x, x\right\rangle \\
&+\frac{1}{2}\left\langle\left|g\left(|T^*|\right)B^*\right|^2\left|f\left(|T|\right)A\right|^2 x, x\right\rangle^r
\end{aligned}
$$

and by (3.4) we get

$$\begin{aligned}|\langle BTVAx,x\rangle|^{2r} &\leq \frac{1}{2}r^{2r}(V)\left\langle\left(\frac{1}{p}\left|f(|T|)A\right|^{2pr}+\frac{1}{q}\left|g(|T^*|)B^*\right|^{2qr}\right)x,x\right\rangle \\ &\quad+\frac{1}{2}r^{2r}(V)\left\langle\left|g(|T^*|)B^*\right|^2\left|f(|T|)A\right|^2x,x\right\rangle^r\end{aligned}$$

or $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, we derive (3.3). □

If $r>0$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\ qr\geq 1$, then by taking $B=A$ in (3.1) we obtain

$$\omega^{2r}(ATVA)\leq r^{2r}(V)\left\|\frac{1}{p}\left|f(|T|)A\right|^{2pr}+\frac{1}{q}\left|g(|T^*|)A^*\right|^{2qr}\right\|,$$

which for $A=I$ gives

$$\omega^{2r}(TV)\leq r^{2r}(V)\left\|\frac{1}{p}\left|f(|T|)\right|^{2pr}+\frac{1}{q}\left|g(|T^*|)\right|^{2qr}\right\|.$$

If $r\geq 1$, then for $B=A$ in (3.2) we get

$$\begin{aligned}&\omega^{2r}(ATVA)\\ &\leq\frac{1}{2}r^{2r}(V)\left(\|f(|T|)A\|^{2r}\|g(|T^*|)A^*\|^{2r}+\left\|\left|g(|T^*|)A^*\right|^2\left|f(|T|)A\right|^2\right\|^r\right),\end{aligned}$$

which for $A=I$ produces

$$\omega^{2r}(TV)\leq\frac{1}{2}r^{2r}(V)\left(\|f(|T|)\|^{2r}\|g(|T^*|)\|^{2r}+\left\|\left|g(|T^*|)\right|^2\left|f(|T|)\right|^2\right\|^r\right).$$

If $r\geq 1$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\ qr\geq 2$, then by taking $B=A$ in (3.3), we derive

$$\begin{aligned}\omega^{2r}(ATVA)&\leq\frac{1}{2}r^{2r}(V)\left(\left\|\frac{1}{p}\left|f(|T|)A\right|^{2pr}+\frac{1}{q}\left|g(|T^*|)A^*\right|^{2qr}\right\|\right.\\ &\quad\left.+\left\|\left|g(|T^*|)A^*\right|^2\left|f(|T|)A\right|^2\right\|^r\right),\end{aligned}$$

which for $A=I$ gives

$$\begin{aligned}\omega^{2r}(TV)&\leq\frac{1}{2}r^{2r}(V)\left(\left\|\frac{1}{p}\left|f(|T|)\right|^{2pr}+\frac{1}{q}\left|g(|T^*|)\right|^{2qr}\right\|\right.\\ &\quad\left.+\left\|\left|g(|T^*|)\right|^2\left|f(|T|)\right|^2\right\|^r\right).\end{aligned}$$

Remark 3. *Consider $f(t)=t^{\lambda}$, $g(t)=t^{1-\lambda}$ with $\lambda\in[0,1]$ in Theorem 3. If $r>0$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\ qr\geq 1$, then*

$$\omega^{2r}(BTVA)\leq r^{2r}(V)\left\|\frac{1}{p}\left||T|^{\lambda}A\right|^{2pr}+\frac{1}{q}\left||T^*|^{1-\lambda}B^*\right|^{2qr}\right\|. \tag{3.5}$$

In particular,

$$\omega^{2r}(BTVA)\leq r^{2r}(V)\left\|\frac{1}{p}\left||T|^{1/2}A\right|^{2pr}+\frac{1}{q}\left||T^*|^{1/2}B^*\right|^{2qr}\right\|.$$

