SCHWARZ TYPE VECTOR INEQUALITIES IN TERMS OF SPECTRAL RADIUS OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. In this paper we show among others that, if T and V are operators in $\mathcal{B}(H)$ such that $|T|V=V^{*}|T|$, then for A, $B\in\mathcal{B}(H)$ we have

$$|\langle BTVAx,y\rangle|\leq r(V)\left\langle |f(|T|)A|^{2}x,x\right\rangle^{1/2}\left\langle |g(|T^{*}|)B^{*}|^{2}y,y\right\rangle^{1/2}$$

for all x, $y\in H$. Also, if X, A, $B\in\mathcal{B}(H)$, then for all $p,q\geq 0$ with $p+q\geq 1$, we have

$$\begin{aligned}&\left|\left\langle BX|X|^{p+q-1}Ax,y\right\rangle\right|\\&\leq\|X\|^{q}\left\langle |f(|X|^{p})A|^{2}x,x\right\rangle^{1/2}\left\langle |g(|X^{*}|^{p})B^{*}|^{2}y,y\right\rangle^{1/2}\end{aligned}$$

for all x, $y\in H$. In particular,

$$|\langle BXAx,y\rangle|\leq\|X\|^{1-\lambda}\left\langle \left|f\left(\left||X|^{\lambda}\right|\right)A\right|^{2}x,x\right\rangle^{1/2}\left\langle \left|g\left(|X^{*}|^{\lambda}\right)B^{*}\right|^{2}y,y\right\rangle^{1/2}$$

for X, A, $B\in\mathcal{B}(H)$ and $\lambda\in[0,1]$. Some applications for the *numerical radius* and *p-Schatten norms* are also provided.

1. Introduction

In 1988, F. Kittaneh obtained the following generalization of Schwarz inequality [10]:

Theorem 1. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Let T, V be operators in $B(H)$ such that $|T|V=V^{*}|T|$, then*

$$|\langle TVx,y\rangle|\leq r(V)\|f(|T|)x\|\|g(|T^{*}|)y\| \tag{1.1}$$

for all x, $y\in H$, where $r(V)$ denotes the spectral radius of V.

If we take $f(t)=t^{\alpha}$ and $g(t)=t^{1-\alpha}$ for $\alpha\in[0,1]$ and $t>0$, then we obtain

$$|\langle TVx,y\rangle|\leq r(V)\left\||T|^{\alpha}x\right\|\left\||T^{*}|^{1-\alpha}y\right\| \tag{1.2}$$

for all x, $y\in H$.

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T)=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.3}$$

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

Key words and phrases. Bounded operators, Aluthge transform, Dougal transform, Partial isometry, Numerical radius.

RGMIA Res. Rep. Coll. **26** (2023), Art. 35, 20 pp. Received 05/06/23

Obviously, by (1.3), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leq w(T) \|x\|^2. \tag{1.4}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T + V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{1.5}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [11], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.5):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.6}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [12] improved the inequality (1.5) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{1.7}$$

for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [8]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{1.8}$$

and

$$\omega^{2r}(T) \leq \left\|\alpha |T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|, \tag{1.9}$$

where $\alpha \in (0, 1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (1.8) that

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{1.10}$$

and from (1.9) that

$$\omega^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{1.11}$$

For more related results, see the recent books on inequalities for numerical radii [7] and [4].

Let $T = U|T|$ be the *polar decomposition* of the bounded linear operator T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T} := |T|^{1/2} U |T|^{1/2}$, see [1].

The following properties of $\widetilde{T}$ are as follows:

(i) $\left\|\widetilde{T}\right\| \leq \|T\|$,
(ii) $w\left(\widetilde{T}\right) \leq \omega(T)$,
(iii) $r\left(\widetilde{T}\right) = \omega(T)$,

(iv) $\omega\left(\widetilde{T}\right)\leq\left\|T^{2}\right\|^{1/2}\left(\leq\|T\|\right)$, [15].

Utilizing this transform T. Yamazaki, [15] obtained in 2007 the following refinement of Kittaneh's inequality (1.6):

$$\omega(T)\leq\frac{1}{2}\left(\|T\|+\omega\left(\widetilde{T}\right)\right)\leq\frac{1}{2}\left(\|T\|+\left\|T^{2}\right\|^{1/2}\right) \tag{1.12}$$

for any operator $T\in B(H)$.

We remark that if $\widetilde{T}=0$, then obviously $w(T)=\frac{1}{2}\|T\|$.

Abu-Omar and Kittaneh [2] improved on inequality (1.12) using generalized Aluthge transform to prove that

$$\omega(T)\leq\frac{1}{2}\left(\|T\|+\min_{t\in[0,1]}\omega\left(\Delta_{t}(T)\right)\right).$$

For $t=1$ this also gives the following result for the *Dougal transform*

$$\omega(T)\leq\frac{1}{2}\left(\|T\|+\omega\left(\widehat{T}\right)\right). \tag{1.13}$$

In [3] Bunia et al. also proved that

$$\omega(T)\leq\min_{t\in[0,1]}\left\{\frac{1}{2}\omega\left(\Delta_{t}(T)\right)+\frac{1}{4}\left(\|T\|^{2t}+\|T\|^{2(1-t)}\right)\right\},$$

which for $t=1/2$ gives (1.12) as well.

Motivated by the above results, we show in this paper, among others that, if f, g and T, V are as in Theorem 1, while A, $B\in\mathcal{B}(H)$, then

$$|\langle BTVAx,y\rangle|\leq r(V)\left\langle\left|f(|T|)A\right|^{2}x,x\right\rangle^{1/2}\left\langle\left|g(|T^{*}|)B^{*}\right|^{2}y,y\right\rangle^{1/2}$$

for all x, $y\in H$. Also, if X, A, $B\in\mathcal{B}(H)$, then for all $p,q\geq0$ with $p+q\geq1$, we have

$$\begin{aligned}&\left|\left\langle BX|X|^{p+q-1}Ax,y\right\rangle\right|\\&\leq\|X\|^{q}\left\langle\left|f(|X|^{p})A\right|^{2}x,x\right\rangle^{1/2}\left\langle\left|g(|X^{*}|^{p})B^{*}\right|^{2}y,y\right\rangle^{1/2}\end{aligned}$$

for all x, $y\in H$. In particular,

$$|\langle BXAx,y\rangle|\leq\|X\|^{1-\lambda}\left\langle\left|f\left(\left||X|^{\lambda}\right|\right)A\right|^{2}x,x\right\rangle^{1/2}\left\langle\left|g\left(|X^{*}|^{\lambda}\right)B^{*}\right|^{2}y,y\right\rangle^{1/2}$$

for X, A, $B\in\mathcal{B}(H)$, x, $y\in H$ and $\lambda\in[0,1]$. Some applications for the *numerical radius* and *p-Schatten norms* are also provided.

2. Main Results

We recall the following vector inequality for positive operators $A\geq0$, obtained by C. A. McCarthy in [13]

$$\langle Ax,x\rangle^{p}\leq\langle A^{p}x,x\rangle,\ p\geq1 \tag{2.1}$$

for $x\in H$, $\|x\|=1$ and *Buzano's inequality* [5],

$$|\langle x,e\rangle\langle e,y\rangle|\leq\frac{1}{2}\left[\|x\|\|y\|+|\langle x,y\rangle|\right]\|e\|^{2} \tag{2.2}$$

that holds for any x, y, $e\in H$.