If $r \geq 1,$ *then*

$$\omega^{2r}\left(BTVA\right) \leq \frac{1}{2}r^{2r}\left(V\right)\left(\left\||T|^{\lambda}A\right\|^{2r}\left\||T^{*}|^{1-\lambda}B^{*}\right\|^{2r}+\left\|\left||T^{*}|^{1-\lambda}B^{*}\right|^{2}\left||T|^{\lambda}A\right|^{2}\right\|^{r}\right). \tag{3.6}$$

In particular,

$$\omega^{2r}\left(BTVA\right) \leq \frac{1}{2}r^{2r}\left(V\right)\left(\left\||T|^{1/2}A\right\|^{2r}\left\||T^{*}|^{1/2}B^{*}\right\|^{2r}+\left\|\left||T^{*}|^{1/2}B^{*}\right|^{2}\left||T|^{1/2}A\right|^{2}\right\|^{r}\right).$$

If $r \geq 1,$ $p,$ $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $pr,$ $qr \geq 2,$ *then also*

$$\omega^{2r}\left(BTVA\right) \leq \frac{1}{2}r^{2r}\left(V\right)\left(\left\|\frac{1}{p}\left||T|^{\lambda}A\right|^{2pr}+\frac{1}{q}\left||T^{*}|^{1-\lambda}B^{*}\right|^{2qr}\right\| + \left\|\left||T^{*}|^{1-\lambda}B^{*}\right|^{2}\left||T|^{\lambda}A\right|^{2}\right\|^{r}\right). \tag{3.7}$$

In particular,

$$\omega^{2r}\left(BTVA\right) \leq \frac{1}{2}r^{2r}\left(V\right)\left(\left\|\frac{1}{p}\left||T|^{1/2}A\right|^{2pr}+\frac{1}{q}\left||T^{*}|^{1/2}B^{*}\right|^{2qr}\right\| + \left\|\left||T^{*}|^{1/2}B^{*}\right|^{2}\left||T|^{1/2}A\right|^{2}\right\|^{r}\right).$$

If we take $p=q=2$ and assume that $r \geq \frac{1}{2}$, then from (3.1) we get

$$\omega^{2r}\left(BTVA\right) \leq \frac{1}{2}r^{2r}\left(V\right)\left\|\left|f\left(|T|\right)A\right|^{4r}+\left|g\left(|T^{*}|\right)B^{*}\right|^{4r}\right\|,$$

which for $r=\frac{1}{2}$ gives

$$\omega\left(BTVA\right) \leq \frac{1}{2}r\left(V\right)\left\|\left|f\left(|T|\right)A\right|^{2}+\left|g\left(|T^{*}|\right)B^{*}\right|^{2}\right\|,$$

while for $r=1$ gives

$$\omega^{2}\left(BTVA\right) \leq \frac{1}{2}r^{2}\left(V\right)\left\|\left|f\left(|T|\right)A\right|^{4}+\left|g\left(|T^{*}|\right)B^{*}\right|^{4}\right\|.$$

If we take $r=1$ in (3.1), then we get

$$\omega^{2}\left(BTVA\right) \leq r^{2}\left(V\right)\left\|\frac{1}{p}\left|f\left(|T|\right)A\right|^{2p}+\frac{1}{q}\left|g\left(|T^{*}|\right)B^{*}\right|^{2q}\right\|$$

for $p,$ $q>1$ with $\frac{1}{p}+\frac{1}{q}=1.$

If we take $r=1$ in (3.2) then we get

$$\omega^{2}\left(BTVA\right) \leq \frac{1}{2}r^{2}\left(V\right)\left(\left\|f\left(|T|\right)A\right\|^{2}\left\|g\left(|T^{*}|\right)B^{*}\right\|^{2}+\left\|\left|g\left(|T^{*}|\right)B^{*}\right|^{2}\left|f\left(|T|\right)A\right|^{2}\right\|\right),$$

while for $r=2,$

$$\omega^{4}\left(BTA\right) \leq \frac{1}{2}r^{4}\left(V\right)\left(\left\|f\left(|T|\right)A\right\|^{8}\left\|g\left(|T^{*}|\right)B^{*}\right\|^{8}+\left\|\left|g\left(|T^{*}|\right)B^{*}\right|^{2}\left|f\left(|T|\right)A\right|^{2}\right\|^{2}\right).$$