If we replace x by $\frac{y}{\|y\|}$, $y \neq 0$ in (2.1), then we get

$$\left\langle A\frac{y}{\|y\|}, \frac{y}{\|y\|}\right\rangle^{p} \leq \left\langle A^{p}\frac{y}{\|y\|}, \frac{y}{\|y\|}\right\rangle,\ p \geq 1,$$

namely

$$\langle Ay, y\rangle^{p} \leq \|y\|^{2(p-1)} \langle A^{p}y, y\rangle,\ p \geq 1, \tag{2.3}$$

for all $y \in H$.

We have:

Theorem 2. *Assume that f and g are non-negative functions on $[0,\infty)$ that are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Let T, V be operators in $\mathcal{B}(H)$ such that $|T|V = V^{*}|T|$, then for A, $B \in \mathcal{B}(H)$*

$$|\langle BTVAx, y\rangle| \leq r(V)\left\langle |f(|T|)A|^{2}x, x\right\rangle^{1/2}\left\langle |g(|T^{*}|)B^{*}|^{2}y, y\right\rangle^{1/2} \tag{2.4}$$

for all x, $y \in H$.

We also have for $s \geq 1$ that

$$|\langle BTVAx, x\rangle| \leq r(V)\|x\|^{2\left(1-\frac{1}{s}\right)}\left\langle \frac{|f(|T|)A|^{2s} + |g(|T^{*}|)B^{*}|^{2s}}{2}x, x\right\rangle^{1/s} \tag{2.5}$$

for all $x \in H$. In particular, for $s = 1$, we obtain

$$|\langle BTVAx, x\rangle| \leq r(V)\left\langle \frac{|f(|T|)A|^{2} + |g(|T^{*}|)B^{*}|^{2}}{2}x, x\right\rangle \tag{2.6}$$

for all $x \in H$.

Moreover, we have

$$\begin{aligned} |\langle BTVAx, x\rangle|^{2} &\leq \frac{1}{2}\|x\|^{2} r^{2}(V)\Big[\left\||f(|T|)A|^{2}x\right\|\left\||g(|T^{*}|)B^{*}|^{2}x\right\| \\ &\quad + \left|\left\langle |g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}x, x\right\rangle\right|\Big] \\ &\leq \frac{1}{2}\|x\|^{2} r^{2}(V)\left[\frac{1}{2}\left\langle\left(|f(|T|)A|^{4} + |g(|T^{*}|)B^{*}|^{4}\right)x, x\right\rangle\right. \\ &\quad \left. + \left|\left\langle |g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}x, x\right\rangle\right|\right] \end{aligned} \tag{2.7}$$

for all $x \in H$.

Proof. Observe that by (1.1) we have

$$\begin{aligned} |\langle TVx, y\rangle|^{2} &\leq r^{2}(V)\|f(|T|)x\|^{2}\|g(|T^{*}|)y\|^{2} \\ &= r^{2}(V)\langle f(|T|)x, f(|T|)x\rangle\langle g(|T^{*}|)y, g(|T^{*}|)y\rangle \\ &= r^{2}(V)\left\langle f^{2}(|T|)x, x\right\rangle\left\langle g^{2}(|T^{*}|)y, y\right\rangle \end{aligned}$$

for all x, $y \in H$.

If we take Ax instead of x and B^*y instead of y, then we get

$$\begin{aligned}|\langle TVAx, B^*y\rangle|^2 &\le r^2(V)\left\langle f^2(|T|)Ax, Ax\right\rangle\left\langle g^2(|T^*|)B^*y, B^*y\right\rangle\\ &= r^2(V)\left\langle A^*f^2(|T|)Ax, x\right\rangle\left\langle Bg^2(|T^*|)B^*y, y\right\rangle\\ &= r^2(V)\left\langle (f(|T|)Ax)^* f(|T|)Ax, x\right\rangle\\ &\times\left\langle (g(|T^*|)B^*)^* g(|T^*|)B^*y, y\right\rangle\\ &= r^2(V)\left\langle |f(|T|)A|^2x, x\right\rangle\left\langle |g(|T^*|)B^*|^2y, y\right\rangle,\end{aligned}$$

namely

$$|\langle BTVAx, y\rangle|^2 \le r^2(V)\left\langle |f(|T|)A|^2x, x\right\rangle\left\langle |g(|T^*|)B^*|^2y, y\right\rangle \tag{2.8}$$

for all x, $y \in H$. This proves (2.4).

From (2.4) for $y = x$ and the A-G *mean inequality* we have

$$\begin{aligned}|\langle BTVAx, x\rangle| &\le r(V)\left\langle |f(|T|)A|^2x, x\right\rangle^{1/2}\left\langle |g(|T^*|)B^*|^2x, x\right\rangle^{1/2}\\ &\le r(V)\frac{\left\langle |f(|T|)A|^2x, x\right\rangle + \left\langle |g(|T^*|)B^*|^2x, x\right\rangle}{2}\end{aligned} \tag{2.9}$$

for all $x \in H$.

If we take the power $s \ge 1$ and use the convexity of the power function, then we get

$$\begin{aligned}|\langle BTVAx, x\rangle|^s &\le r^s(V)\left(\frac{\left\langle |f(|T|)A|^2x, x\right\rangle + \left\langle |g(|T^*|)B^*|^2x, x\right\rangle}{2}\right)^s\\ &\le r^s(V)\frac{\left\langle |f(|T|)A|^2x, x\right\rangle^s + \left\langle |g(|T^*|)B^*|^2x, x\right\rangle^s}{2}\end{aligned} \tag{2.10}$$

for all $x \in H$.

If we use McCarthy's inequality (2.3) we have

$$\left\langle |f(|T|)A|^2x, x\right\rangle^s \le \|x\|^{2(s-1)}\left\langle |f(|T|)A|^{2s}x, x\right\rangle,\ s \ge 1,$$

and

$$\left\langle |g(|T^*|)B^*|^2x, x\right\rangle^s \le \|x\|^{2(s-1)}\left\langle |g(|T^*|)B^*|^{2s}x, x\right\rangle,\ s \ge 1$$

for all $x \in H$.

By utilizing (2.10) we then get

$$\begin{aligned}&|\langle BTVAx, x\rangle|^s\\ &\le r^s(V)\frac{\|x\|^{2(s-1)}\left\langle |f(|T|)A|^{2s}x, x\right\rangle + \|x\|^{2(s-1)}\left\langle |g(|T^*|)B^*|^{2s}x, x\right\rangle}{2}\\ &= r^s(V)\|x\|^{2(s-1)}\left\langle \frac{|f(|T|)A|^{2s} + |g(|T^*|)B^*|^{2s}}{2}x, x\right\rangle\end{aligned}$$

for all $x \in H$, which proves (2.5).

If we use (2.8) for $y=x$ and then Buzano's inequality, then we get

$$\begin{aligned}
&|\langle BTVAx,x\rangle|^2\\
&\leq r^2(V)\left\langle |f(|T|)A|^2x,x\right\rangle\left\langle x,|g(|T^*|)B^*|^2x\right\rangle\\
&\leq \frac{1}{2}\|x\|^2 r^2(V)\\
&\times\left[\left\||f(|T|)A|^2x\right\|\left\||g(|T^*|)B^*|^2x\right\|+\left|\left\langle |f(|T|)A|^2x,|g(|T^*|)B^*|^2x\right\rangle\right|\right]\\
&=\frac{1}{2}\|x\|^2 r^2(V)\\
&\times\left[\left\||f(|T|)A|^2x\right\|\left\||g(|T^*|)B^*|^2x\right\|+\left|\left\langle |g(|T^*|)B^*|^2|f(|T|)A|^2x,x\right\rangle\right|\right]
\end{aligned}$$

for all $x\in H$, which proves the first part of (2.7).

Further, observe that by the *A-G mean inequality*

$$\begin{aligned}
\left\||f(|T|)A|^2x\right\|\left\||g(|T^*|)B^*|^2x\right\| &\leq \frac{1}{2}\left[\left\||f(|T|)A|^2x\right\|^2+\left\||g(|T^*|)B^*|^2x\right\|^2\right]\\
&=\left\langle\left(\frac{|f(|T|)A|^4+|g(|T^*|)B^*|^4}{2}\right)x,x\right\rangle,
\end{aligned}$$

which proves the second part of (2.7). □

Corollary 1. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Let X, A, $B\in\mathcal{B}(H)$, then for all p, $q\geq 0$ with $p+q\geq 1$, we have*

$$\begin{aligned}
&\left|\left\langle BX|X|^{p+q-1}Ax,y\right\rangle\right| \qquad (2.11)\\
&\leq \|X\|^q\left\langle |f(|X|^p)A|^2x,x\right\rangle^{1/2}\left\langle |g(|X^*|^p)B^*|^2y,y\right\rangle^{1/2}
\end{aligned}$$

for all x, $y\in H$.

We also have

$$\begin{aligned}
&\left|\left\langle BX|X|^{p+q-1}Ax,x\right\rangle\right| \qquad (2.12)\\
&\leq \|X\|^q\|x\|^{2\left(1-\frac{1}{s}\right)}\left\langle\frac{|f(|X|^p)A|^{2s}+|g(|X^*|^p)B^*|^{2s}}{2}x,x\right\rangle^{1/s}
\end{aligned}$$

for all $x\in H$. In particular for $s=1$, we obtain

$$\left|\left\langle BX|X|^{p+q-1}Ax,x\right\rangle\right|\leq\|X\|^q\left\langle\frac{|f(|X|^p)A|^2+|g(|X^*|^p)B^*|^2}{2}x,x\right\rangle \qquad (2.13)$$

for all $x\in H$.

Moreover, we have

$$(2.14)\qquad \begin{aligned} &\left|\left\langle BX\left|X\right|^{p+q-1}Ax,x\right\rangle\right|^2 \\ &\leq \frac{1}{2}\left\|x\right\|^2\left\|X\right\|^{2q}\left[\left\|\left|f\left(\left|X\right|^p\right)A\right|^2 x\right\|\left\|\left|g\left(\left|X^*\right|^p\right)B^*\right|^2 x\right\|\right. \\ &\left.+\left|\left\langle\left|g\left(\left|X^*\right|^p\right)B^*\right|^2\left|f\left(\left|X\right|^p\right)A\right|^2 x,x\right\rangle\right|\right] \\ &\leq \frac{1}{2}\left\|x\right\|^2\left\|X\right\|^{2q}\left[\frac{1}{2}\left\langle\left(\left|f\left(\left|X\right|^p\right)A\right|^4+\left|g\left(\left|X^*\right|^p\right)B^*\right|^4\right)x,x\right\rangle\right. \\ &\left.+\left|\left\langle\left|g\left(\left|X^*\right|^p\right)B^*\right|^2\left|f\left(\left|X\right|^p\right)A\right|^2 x,x\right\rangle\right|\right] \end{aligned}$$

for all $x\in H$.

Proof. Let $X=U\left|X\right|$ be the *polar decomposition* of the bounded linear operator X, with U a partial isometry. If we take $T=U\left|X\right|^p$ and $V=\left|X\right|^q$, then we have

$$TV=U\left|X\right|^{p+q}=X\left|X\right|^{p+q-1},\ \ \left|T\right|=\left|X\right|^p\ \text{ and }\ \left|T^*\right|=\left|X^*\right|^p$$

and since

$$\left|T\right|V=\left|X\right|^{p+q}=V^*\left|T\right|$$

and

$$r\left(V\right)=r\left(\left|X\right|^q\right)=\left\|\left|X\right|^q\right\|=\left\|X\right\|^q,$$

hence by Theorem 2 we derive the desired inequalities (2.11)-(2.14). □

Corollary 2. *With the assumptions of Corollary 1 we have for all* X, A, $B\in\mathcal{B}\left(H\right)$ *and* $\lambda\in[0,1]$ *that*

$$(2.15)\qquad \begin{aligned} &\left|\left\langle BXAx,y\right\rangle\right| \\ &\leq\left\|X\right\|^{1-\lambda}\left\langle\left|f\left(\left|\left|X\right|^{\lambda}\right|\right)A\right|^2 x,x\right\rangle^{1/2}\left\langle\left|g\left(\left|X^*\right|^{\lambda}\right)B^*\right|^2 y,y\right\rangle^{1/2} \end{aligned}$$

for all x, $y\in H$.

We also have

$$(2.16)\qquad \begin{aligned} &\left|\left\langle BXAx,x\right\rangle\right| \\ &\leq\left\|X\right\|^{1-\lambda}\left\|x\right\|^{2\left(1-\frac{1}{s}\right)}\left\langle\frac{\left|f\left(\left|X\right|^{\lambda}\right)A\right|^{2s}+\left|g\left(\left|X^*\right|^{\lambda}\right)B^*\right|^{2s}}{2}x,x\right\rangle^{1/s} \end{aligned}$$

for all $x\in H$. *In particular, for* $s=1$, *we obtain*

$$(2.17)\qquad \left|\left\langle BXAx,x\right\rangle\right|\leq\left\|X\right\|^{1-\lambda}\left\langle\frac{\left|f\left(\left|X\right|^{\lambda}\right)A\right|^{2}+\left|g\left(\left|X^*\right|^{\lambda}\right)B^*\right|^{2}}{2}x,x\right\rangle$$

for all $x\in H$.

Moreover, we have

$$|\langle BXAx,x\rangle|^2 \le \frac{1}{2}\|x\|^2\|X\|^{2(1-\lambda)}\left[\left\|\left|f\left(|X|^{\lambda}\right)A\right|^2 x\right\|\left\|\left|g\left(|X^*|^{\lambda}\right)B^*\right|^2 x\right\| + \left|\left\langle \left|g\left(|X^*|^{\lambda}\right)B^*\right|^2\left|f\left(|X|^{\lambda}\right)A\right|^2 x,x\right\rangle\right|\right]$$
$$\le \frac{1}{2}\|x\|^2\|X\|^{2(1-\lambda)}\left[\frac{1}{2}\left\langle\left(\left|f\left(|X|^{\lambda}\right)A\right|^4+\left|g\left(|X^*|^{\lambda}\right)B^*\right|^4\right)x,x\right\rangle + \left|\left\langle \left|g\left(|X^*|^{\lambda}\right)B^*\right|^2\left|f\left(|X|^{\lambda}\right)A\right|^2 x,x\right\rangle\right|\right] \tag{2.18}$$

for all $x\in H$.

Remark 1. *If we take* $f(t)=t^{\alpha}$, $g(t)=t^{1-\alpha}$ *with* $\alpha\in[0,1]$ *in Theorem 2, then we get*

$$|\langle BTVAx,y\rangle| \le r(V)\left\langle \left||T|^{\alpha}A\right|^2 x,x\right\rangle^{1/2}\left\langle\left||T^*|^{1-\alpha}B^*\right|^2 y,y\right\rangle^{1/2} \tag{2.19}$$

for all x, $y\in H$.

We also have

$$|\langle BTVAx,x\rangle| \le r(V)\|x\|^{2\left(1-\frac{1}{s}\right)}\left\langle\frac{\left||T|^{\alpha}A\right|^{2s}+\left||T^*|^{1-\alpha}B^*\right|^{2s}}{2}x,x\right\rangle^{1/s} \tag{2.20}$$

for all $x\in H$. *In particular, for* $s=1$, *we obtain*

$$|\langle BTVAx,x\rangle| \le r(V)\left\langle\frac{\left||T|^{\alpha}A\right|^{2}+\left||T^*|^{1-\alpha}B^*\right|^{2}}{2}x,x\right\rangle \tag{2.21}$$

for all $x\in H$.