Also, if we take $p=q=2$ and $r\geq 1$ in (3.3), then we get

$$\omega^{2r}(BTVA)\leq \frac{1}{2}r^{2r}(V)\left(\frac{1}{2}\left\Vert \left\vert f(\left\vert T\right\vert)A\right\vert ^{4r}+\left\vert g(\left\vert T^{\ast }\right\vert)B^{\ast }\right\vert ^{4r}\right\Vert \right.$$
$$\left. +\left\Vert \left\vert g(\left\vert T^{\ast }\right\vert)B^{\ast }\right\vert ^{2}\left\vert f(\left\vert T\right\vert)A\right\vert ^{2}\right\Vert ^{r}\right) .$$

In particular, for $r=1$ we derive

$$\omega^{2}(BTVA)\leq \frac{1}{2}r^{2}(V)\left(\frac{1}{2}\left\Vert \left\vert f(\left\vert T\right\vert)A\right\vert ^{4}+\left\vert g(\left\vert T^{\ast }\right\vert)B^{\ast }\right\vert ^{4}\right\Vert \right.$$
$$\left. +\left\Vert \left\vert g(\left\vert T^{\ast }\right\vert)B^{\ast }\right\vert ^{2}\left\vert f(\left\vert T\right\vert)A\right\vert ^{2}\right\Vert \right) .$$

Moreover, if $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and take $r=2$ in (3.3), then we derive the inequality

$$\omega^{4}(BTA)\leq \frac{1}{2}r^{4}(V)\left(\left\Vert \frac{1}{p}\left\vert f(\left\vert T\right\vert)A\right\vert ^{4p}+\frac{1}{q}\left\vert g(\left\vert T^{\ast }\right\vert)B^{\ast }\right\vert ^{4q}\right\Vert \right.$$
$$\left. +\left\Vert \left\vert g(\left\vert T^{\ast }\right\vert)B^{\ast }\right\vert ^{2}\left\vert f(\left\vert T\right\vert)A\right\vert ^{2}\right\Vert ^{2}\right) ,$$

which for $p=q=2$ provides

$$\omega^{4}(BTVA)\leq \frac{1}{2}r^{4}(V)\left(\frac{1}{2}\left\Vert \left\vert f(\left\vert T\right\vert)A\right\vert ^{8}+\left\vert g(\left\vert T^{\ast }\right\vert)B^{\ast }\right\vert ^{8}\right\Vert \right.$$
$$\left. +\left\Vert \left\vert g(\left\vert T^{\ast }\right\vert)B^{\ast }\right\vert ^{2}\left\vert f(\left\vert T\right\vert)A\right\vert ^{2}\right\Vert ^{2}\right) .$$

Further, if we take $f(t)=t^{\lambda}$, $g(t)=t^{1-\lambda}$ with $\lambda\in[0,1]$ then we can get other similar inequalities. The details are omitted.

Corollary 2. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Let A, B, $X\in\mathcal{B}(H)$, then for all $\alpha\in[0,1]$,*

$$\omega^{2r}(BXA)\leq \left\Vert X\right\Vert ^{2r\alpha}\left\Vert \frac{1}{p}\left\vert f\left(\left\vert X\right\vert ^{1-\alpha}\right)A\right\vert ^{2pr}+\frac{1}{q}\left\vert g\left(\left\vert X^{\ast }\right\vert ^{1-\alpha}\right)B^{\ast }\right\vert ^{2qr}\right\Vert . \tag{3.8}$$