Moreover, we have

$$|\langle BTVAx,x\rangle|^2 \le \frac{1}{2}\|x\|^2 r^2(V)\left[\left\|\left||T|^{\alpha}A\right|^2x\right\|\left\|\left||T^*|^{1-\alpha}B^*\right|^2x\right\| + \left|\left\langle\left||T^*|^{1-\alpha}B^*\right|^2\left||T|^{\alpha}A\right|^2x,x\right\rangle\right|\right]$$
$$\le \frac{1}{2}\|x\|^2 r^2(V)\left[\frac{1}{2}\left\langle\left(\left||T|^{\alpha}A\right|^4+\left||T^*|^{1-\alpha}B^*\right|^4\right)x,x\right\rangle + \left|\left\langle\left||T^*|^{1-\alpha}B^*\right|^2\left||T|^{\alpha}A\right|^2x,x\right\rangle\right|\right] \tag{2.22}$$

for all $x\in H$.

From Corollary 1 we have for all p, $q\ge 0$ *with* $p+q\ge 1$ *that*

$$\left|\left\langle BX|X|^{p+q-1}Ax,y\right\rangle\right| \le \|X\|^q\left\langle\left||X|^{\alpha p}A\right|^2x,x\right\rangle^{1/2}\left\langle\left||X^*|^{(1-\alpha)p}B^*\right|^2y,y\right\rangle^{1/2} \tag{2.23}$$

for all x, $y\in H$.

We also have

$$\left|\left\langle BX\left|X\right|^{p+q-1}Ax,x\right\rangle\right| \le \|X\|^{q}\|x\|^{2\left(1-\frac{1}{s}\right)}\left\langle \frac{\left|\left|X\right|^{\alpha p}A\right|^{2s}+\left|\left|X^{*}\right|^{(1-\alpha)p}B^{*}\right|^{2s}}{2}x,x\right\rangle^{1/s} \tag{2.24}$$

for all $x\in H$. In particular for $s=1$, we obtain

$$\left|\left\langle BX\left|X\right|^{p+q-1}Ax,x\right\rangle\right|\le\|X\|^{q}\left\langle \frac{\left|\left|X\right|^{\alpha p}A\right|^{2}+\left|\left|X^{*}\right|^{(1-\alpha)p}B^{*}\right|^{2}}{2}x,x\right\rangle \tag{2.25}$$

for all $x\in H$.

Moreover, we have

$$\begin{aligned}&\left|\left\langle BX\left|X\right|^{p+q-1}Ax,x\right\rangle\right|^{2}\\ &\le\frac{1}{2}\|x\|^{2}\|X\|^{2q}\left[\left\|\left|\left|X\right|^{\alpha p}A\right|^{2}x\right\|\left\|\left|\left|X^{*}\right|^{(1-\alpha)p}B^{*}\right|^{2}x\right\|\right.\\ &\left.+\left|\left\langle\left|\left|X^{*}\right|^{(1-\alpha)p}B^{*}\right|^{2}\left|\left|X\right|^{\alpha p}A\right|^{2}x,x\right\rangle\right|\right]\\ &\le\frac{1}{2}\|x\|^{2}\|X\|^{2q}\left[\frac{1}{2}\left\langle\left(\left|\left|X\right|^{\alpha p}A\right|^{4}+\left|\left|X^{*}\right|^{(1-\alpha)p}B^{*}\right|^{4}\right)x,x\right\rangle\right.\\ &\left.+\left|\left\langle\left|\left|X^{*}\right|^{(1-\alpha)p}B^{*}\right|^{2}\left|\left|X\right|^{\alpha p}A\right|^{2}x,x\right\rangle\right|\right]\end{aligned} \tag{2.26}$$

for all $x\in H$.

From Corollary 2 we have for all $X,\ A,\ B\in\mathcal{B}(H)$ and $\lambda,\ \alpha\in[0,1]$ that

$$\begin{aligned}&\left|\langle BXAx,y\rangle\right|\\ &\le\|X\|^{1-\lambda}\left\langle\left|\left|\left|X\right|^{\alpha\lambda}\right|A\right|^{2}x,x\right\rangle^{1/2}\left\langle\left|\left|X^{*}\right|^{(1-\alpha)\lambda}B^{*}\right|^{2}y,y\right\rangle^{1/2}\end{aligned} \tag{2.27}$$

for all $x,\ y\in H$.

We also have

$$\begin{aligned}&\left|\langle BXAx,x\rangle\right|\\ &\le\|X\|^{1-\lambda}\|x\|^{2\left(1-\frac{1}{s}\right)}\left\langle\frac{\left|\left|X\right|^{\alpha\lambda}A\right|^{2s}+\left|\left|X^{*}\right|^{(1-\alpha)\lambda}B^{*}\right|^{2s}}{2}x,x\right\rangle^{1/s}\end{aligned} \tag{2.28}$$

for all $x\in H$. In particular for $s=1$, we obtain

$$\left|\langle BXAx,x\rangle\right|\le\|X\|^{1-\lambda}\left\langle\frac{\left|\left|X\right|^{\alpha\lambda}A\right|^{2}+\left|\left|X^{*}\right|^{(1-\alpha)\lambda}B^{*}\right|^{2}}{2}x,x\right\rangle \tag{2.29}$$

for all $x\in H$.

Moreover, we have

$$
\begin{aligned}
&\left|\left\langle BXAx,x\right\rangle\right|^{2} \\
&\leq\frac{1}{2}\left\|x\right\|^{2}\left\|X\right\|^{2(1-\lambda)}\left[\left\|\left|\left|X\right|^{\alpha\lambda}A\right|^{2}x\right\|\left\|\left|\left|X^{*}\right|^{(1-\alpha)\lambda}B^{*}\right|^{2}x\right\|\right. \\
&\left.+\left|\left\langle\left|\left|X^{*}\right|^{(1-\alpha)\lambda}B^{*}\right|^{2}\left|\left|X\right|^{\alpha\lambda}A\right|^{2}x,x\right\rangle\right|\right] \\
&\leq\frac{1}{2}\left\|x\right\|^{2}\left\|X\right\|^{2(1-\lambda)}\left[\frac{1}{2}\left\langle\left(\left|\left|X\right|^{\alpha\lambda}A\right|^{4}+\left|\left|X^{*}\right|^{(1-\alpha)\lambda}B^{*}\right|^{4}\right)x,x\right\rangle\right. \\
&\left.+\left|\left\langle\left|\left|X^{*}\right|^{(1-\alpha)\lambda}B^{*}\right|^{2}\left|\left|X\right|^{\alpha\lambda}A\right|^{2}x,x\right\rangle\right|\right]
\end{aligned}
\tag{2.30}
$$

for all $x\in H$.

3. Numerical Radius Inequalities

We can state the following result:

Proposition 1. *Assume that f and g are non-negative functions on $[0,\infty)$ that are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Let T, V be operators in $\mathcal{B}(H)$ such that $|T|V=V^{*}|T|$, then for A, $B\in\mathcal{B}(H)$ we have the norm inequality*

$$
\left\|BTVA\right\|\leq r\left(V\right)\left\|f\left(\left|T\right|\right)A\right\|\left\|g\left(\left|T^{*}\right|\right)B^{*}\right\|. \tag{3.1}
$$

We also have the numerical radius inequalities

$$
\omega\left(BTVA\right)\leq r\left(V\right)\left\|\frac{\left|f\left(\left|T\right|\right)A\right|^{2s}+\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2s}}{2}\right\|^{1/s}. \tag{3.2}
$$

In particular, for $s=1$, we obtain

$$
\omega\left(BTVA\right)\leq r\left(V\right)\left\|\frac{\left|f\left(\left|T\right|\right)A\right|^{2}+\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}}{2}\right\|. \tag{3.3}
$$

Moreover, we have

$$
\begin{aligned}
\omega^{2}\left(BTVA\right)&\leq\frac{1}{2}r^{2}\left(V\right)\left[\left\|f\left(\left|T\right|\right)A\right\|^{2}\left\|g\left(\left|T^{*}\right|\right)B^{*}\right\|^{2}\right. \\
&\left.+\omega\left(\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}\left|f\left(\left|T\right|\right)A\right|^{2}\right)\right] \\
&\leq\frac{1}{2}r^{2}\left(V\right)\left[\frac{1}{2}\left\|\left|f\left(\left|T\right|\right)A\right|^{4}+\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{4}\right\|\right. \\
&\left.+\omega\left(\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}\left|f\left(\left|T\right|\right)A\right|^{2}\right)\right].
\end{aligned}
\tag{3.4}
$$

Proof. If we take the supremum over $\|x\| = \|y\| = 1$ in (2.4), then we have

$$
\begin{aligned}
\|BTVA\| &= \sup_{\|x\|=\|y\|=1} |\langle BTVAx, y\rangle| \\
&\le r(V) \sup_{\|x\|=\|y\|=1} \left[\left\langle |f(|T|)A|^2 x, x \right\rangle^{1/2} \left\langle |g(|T^*|)B^*|^2 y, y \right\rangle^{1/2} \right] \\
&= r(V) \sup_{\|x\|=1} \left\langle |f(|T|)A|^2 x, x \right\rangle^{1/2} \sup_{\|y\|=1} \left\langle |g(|T^*|)B^*|^2 y, y \right\rangle^{1/2} \\
&= r(V) \left\| |f(|T|)A|^2 \right\|^{1/2} \left\| |g(|T^*|)B^*|^2 \right\|^{1/2} \\
&= r(V) \|f(|T|)A\| \|g(|T^*|)B^*\|,
\end{aligned}
$$

which proves (3.1).

Now, if we take the supremum over $\|x\| = 1$ in (2.5), then we get

$$
\begin{aligned}
\omega(BTVA) &= \sup_{\|x\|=1} |\langle BTVAx, x\rangle| \\
&\le r(V) \sup_{\|x\|=1} \left\{ \|x\|^{2\left(1-\frac{1}{s}\right)} \left\langle \frac{|f(|T|)A|^{2s} + |g(|T^*|)B^*|^{2s}}{2} x, x \right\rangle^{1/s} \right\} \\
&= r(V) \sup_{\|x\|=1} \left\langle \frac{|f(|T|)A|^{2s} + |g(|T^*|)B^*|^{2s}}{2} x, x \right\rangle^{1/s} \\
&= r(V) \left\| \frac{|f(|T|)A|^{2s} + |g(|T^*|)B^*|^{2s}}{2} \right\|^{1/s},
\end{aligned}
$$

which proves (3.2).

The other inequalities follow in a similar manner from the corresponding vector inequalities from Theorem 2 and we omit the details. □

We have the following particular results of interest:

Corollary 3. *Assume that f and g are non-negative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Let X, A, $B \in \mathcal{B}(H)$, then for all $p, q \ge 0$ with $p + q \ge 1$, we have the norm inequality*

$$
\left\| BX |X|^{p+q-1} A \right\| \le \|X\|^q \|f(||X|^p|)A\| \|g(|X^*|^p)B^*\|. \tag{3.5}
$$

We also have the numerical radius inequalities

$$
\omega\left(BX |X|^{p+q-1} A\right) \le \|X\|^q \left\| \frac{|f(|X|^p)A|^{2s} + |g(|X^*|^p)B^*|^{2s}}{2} \right\|^{1/s}. \tag{3.6}
$$

In particular for $s = 1$, we obtain

$$
\omega\left(BX |X|^{p+q-1} A\right) \le \|X\|^q \left\| \frac{|f(|X|^p)A|^2 + |g(|X^*|^p)B^*|^2}{2} \right\| \tag{3.7}
$$

Moreover, we have

$$\omega^2\left(BX\left|X\right|^{p+q-1}A\right)\leq\frac{1}{2}\left\|X\right\|^{2q}\left[\left\|f\left(\left|X\right|^p\right)A\right\|^2\left\|g\left(\left|X^*\right|^p\right)B^*\right\|^2\right.$$
$$\left.+\omega\left(\left|g\left(\left|X^*\right|^p\right)B^*\right|^2\left|f\left(\left|X\right|^p\right)A\right|^2\right)\right]$$
$$\leq\frac{1}{2}\left\|X\right\|^{2q}\left[\frac{1}{2}\left\|\left|f\left(\left|X\right|^p\right)A\right|^4+\left|g\left(\left|X^*\right|^p\right)B^*\right|^4\right\|\right.$$
$$\left.+\omega\left(\left|g\left(\left|X^*\right|^p\right)B^*\right|^2\left|f\left(\left|X\right|^p\right)A\right|^2\right)\right]. \tag{3.8}$$

We also have:

Corollary 4. *With the assumptions of Corollary 3 we have for all X, A, $B\in\mathcal{B}(H)$ and $\lambda\in[0,1]$ that*

$$\left\|BXA\right\|\leq\left\|X\right\|^{1-\lambda}\left\|f\left(\left|X\right|^\lambda\right)A\right\|\left\|g\left(\left|X^*\right|^\lambda\right)B^*\right\|. \tag{3.9}$$

We also have

$$\omega\left(BXA\right)\leq\left\|X\right\|^{1-\lambda}\left\|\frac{\left|f\left(\left|X\right|^\lambda\right)A\right|^{2s}+\left|g\left(\left|X^*\right|^\lambda\right)B^*\right|^{2s}}{2}\right\|^{1/s} \tag{3.10}$$

In particular for $s=1$, we obtain

$$\omega\left(BXA\right)\leq\left\|X\right\|^{1-\lambda}\left\|\frac{\left|f\left(\left|X\right|^\lambda\right)A\right|^{2}+\left|g\left(\left|X^*\right|^\lambda\right)B^*\right|^{2}}{2}\right\|. \tag{3.11}$$

Moreover, we have

$$\omega^2\left(BXA\right)\leq\frac{1}{2}\left\|X\right\|^{2(1-\lambda)}\left[\left\|f\left(\left|X\right|^\lambda\right)A\right\|^2\left\|g\left(\left|X^*\right|^\lambda\right)B^*\right\|^2\right.$$
$$\left.+\omega\left(\left|g\left(\left|X^*\right|^\lambda\right)B^*\right|^2\left|f\left(\left|X\right|^\lambda\right)A\right|^2\right)\right]$$
$$\leq\frac{1}{2}\left\|X\right\|^{2(1-\lambda)}\left[\frac{1}{2}\left\|\left|f\left(\left|X\right|^\lambda\right)A\right|^4+\left|g\left(\left|X^*\right|^\lambda\right)B^*\right|^4\right\|\right.$$
$$\left.+\omega\left(\left|g\left(\left|X^*\right|^\lambda\right)B^*\right|^2\left|f\left(\left|X\right|^\lambda\right)A\right|^2\right)\right]. \tag{3.12}$$

If we take $f(t)=t^\alpha$, $g(t)=t^{1-\alpha}$ with $\alpha\in[0,1]$ in Proposition 1, then we get

$$\left\|BTVA\right\|\leq r\left(V\right)\left\|\left|T\right|^\alpha A\right\|\left\|\left|T^*\right|^{1-\alpha}B^*\right\|. \tag{3.13}$$