If $r\geq 1$, then

$$\omega^{2r}(BXA)\leq \frac{1}{2}\left\Vert X\right\Vert ^{2r\alpha}\left[\left\Vert f\left(\left\vert X\right\vert ^{1-\alpha}\right)A\right\Vert ^{2r}\left\Vert g\left(\left\vert X^{\ast }\right\vert ^{1-\alpha}\right)B^{\ast }\right\Vert ^{2r}\right. \tag{3.9}$$
$$\left. +\left\Vert \left\vert g\left(\left\vert X^{\ast }\right\vert ^{1-\alpha}\right)B^{\ast }\right\vert ^{2}\left\vert f\left(\left\vert X\right\vert ^{1-\alpha}\right)A\right\vert ^{2}\right\Vert ^{r}\right] .$$

If $r\geq 1$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\geq 2$, then also

$$\omega^{2r}(BXA)\leq \frac{1}{2}\left\Vert X\right\Vert ^{2r\alpha}\left(\left\Vert \frac{1}{p}\left\vert f\left(\left\vert X\right\vert ^{1-\alpha}\right)A\right\vert ^{2pr}+\frac{1}{q}\left\vert g\left(\left\vert X^{\ast }\right\vert ^{1-\alpha}\right)B^{\ast }\right\vert ^{2qr}\right\Vert \right. \tag{3.10}$$
$$\left. +\left\Vert \left\vert g\left(\left\vert X^{\ast }\right\vert ^{1-\alpha}\right)B^{\ast }\right\vert ^{2}\left\vert f\left(\left\vert X\right\vert ^{1-\alpha}\right)A\right\vert ^{2}\right\Vert ^{r}\right) .$$

Remark 4. *If we take* $f(t) = t^{\lambda}$, $g(t) = t^{1-\lambda}$ *with* $\lambda \in [0,1]$ *in Corollary 2, then we get*

$$\omega^{2r}(BXA) \leq \|X\|^{2r\alpha} \left\| \frac{1}{p} \left| |X|^{\lambda(1-\alpha)} A \right|^{2pr} + \frac{1}{q} \left| |X^*|^{(1-\lambda)(1-\alpha)} B^* \right|^{2qr} \right\|. \tag{3.11}$$

If $r \geq 1$, *then*

$$\omega^{2r}(BXA) \leq \frac{1}{2} \|X\|^{2r\alpha} \left[\left\| |X|^{\lambda(1-\alpha)} A \right\|^{2r} \left\| |X^*|^{(1-\lambda)(1-\alpha)} B^* \right\|^{2r} + \left\| \left| |X^*|^{(1-\lambda)(1-\alpha)} B^* \right|^2 \left| |X|^{\lambda(1-\alpha)} A \right|^2 \right\|^r \right]. \tag{3.12}$$

If $r \geq 1$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $pr, qr \geq 2$, *then also*

$$\omega^{2r}(BXA) \leq \frac{1}{2} \|X\|^{2r\alpha} \left(\left\| \frac{1}{p} \left| |X|^{\lambda(1-\alpha)} A \right|^{2pr} + \frac{1}{q} \left| |X^*|^{(1-\lambda)(1-\alpha)} B^* \right|^{2qr} \right\| + \left\| \left| |X^*|^{(1-\lambda)(1-\alpha)} B^* \right|^2 \left| |X|^{\lambda(1-\alpha)} A \right|^2 \right\|^r \right). \tag{3.13}$$

4. Some Results Via A-G Inequality

We also have:

Theorem 4. *With the assumptions of Theorem 2, we have for* $r \geq 1$ *that*

$$\omega^2(BTVA) \leq r^2(V) \left\| (1-\alpha) |f(|T|) A|^{2r} + \alpha |g(|T^*|) B^*|^{2r} \right\|^{1/r} \times \|f(|T|) A\|^{2\alpha} \|g(|T^*|) B^*\|^{2(1-\alpha)} \tag{4.1}$$

for all $\alpha \in [0,1]$.

Also, we have

$$\omega^2(BTVA) \leq r^2(V) \left\| (1-\alpha) |f(|T|) A|^{2r} + \alpha |g(|T^*|) B^*|^{2r} \right\|^{1/r} \times \left\| \alpha |f(|T|) A|^{2r} + (1-\alpha) |g(|T^*|) B^*|^{2r} \right\|^{1/r} \tag{4.2}$$

for all $\alpha \in [0,1]$ *and* $r \geq 1$.