We also have

$$\omega\left(BTVA\right)\leq r\left(V\right)\left\|\frac{\left|\left|T\right|^\alpha A\right|^{2s}+\left|\left|T^*\right|^{1-\alpha}B^*\right|^{2s}}{2}\right\|^{1/s}. \tag{3.14}$$

In particular for $s = 1$, we obtain

$$\omega(BTVA) \leq r(V) \left\| \frac{\left|\left|T\right|^{\alpha} A\right|^{2} + \left|\left|T^{*}\right|^{1-\alpha} B^{*}\right|^{2}}{2} \right\|. \tag{3.15}$$

Moreover, we have

$$\begin{aligned} \omega^{2}(BTVA) &\leq \frac{1}{2} r^{2}(V) \left[\left\|\left|T\right|^{\alpha} A\right\|^{2} \left\|\left|T^{*}\right|^{1-\alpha} B^{*}\right\|^{2} \right. \\ &\left. + \omega\left(\left|\left|T^{*}\right|^{1-\alpha} B^{*}\right|^{2} \left|\left|T\right|^{\alpha} A\right|^{2} \right) \right] \\ &\leq \frac{1}{2} r^{2}(V) \left[\frac{1}{2} \left\| \left|\left|T\right|^{\alpha} A\right|^{4} + \left|\left|T^{*}\right|^{1-\alpha} B^{*}\right|^{4} \right\| \right. \\ &\left. + \omega\left(\left|\left|T^{*}\right|^{1-\alpha} B^{*}\right|^{2} \left|\left|T\right|^{\alpha} A\right|^{2} \right) \right]. \end{aligned} \tag{3.16}$$

From Corollary 1 we have for all $p, q \geq 0$ with $p + q \geq 1$ that

$$\left\| BX\left|X\right|^{p+q-1} A \right\| \leq \left\|X\right\|^{q} \left\|\left|X\right|^{\alpha p} A\right\| \left\|\left|X^{*}\right|^{(1-\alpha)p} B^{*}\right\|. \tag{3.17}$$

We also have

$$\omega\left(BX\left|X\right|^{p+q-1} A\right) \leq \left\|X\right\|^{q} \left\| \frac{\left|\left|X\right|^{\alpha p} A\right|^{2s} + \left|\left|X^{*}\right|^{(1-\alpha)p} B^{*}\right|^{2s}}{2} \right\|^{1/s}. \tag{3.18}$$

In particular for $s = 1$, we obtain

$$\omega\left(BX\left|X\right|^{p+q-1} A\right) \leq \left\|X\right\|^{q} \left\| \frac{\left|\left|X\right|^{\alpha p} A\right|^{2} + \left|\left|X^{*}\right|^{(1-\alpha)p} B^{*}\right|^{2}}{2} \right\|. \tag{3.19}$$

Moreover, we have

$$\begin{aligned} &\omega^{2}\left(BX\left|X\right|^{p+q-1} A\right) \\ &\leq \frac{1}{2} \left\|X\right\|^{2q} \left[\left\|\left|X\right|^{\alpha p} A\right\|^{2} \left\|\left|X^{*}\right|^{(1-\alpha)p} B^{*}\right\|^{2} \right. \\ &\left. + \omega\left(\left|\left|X^{*}\right|^{(1-\alpha)p} B^{*}\right|^{2} \left|\left|X\right|^{\alpha p} A\right|^{2} \right) \right] \\ &\leq \frac{1}{2} \left\|X\right\|^{2q} \left[\frac{1}{2} \left\| \left|\left|X\right|^{\alpha p} A\right|^{4} + \left|\left|X^{*}\right|^{(1-\alpha)p} B^{*}\right|^{4} \right\| \right. \\ &\left. + \omega\left(\left|\left|X^{*}\right|^{(1-\alpha)p} B^{*}\right|^{2} \left|\left|X\right|^{\alpha p} A\right|^{2} \right) \right]. \end{aligned} \tag{3.20}$$

From Corollary 2 we have for all X, A, $B \in \mathcal{B}(H)$ and λ, $\alpha \in [0, 1]$ that

$$\left\|BXA\right\| \leq \left\|X\right\|^{1-\lambda} \left\| \left|\left|X\right|^{\alpha\lambda}\right| A \right\| \left\|\left|X^{*}\right|^{(1-\alpha)\lambda} B^{*}\right\|. \tag{3.21}$$

We also have

$$\omega(BXA) \leq \|X\|^{1-\lambda} \left\| \frac{\left||X|^{\alpha\lambda} A\right|^{2s} + \left||X^*|^{(1-\alpha)\lambda} B^*\right|^{2s}}{2} \right\|^{1/s}. \tag{3.22}$$

In particular for $s = 1$, we obtain

$$\omega(BXA) \leq \|X\|^{1-\lambda} \left\| \frac{\left||X|^{\alpha\lambda} A\right|^{2} + \left||X^*|^{(1-\alpha)\lambda} B^*\right|^{2}}{2} \right\|. \tag{3.23}$$

Moreover, we have

$$\begin{aligned} &\omega^2(BXA) \\ &\leq \frac{1}{2} \|X\|^{2(1-\lambda)} \left[\left\| |X|^{\alpha\lambda} A \right\|^2 \left\| |X^*|^{(1-\alpha)\lambda} B^* \right\|^2 \right. \\ &\left. +\omega \left(\left| |X^*|^{(1-\alpha)\lambda} B^* \right|^2 \left| |X|^{\alpha\lambda} A \right|^2 \right) \right] \\ &\leq \frac{1}{2} \|X\|^{2(1-\lambda)} \left[\frac{1}{2} \left\| \left| |X|^{\alpha\lambda} A \right|^4 + \left| |X^*|^{(1-\alpha)\lambda} B^* \right|^4 \right\| \right. \\ &\left. +\omega \left(\left| |X^*|^{(1-\alpha)\lambda} B^* \right|^2 \left| |X|^{\alpha\lambda} A \right|^2 \right) \right]. \end{aligned} \tag{3.24}$$

If the operator T has the polar decomposition $T = U|T|$ with U a partial isometry, we define the transform

$$\Delta_{p,q}(T) := |T|^p U |T|^q,$$

for $p, q \geq 0$. Here we assume that $|T|^0 = I$.

The *p-generalized Dougal transform* is defined by

$$\widehat{T}_p := |T|^p U,$$

the usual *Dougal transform* is then

$$\widehat{T} := |T| U,$$

and the *p-generalized Aluthge transform*

$$\widetilde{T}_p := |T|^p U |T|^p,$$

which for $p = 1/2$ gives the usual Aluthge transform

$$\widetilde{T} := |T|^{1/2} U |T|^{1/2},$$

Also

$$T_q := U |T|^q,$$

which gives for $q = 1$ the usual polar decomposition $T = U|T|$.

For $p = t$, $q = 1 - t$, where $t \in [0, 1]$ we have

$$\Delta_t(T) := \Delta_{t,1-t}(T) = |T|^t V |T|^{1-t}.$$

The transform $\Delta_t(T)$ was introduced and studied in [6].