Proof. From (2.5) we have for all $\alpha \in [0,1]$ that

$$\begin{aligned} |\langle BTVAx, x\rangle|^2 &\leq r^2(V) \left\langle |f(|T|) A|^2 x, x \right\rangle \left\langle |g(|T^*|) B^*|^2 x, x \right\rangle \\ &= r^2(V) \left\langle |f(|T|) A|^2 x, x \right\rangle^{1-\alpha} \left\langle |g(|T^*|) B^*|^2 x, x \right\rangle^{\alpha} \\ &\times \left\langle |f(|T|) A|^2 x, x \right\rangle^{\alpha} \left\langle |g(|T^*|) B^*|^2 x, x \right\rangle^{1-\alpha} \\ &\leq r^2(V) \left[(1-\alpha) \left\langle |f(|T|) A|^2 x, x \right\rangle + \alpha \left\langle |g(|T^*|) B^*|^2 x, x \right\rangle \right] \\ &\times \left\langle |f(|T|) A|^2 x, x \right\rangle^{\alpha} \left\langle |g(|T^*|) B^*|^2 x, x \right\rangle^{1-\alpha} \end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

If we take the power $r \geq 1$, then we get by the convexity of power r that

$$
\begin{aligned}
(4.3)\qquad & \left|\langle BTVAx, x\rangle\right|^{2r} \\
& \leq r^{2r}(V)\left[(1-\alpha)\left\langle \left|f(|T|)A\right|^{2}x, x\right\rangle + \alpha\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}x, x\right\rangle\right]^{r} \\
& \times \left\langle \left|f(|T|)A\right|^{2}x, x\right\rangle^{r\alpha}\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}x, x\right\rangle^{r(1-\alpha)} \\
& \leq r^{2r}(V)\left[(1-\alpha)\left\langle \left|f(|T|)A\right|^{2}x, x\right\rangle^{r} + \alpha\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}x, x\right\rangle^{r}\right] \\
& \times \left\langle \left|f(|T|)A\right|^{2}x, x\right\rangle^{r\alpha}\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}x, x\right\rangle^{r(1-\alpha)}
\end{aligned}
$$

for all $x \in H$, $\|x\| = 1$.

If we use McCarthy inequality for power $r \geq 1$, then we get

$$
\begin{aligned}
& (1-\alpha)\left\langle \left|f(|T|)A\right|^{2}x, x\right\rangle^{r} + \alpha\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}x, x\right\rangle^{r} \\
& \leq (1-\alpha)\left\langle \left|f(|T|)A\right|^{2r}x, x\right\rangle + \alpha\left\langle \left|g(|T^{*}|)B^{*}\right|^{2r}x, x\right\rangle \\
& = \left\langle \left[(1-\alpha)\left|f(|T|)A\right|^{2r} + \alpha\left|g(|T^{*}|)B^{*}\right|^{2r}\right]x, x\right\rangle
\end{aligned}
$$

and by (4.3)

$$
\begin{aligned}
\left|\langle BTVAx, x\rangle\right|^{2r} & \leq r^{2r}(V)\left\langle \left[(1-\alpha)\left|f(|T|)A\right|^{2r} + \alpha\left|g(|T^{*}|)B^{*}\right|^{2r}\right]x, x\right\rangle \\
& \times \left\langle \left|f(|T|)A\right|^{2}x, x\right\rangle^{r\alpha}\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}x, x\right\rangle^{r(1-\alpha)}
\end{aligned}
$$

for all $x \in H$, $\|x\| = 1$.

If we take the supremum over $\|x\| = 1$, then we get

$$
\begin{aligned}
\omega^{2r}(BTVA) & = \sup_{\|x\|=1}\left|\langle BTVAx, x\rangle\right|^{2r} \\
& \leq r^{2r}(V)\sup_{\|x\|=1}\left\langle \left[(1-\alpha)\left|f(|T|)A\right|^{2r} + \alpha\left|g(|T^{*}|)B^{*}\right|^{2r}\right]x, x\right\rangle \\
& \times \sup_{\|x\|=1}\left\langle \left|f(|T|)A\right|^{2}x, x\right\rangle^{r\alpha}\sup_{\|x\|=1}\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}x, x\right\rangle^{r(1-\alpha)} \\
& = r^{2r}(V)\left\|(1-\alpha)\left|f(|T|)A\right|^{2r} + \alpha\left|g(|T^{*}|)B^{*}\right|^{2r}\right\| \\
& \times \left\|\left|f(|T|)A\right|^{2}\right\|^{r\alpha}\left\|\left|g(|T^{*}|)B^{*}\right|^{2}\right\|^{r(1-\alpha)} \\
& = r^{2r}(V)\left\|(1-\alpha)\left|f(|T|)A\right|^{2r} + \alpha\left|g(|T^{*}|)B^{*}\right|^{2r}\right\| \\
& \times \left\|f(|T|)A\right\|^{2r\alpha}\left\|g(|T^{*}|)B^{*}\right\|^{2r(1-\alpha)},
\end{aligned}
$$

which proves (4.1).

We also have

$$
\begin{aligned}
|\langle BTVAx,x\rangle|^2 &\le r^2(V)\left\langle |f(|T|)A|^2 x,x\right\rangle^{1-\alpha}\left\langle |g(|T^*|)B^*|^2 x,x\right\rangle^{\alpha}\\
&\quad\times\left\langle |f(|T|)A|^2 x,x\right\rangle^{\alpha}\left\langle |g(|T^*|)B^*|^2 x,x\right\rangle^{1-\alpha}\\
&\le r^2(V)\left[(1-\alpha)\left\langle |f(|T|)A|^2 x,x\right\rangle+\alpha\left\langle |g(|T^*|)B^*|^2 x,x\right\rangle\right]\\
&\quad\times\left[\alpha\left\langle |f(|T|)A|^2 x,x\right\rangle+(1-\alpha)\left\langle |g(|T^*|)B^*|^2 x,x\right\rangle\right],
\end{aligned}
$$

for all $x\in H$, $\|x\|=1$.

This implies in the same way that

$$
\begin{aligned}
|\langle BTAx,x\rangle|^{2r} &\le r^{2r}(V)\left\langle\left[(1-\alpha)|f(|T|)A|^{2r}+\alpha|g(|T^*|)B^*|^{2r}\right]x,x\right\rangle\\
&\quad\times\left\langle\left[\alpha|f(|T|)A|^{2r}+(1-\alpha)|g(|T^*|)B^*|^{2r}\right]x,x\right\rangle
\end{aligned}
$$

for all $x\in H$, $\|x\|=1$, which proves (4.2). □

If we take $B=A$ in Theorem 4, then we get that

$$
\begin{aligned}
\omega^2(ATVA) &\le r^2(V)\left\|(1-\alpha)|f(|T|)A|^{2r}+\alpha|g(|T^*|)A^*|^{2r}\right\|^{1/r}\\
&\quad\times\|f(|T|)A\|^{2\alpha}\|g(|T^*|)A^*\|^{2(1-\alpha)}
\end{aligned}
$$

and

$$
\begin{aligned}
\omega^2(ATVA) &\le r^2(V)\left\|(1-\alpha)|f(|T|)A|^{2r}+\alpha|g(|T^*|)A^*|^{2r}\right\|^{1/r}\\
&\quad\times\left\|\alpha|f(|T|)A|^{2r}+(1-\alpha)|g(|T^*|)A^*|^{2r}\right\|^{1/r}
\end{aligned}
$$

for all $\alpha\in[0,1]$ and $r\ge1$. Moreover, if we choose $A=I$ in these inequalities, then we obtain

$$
\begin{aligned}
&\omega^2(TV)\\
&\le r^2(V)\left\|(1-\alpha)|f(|T|)|^{2r}+\alpha|g(|T^*|)|^{2r}\right\|^{1/r}\|f(|T|)\|^{2\alpha}\|g(|T^*|)\|^{2(1-\alpha)}
\end{aligned}
$$

and

$$
\begin{aligned}
\omega^2(TV) &\le r^2(V)\left\|(1-\alpha)|f(|T|)|^{2r}+\alpha|g(|T^*|)|^{2r}\right\|^{1/r}\\
&\quad\times\left\|\alpha|f(|T|)|^{2r}+(1-\alpha)|g(|T^*|)|^{2r}\right\|^{1/r}
\end{aligned}
$$

for all $\alpha\in[0,1]$ and $r\ge1$.