Now, if we use Corollary 4 for $X = \Delta_{p,q}(T)$ and $A = |T|^m$ and $B = |T|^n$ for $p, q, m, n \geq 0$, then we get

$$\begin{aligned} &\|\Delta_{p+m,q+n}(T)\| \\ &\leq \|\Delta_{p,q}(T)\|^{1-\lambda} \left\| f\left(|\Delta_{p,q}(T)|^{\lambda}\right) |T|^m \right\| \left\| g\left(\left|(\Delta_{p,q}(T))^*\right|^{\lambda}\right) |T|^n \right\|. \end{aligned} \tag{3.25}$$

We also have

$$\begin{aligned} &\omega(\Delta_{p+m,q+n}(T)) \\ &\leq \|\Delta_{p,q}(T)\|^{1-\lambda} \\ &\times \left\| \frac{\left| f\left(|\Delta_{p,q}(T)|^{\lambda}\right) |T|^m \right|^{2s} + \left| g\left(\left|(\Delta_{p,q}(T))^*\right|^{\lambda}\right) |T|^n \right|^{2s}}{2} \right\|^{1/s}. \end{aligned} \tag{3.26}$$

In particular, for $s = 1$, we obtain

$$\begin{aligned} &\omega(\Delta_{p+m,q+n}(T)) \\ &\leq \|\Delta_{p,q}(T)\|^{1-\lambda} \left\| \frac{\left| f\left(|\Delta_{p,q}(T)|^{\lambda}\right) |T|^m \right|^{2} + \left| g\left(\left|(\Delta_{p,q}(T))^*\right|^{\lambda}\right) |T|^n \right|^{2}}{2} \right\|. \end{aligned} \tag{3.27}$$

Moreover, we have

$$\begin{aligned} &\omega^2(\Delta_{p+m,q+n}(T)) \\ &\leq \frac{1}{2} \|\Delta_{p,q}(T)\|^{2(1-\lambda)} \left[\left\| f\left(|\Delta_{p,q}(T)|^{\lambda}\right) |T|^m \right\|^2 \left\| g\left(\left|(\Delta_{p,q}(T))^*\right|^{\lambda}\right) |T|^n \right\|^2 \right. \\ &\left. + \omega\left(\left| g\left(\left|(\Delta_{p,q}(T))^*\right|^{\lambda}\right) |T|^n \right|^2 \left| f\left(|\Delta_{p,q}(T)|^{\lambda}\right) |T|^m \right|^2 \right) \right] \\ &\leq \frac{1}{2} \|\Delta_{p,q}(T)\|^{2(1-\lambda)} \\ &\times \left[\frac{1}{2} \left\| \left| f\left(|\Delta_{p,q}(T)|^{\lambda}\right) |T|^m \right|^4 + \left| g\left(\left|(\Delta_{p,q}(T))^*\right|^{\lambda}\right) |T|^n \right|^4 \right\| \right. \\ &\left. + \omega\left(\left| g\left(\left|(\Delta_{p,q}(T))^*\right|^{\lambda}\right) |T|^n \right|^2 \left| f\left(|\Delta_{p,q}(T)|^{\lambda}\right) |T|^m \right|^2 \right) \right]. \end{aligned} \tag{3.28}$$

By taking some particular values for $p, q, m, n \geq 0$ we can obtain certain inequalities for the Aluthge and Dougal transforms. The details are omitted.

4. Inequalities for p-Schatten Norms

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H; \langle ., . \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{4.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{4.2}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (4.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{4.3}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT,\ TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \ \text{ and } \ |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{4.4}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A,\ B \in \mathcal{B}_2(H)$ *then* $AB,\ BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite [16, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} = \left(\sum_{i \in I} \langle |A|^p e_i, e_i \rangle \right)^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{4.5}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{4.6}$$

For $p \ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [16, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{4.7}$$

$$\|AB\|_p \le \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{4.8}$$

and

$$\|AB\|_p \le \|A\|_p \|B\|, \ \|BA\|_p \le \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{4.9}$$

This implies that

$$\|CAB\|_p \le \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{4.10}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \le) \|AB\|_1 \le \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{4.11}$$

For the theory of trace functionals and their applications the reader is referred to [14] and [16].

Proposition 2. *Assume that f and g are non-negative functions on $[0,\infty)$ that are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Let T, V be operators in $\mathcal{B}(H)$ such that $|T|V=V^*|T|$, then for A, $B\in\mathcal{B}(H)$ with $f(|T|)A$, $g(|T^*|)B^*\in\mathcal{B}_2(H)$, we have that $BTVA\in\mathcal{B}_1(H)$ and*

$$|\mathrm{tr}(BTVA)|\le r(V)\|f(|T|)A\|_2\|g(|T^*|)B^*\|_2. \tag{4.12}$$

Proof. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, then by (2.4) for $y=x=e_i$, $i\in I$, we get $i\in I$

$$|\langle BTVAe_i,e_i\rangle|\le r(V)\left\langle|f(|T|)A|^2e_i,e_i\right\rangle^{1/2}\left\langle|g(|T^*|)B^*|^2e_i,e_i\right\rangle^{1/2}$$

for $i\in I$.

Therefore, by the triangle and Cauchy-Schwarz inequality, we derive

$$\begin{aligned}|\mathrm{tr}(BTVA)|&=\left|\sum_{i\in I}\langle BTVAe_i,e_i\rangle\right|\le\sum_{i\in I}|\langle BTVAe_i,e_i\rangle|\\&\le r(V)\sum_{i\in I}\left\langle|f(|T|)A|^2e_i,e_i\right\rangle^{1/2}\left\langle|g(|T^*|)B^*|^2e_i,e_i\right\rangle^{1/2}\\&\le r(V)\left(\sum_{i\in I}\left\langle|f(|T|)A|^2e_i,e_i\right\rangle\right)^{1/2}\left\langle\sum_{i\in I}|g(|T^*|)B^*|^2e_i,e_i\right\rangle^{1/2}\\&=r(V)\|f(|T|)A\|_2\|g(|T^*|)B^*\|_2,\end{aligned}$$

which proves (4.12). □

Corollary 5. *Let $p,q\ge0$ with $p+q\ge1$ and such that X, A, $B\in\mathcal{B}(H)$ with $f(|X|^p)A$, $g(|X^*|^p)B^*\in\mathcal{B}_2(H)$, then $BX|X|^{p+q-1}A\in\mathcal{B}_1(H)$ and*

$$\left|\mathrm{tr}\left(BX|X|^{p+q-1}A\right)\right|\le\|X\|^q\|f(|X|^p)A\|_2\|g(|X^*|^p)B^*\|_2. \tag{4.13}$$

If X, A, $B\in\mathcal{B}(H)$ and $\lambda\in[0,1]$ such that $f\left(|X|^\lambda\right)A$, $g\left(|X^*|^\lambda\right)B^*\in\mathcal{B}_2(H)$, then $BXA\in\mathcal{B}_1(H)$ and

$$|\mathrm{tr}(BXA)|\le\|X\|^{1-\lambda}\left\|f\left(|X|^\lambda\right)A\right\|_2\left\|g\left(|X^*|^\lambda\right)B^*\right\|_2. \tag{4.14}$$

If we take $f(t)=t^\alpha$, $g(t)=t^{1-\alpha}$ with $\alpha\in[0,1]$, then we have for $|T|^\alpha A$, $|T^*|^{1-\alpha}B^*\in\mathcal{B}_2(H)$, that

$$|\mathrm{tr}(BTVA)|\le r(V)\||T|^\alpha A\|_2\left\||T^*|^{1-\alpha}B^*\right\|_2 \tag{4.15}$$

provided that $|T|V=V^*|T|$.