Remark 5. *Consider $f(t)=t^{\lambda}$, $g(t)=t^{1-\lambda}$ with $\lambda\in[0,1]$ in Theorem 4, then*

$$
\begin{aligned}
\omega^2(BTVA) &\le r^2(V)\left\|(1-\alpha)\left||T|^{\lambda}A\right|^{2r}+\alpha\left||T^*|^{1-\lambda}B^*\right|^{2r}\right\|^{1/r}\\
&\quad\times\left\||T|^{\lambda}A\right\|^{2\alpha}\left\||T^*|^{1-\lambda}B^*\right\|^{2(1-\alpha)}
\end{aligned}
$$

and

$$\omega^2(BTVA) \le r^2(V) \left\| (1-\alpha) \left| |T|^{\lambda} A \right|^{2r} + \alpha \left| |T^*|^{1-\lambda} B^* \right|^{2r} \right\|^{1/r} \times \left\| \alpha \left| |T|^{\lambda} A \right|^{2r} + (1-\alpha) \left| |T^*|^{1-\lambda} B^* \right|^{2r} \right\|^{1/r}$$

for all $\alpha \in [0,1]$ *and* $r \ge 1$.

In particular, for $\lambda = 1/2$ *we obtain*

$$\omega^2(BTVA) \le r^2(V) \left\| (1-\alpha) \left| |T|^{1/2} A \right|^{2r} + \alpha \left| |T^*|^{1/2} B^* \right|^{2r} \right\|^{1/r} \times \left\| |T|^{1/2} A \right\|^{2\alpha} \left\| |T^*|^{1/2} B^* \right\|^{2(1-\alpha)}$$

and

$$\omega^2(BTVA) \le r^2(V) \left\| (1-\alpha) \left| |T|^{1/2} A \right|^{2r} + \alpha \left| |T^*|^{1/2} B^* \right|^{2r} \right\|^{1/r} \times \left\| \alpha \left| |T|^{1/2} A \right|^{2r} + (1-\alpha) \left| |T^*|^{1/2} B^* \right|^{2r} \right\|^{1/r}$$

for all $\alpha \in [0,1]$ *and* $r \ge 1$.

Corollary 3. *With the assumptions of Theorem 2 we have*

$$\omega^2(BTVA) \le \frac{1}{2^{1/r}} r^2(V) \left\| |f(|T|)A|^{2r} + |g(|T^*|)B^*|^{2r} \right\|^{1/r} \|f(|T|)A\| \, \|g(|T^*|)B^*\|$$

for all $r \ge 1$.

Also, we have

$$\omega(BTVA) \le \frac{1}{2^{1/r}} r(V) \left\| |f(|T|)A|^{2r} + |g(|T^*|)B^*|^{2r} \right\|^{1/r}$$

for all $r \ge 1$.

In particular, we have

$$\omega^2(BTVA) \le \frac{1}{2} r^2(V) \left\| |f(|T|)A|^{2} + |g(|T^*|)B^*|^{2} \right\| \|f(|T|)A\| \, \|g(|T^*|)B^*\|$$

and

$$\omega(BTVA) \le \frac{1}{2} r(V) \left\| |f(|T|)A|^{2} + |g(|T^*|)B^*|^{2} \right\|.$$

If we take $f(t) = t^{\lambda}$, $g(t) = t^{1-\lambda}$ with $\lambda \in [0,1]$ in Corollary 1, then we get

$$\omega^2(BTVA) \le \frac{1}{2^{1/r}} r^2(V) \left\| \left| |T|^{\lambda} A \right|^{2r} + \left| |T^*|^{1-\lambda} B^* \right|^{2r} \right\|^{1/r} \left\| |T|^{\lambda} A \right\| \left\| |T^*|^{1-\lambda} B^* \right\|$$

for all $r \ge 1$.