Also, if $|X|^{\alpha p}A$, $|X^*|^{(1-\alpha)p}B^*\in\mathcal{B}_2(H)$, then $BX|X|^{p+q-1}A\in\mathcal{B}_1(H)$ and

$$\left|\mathrm{tr}\left(BX|X|^{p+q-1}A\right)\right|\le\|X\|^q\||X|^{\alpha p}A\|_2\left\||X^*|^{(1-\alpha)p}B^*\right\|_2. \tag{4.16}$$

Moreover, if $|X|^{\alpha\lambda}A$, $|X^*|^{(1-\alpha)\lambda}B^*\in\mathcal{B}_2(H)$, then

$$|\mathrm{tr}(BXA)|\le\|X\|^{1-\lambda}\left\||X|^{\alpha\lambda}A\right\|_2\left\||X^*|^{(1-\alpha)\lambda}B^*\right\|_2. \tag{4.17}$$

For $\mathcal{E} := \{e_i\}_{i\in I}$ an orthonormal basis of H we define for $A \in \mathcal{B}_p(H)$, $p \geq 1$

$$\|A\|_{\mathcal{E},p} := \left(\sum_{i\in I} |\langle Ae_i, e_i\rangle|^p\right)^{1/p}.$$

We observe that $\|\cdot\|_{\mathcal{E},p}$ is a norm on $\mathcal{B}_p(H)$ and $\|A\|_{\mathcal{E},p} \leq \|A\|_p$ for $A \in \mathcal{B}_p(H)$.

We also have:

Proposition 3. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Let T, V be operators in $\mathcal{B}(H)$ such that $|T|V = V^*|T|$, then for A, $B \in \mathcal{B}(H)$ with $f(|T|)A$, $g(|T^*|)B^* \in \mathcal{B}_{2p}(H)$, $s \geq 1$, we have that*

$$\|BTVA\|_{\mathcal{E},p}^s \leq \frac{1}{2}r^s(V)\left(\|f(|T|)A\|_{2s}^{2s} + \|g(|T^*|)B^*\|_{2s}^{2s}\right). \tag{4.18}$$

In particular, for $s = 1$, we get

$$\|BTVA\|_{\mathcal{E},1} \leq \frac{1}{2}r(V)\left(\|f(|T|)A\|_2^2 + \|g(|T^*|)B^*\|_2^2\right). \tag{4.19}$$

Proof. If $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis of H we have by (2.5) that

$$|\langle BTVAe_i, e_i\rangle|^s \leq r^s(V)\left\langle \frac{|f(|T|)A|^{2s} + |g(|T^*|)B^*|^{2s}}{2}e_i, e_i\right\rangle$$

for all $i \in I$.

If we sum over $i \in I$, then we get

$$\begin{aligned}
\|BTVA\|_{\mathcal{E},p}^s &= \sum_{i\in I}|\langle BTVAe_i, e_i\rangle|^s \\
&\leq r^s(V)\sum_{i\in I}\left\langle \frac{|f(|T|)A|^{2s} + |g(|T^*|)B^*|^{2s}}{2}e_i, e_i\right\rangle \\
&= \frac{1}{2}r^s(V)\left[\sum_{i\in I}\left\langle |f(|T|)A|^{2s}e_i, e_i\right\rangle + \sum_{i\in I}\left\langle |g(|T^*|)B^*|^{2s}e_i, e_i\right\rangle\right] \\
&= \frac{1}{2}r^s(V)\left(\|f(|T|)A\|_{2s}^{2s} + \|g(|T^*|)B^*\|_{2s}^{2s}\right),
\end{aligned}$$

which proves (4.18). □

Corollary 6. *Let p, $q \geq 0$ with $p + q \geq 1$ and such that X, A, $B \in \mathcal{B}(H)$ with $f(|X|^p)A$, $g(|X^*|^p)B^* \in \mathcal{B}_{2s}(H)$, $s \geq 1$, then*

$$\left\|BX|X|^{p+q-1}A\right\|_{\mathcal{E},s}^s \leq \frac{1}{2}\|X\|^{sq}\left(\|f(|X|^p)A\|_{2s}^{2s} + \|g(|X^*|^p)B^*\|_{2s}^{2s}\right) \tag{4.20}$$

and, in particular,

$$\left\|BX|X|^{p+q-1}A\right\|_{\mathcal{E},1} \leq \frac{1}{2}\|X\|^q\left(\|f(|X|^p)A\|_2^2 + \|g(|X^*|^p)B^*\|_2^2\right). \tag{4.21}$$

If X, A, $B \in \mathcal{B}(H)$ and $\lambda \in [0,1]$ such that $f\left(|X|^\lambda\right)A$, $g\left(|X^*|^\lambda\right)B^* \in \mathcal{B}_{2s}(H)$, $s \geq 1$, then

$$\|BXA\|_{\mathcal{E},s}^s \leq \|X\|^{(1-\lambda)s}\left(\left\|f\left(|X|^\lambda\right)A\right\|_{2s}^{2s} + \left\|g\left(|X^*|^\lambda\right)B^*\right\|_{2s}^{2s}\right) \tag{4.22}$$

and, in particular

$$\|BXA\|_{\mathcal{E},1} \leq \|X\|^{1-\lambda} \left(\left\| f\left(|X|^{\lambda}\right) A \right\|_2^2 + \left\| g\left(|X^*|^{\lambda}\right) B^* \right\|_2^2 \right). \tag{4.23}$$

If we take $f(t) = t^{\alpha}$, $g(t) = t^{1-\alpha}$ with $\alpha \in [0,1]$ in Proposition 3, then we have for $|T|^{\alpha} A$, $|T^*|^{1-\alpha} B^* \in \mathcal{B}_{2s}(H)$, that

$$\|BTVA\|_{\mathcal{E},p}^s \leq \frac{1}{2} r^s(V) \left(\||T|^{\alpha} A\|_{2s}^{2s} + \left\| |T^*|^{1-\alpha} B^* \right\|_{2s}^{2s} \right) \tag{4.24}$$

provided that $|T| V = V^* |T|$. For $s = 1$ we derive that

$$\|BTVA\|_{\mathcal{E},p} \leq \frac{1}{2} r(V) \left(\||T|^{\alpha} A\|_2^2 + \left\| |T^*|^{1-\alpha} B^* \right\|_2^2 \right). \tag{4.25}$$

If we take these functions in Corollary 6, then we get the power inequalities

$$\left\| BX |X|^{p+q-1} A \right\|_{\mathcal{E},s}^s \leq \frac{1}{2} \|X\|^{sq} \left(\||X|^{\alpha p} A\|_{2s}^{2s} + \left\| |X^*|^{(1-\alpha)p} B^* \right\|_{2s}^{2s} \right) \tag{4.26}$$

and, in particular,

$$\left\| BX |X|^{p+q-1} A \right\|_{\mathcal{E},1} \leq \frac{1}{2} \|X\|^{q} \left(\||X|^{\alpha p} A\|_{2}^{2} + \left\| |X^*|^{(1-\alpha)p} B^* \right\|_{2}^{2} \right). \tag{4.27}$$

If X, A, $B \in \mathcal{B}(H)$ and $\lambda \in [0,1]$ such that $|X|^{\alpha\lambda} A$, $|X^*|^{(1-\alpha)\lambda} B^* \in \mathcal{B}_{2s}(H)$, $s \geq 1$, then

$$\|BXA\|_{\mathcal{E},s}^s \leq \|X\|^{(1-\lambda)s} \left(\left\| |X|^{\alpha\lambda} A \right\|_{2s}^{2s} + \left\| |X^*|^{(1-\alpha)\lambda} B^* \right\|_{2s}^{2s} \right) \tag{4.28}$$

and, in particular

$$\|BXA\|_{\mathcal{E},1} \leq \|X\|^{1-\lambda} \left(\left\| |X|^{\alpha\lambda} A \right\|_{2}^{2} + \left\| |X^*|^{(1-\alpha)\lambda} B^* \right\|_{2}^{2} \right). \tag{4.29}$$

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