Also, we have

$$\omega(BTVA) \le \frac{1}{2^{1/r}} r(V) \left\| \left| |T|^{\lambda} A \right|^{2r} + \left| |T^*|^{1-\lambda} B^* \right|^{2r} \right\|^{1/r}$$

for all $r \geq 1$.

In particular, we obtain

$$\omega^2(BTVA) \leq \frac{1}{2} r^2(V) \left\| \left| |T|^{\lambda} A \right|^2 + \left| |T^*|^{1-\lambda} B^* \right|^2 \right\| \left\| |T|^{\lambda} A \right\| \left\| |T^*|^{1-\lambda} B^* \right\| \tag{4.4}$$

and

$$\omega(BTVA) \leq \frac{1}{2} r(V) \left\| \left| |T|^{\lambda} A \right|^2 + \left| |T^*|^{1-\lambda} B^* \right|^2 \right\|.$$

We also have:

Corollary 4. *Assume that f and g are non-negative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Let A, B, $X \in \mathcal{B}(H)$, then for all $\alpha \in [0,1]$, we have for $r \geq 1$ that*

$$\omega^2(BXA) \leq \|X\|^{2\alpha} \left\| (1-\alpha) \left| f\left(|X|^{1-\alpha}\right) A \right|^{2r} + \alpha \left| g\left(|X^*|^{1-\alpha}\right) B^* \right|^{2r} \right\|^{1/r} \times \left\| f\left(|X|^{1-\alpha}\right) A \right\|^{2\alpha} \left\| g\left(|X^*|^{1-\alpha}\right) B^* \right\|^{2(1-\alpha)} \tag{4.5}$$

for all $\alpha \in [0,1]$.

Also, we have

$$\omega^2(BXA) \leq \|X\|^{2\alpha} \left\| (1-\alpha) \left| f\left(|X|^{1-\alpha}\right) A \right|^{2r} + \alpha \left| g\left(|X^*|^{1-\alpha}\right) B^* \right|^{2r} \right\|^{1/r} \times \left\| \alpha \left| f\left(|X|^{1-\alpha}\right) A \right|^{2r} + (1-\alpha) \left| g\left(|X^*|^{1-\alpha}\right) B^* \right|^{2r} \right\|^{1/r} \tag{4.6}$$

for all $\alpha \in [0,1]$ and $r \geq 1$.

If we take $f(t) = t^{\lambda}$, $g(t) = t^{1-\lambda}$ with $\lambda \in [0,1]$ in Corollary 4, then for $r \geq 1$, $\alpha \in [0,1]$, we get

$$\omega^2(BXA) \leq \|X\|^{2\alpha} \left\| (1-\alpha) \left| |X|^{\lambda(1-\alpha)} A \right|^{2r} + \alpha \left| |X^*|^{(1-\lambda)(1-\alpha)} B^* \right|^{2r} \right\|^{1/r} \times \left\| |X|^{\lambda(1-\alpha)} A \right\|^{2\alpha} \left\| |X^*|^{(1-\lambda)(1-\alpha)} B^* \right\|^{2(1-\alpha)} \tag{4.7}$$

and

$$\omega^2(BXA) \leq \|X\|^{2\alpha} \left\| (1-\alpha) \left| |X|^{\lambda(1-\alpha)} A \right|^{2r} + \alpha \left| |X^*|^{(1-\lambda)(1-\alpha)} B^* \right|^{2r} \right\|^{1/r} \times \left\| \alpha \left| |X|^{\lambda(1-\alpha)} A \right|^{2r} + (1-\alpha) \left| |X^*|^{(1-\lambda)(1-\alpha)} B^* \right|^{2r} \right\|^{1/r}. \tag{4.8}$$

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