ON SOME INEQUALITIES FOR NUMERICAL RADIUS OF OPERATOR PRODUCTS IN HILBERT SPACES

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Abstract. Let A be a nonnegative operator on H and D, B, C three bounded operators on H. Then we have the inequality

$$w\left(C^*ADB^*AC\right) \leq \frac{1}{2}\left\|A^{1/2}C\right\|^2\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\| + \|B^*AD\|\right].$$

If $B^*AC = C^*AD$, then also

$$w^2\left(C^*AD\right) \leq \frac{1}{2}\left\|A^{1/2}C\right\|^2\left[\left\|\frac{\left|A^{1/2}D\right|^2 + \left|A^{1/2}B\right|^2}{2}\right\| + w\left(B^*AD\right)\right].$$

Some applications for the trace of operators are also given.

1. Introduction

Let $\mathbb{K}$ be the field of real or complex numbers, i.e., $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and X be a linear space over $\mathbb{K}$.

Definition 1. *A functional $(\cdot,\cdot) : X \times X \to \mathbb{K}$ is said to be a Hermitian form on X if*

(H1) $(ax + by, z) = a(x, z) + b(y, z)$ *for $a, b \in \mathbb{K}$ and $x, y, z \in X$;*
(H2) $(x, y) = \overline{(y, x)}$ *for all $x, y \in X$.*

The functional $(\cdot,\cdot)$ is said to be *positive semi-definite* on a subspace Y of X if

(H3) $(y, y) \geq 0$ for every $y \in Y$,

and *positive definite* on Y if it is positive semi-definite on Y and

(H4) $(y, y) = 0$, $y \in Y$ implies $y = 0$.

The functional $(\cdot,\cdot)$ is said to be *definite* on Y provided that either $(\cdot,\cdot)$ or $-(\cdot,\cdot)$ is positive semi-definite on Y.

When a Hermitian functional $(\cdot,\cdot)$ is positive-definite on the whole space X, then, as usual, we will call it an *inner product* on X and will denote it by $\langle\cdot,\cdot\rangle$.

We use the following notations related to a given Hermitian form $(\cdot,\cdot)$ on X :

$$X_0 := \{x \in X \mid (x, x) = 0\},\ K := \{x \in X \mid (x, x) < 0\}$$

and, for a given $z \in X$,

$$X^{(z)} := \{x \in X \mid (x, z) = 0\} \quad \text{and} \quad L(z) := \{az \mid a \in \mathbb{K}\}.$$

The following fundamental facts concerning Hermitian forms hold:

Theorem 1 (Kurepa, 1968 [15]). *Let X and $(\cdot,\cdot)$ be as above.*

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(1) *If* $e \in X$ *is such that* $(e,e) \neq 0$, *then we have the decomposition*

$$X = L(e) \bigoplus X^{(e)}, \tag{1.1}$$

where $\bigoplus$ *denotes the direct sum of the linear subspaces* $X^{(e)}$ *and* $L(e)$;

(2) *If the functional* $(\cdot,\cdot)$ *is positive semi-definite on* $X^{(e)}$ *for at least one* $e \in K$, *then* $(\cdot,\cdot)$ *is positive semi-definite on* $X^{(f)}$ *for each* $f \in K$;

(3) *The functional* $(\cdot,\cdot)$ *is positive semi-definite on* $X^{(e)}$ *with* $e \in K$ *if and only if the inequality*

$$|(x,y)|^2 \geq (x,x)(y,y) \tag{1.2}$$

holds for all $x \in K$ *and all* $y \in X$;

(4) *The functional* $(\cdot,\cdot)$ *is semi-definite on* X *if and only if the Schwarz's inequality*

$$|(x,y)|^2 \leq (x,x)(y,y) \tag{1.3}$$

holds for all $x, y \in X$;

(5) *The case of equality holds in (1.3) for* $x, y \in X$ *and in (1.2), for* $x \in K$, $y \in X$, *respectively; if and only if there exists a scalar* $a \in \mathbb{K}$ *such that*

$$y - ax \in X_0^{(x)} := X_0 \cap X^{(x)}.$$

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, *nonnegative* forms on X.

If $(\cdot,\cdot) \in \mathcal{H}(X)$, then the functional $\|\cdot\| = (\cdot,\cdot)^{\frac{1}{2}}$ is a *semi-norm* on X and the following equivalent versions of Schwarz's inequality hold:

$$\|x\|^2 \|y\|^2 \geq |(x,y)|^2 \quad \text{or} \quad \|x\| \|y\| \geq |(x,y)| \tag{1.4}$$

for any $x, y \in X$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$, i.e.,

(e) $(\cdot,\cdot)_1, (\cdot,\cdot)_2 \in \mathcal{H}(X)$ implies that $(\cdot,\cdot)_1 + (\cdot,\cdot)_2 \in \mathcal{H}(X)$;
(ee) $\alpha \geq 0$ and $(\cdot,\cdot) \in \mathcal{H}(X)$ implies that $\alpha(\cdot,\cdot) \in \mathcal{H}(X)$.

The following simple result is of interest in itself as well:

Lemma 1. *Let* X *be a linear space over the real or complex number field* $\mathbb{K}$ *and* $(\cdot,\cdot)$ *a nonnegative Hermitian form on* X. *If* $y \in X$ *is such that* $(y,y) \neq 0$, *then*

$$p_y : H \times H \to \mathbb{K}, \; p_y(x,z) = (x,z)\|y\|^2 - (x,y)(y,z) \tag{1.5}$$

is also a nonnegative Hermitian form on X.

We have the inequalities

$$\left(\|x\|^2 \|y\|^2 - |(x,y)|^2\right)\left(\|y\|^2 \|z\|^2 - |(y,z)|^2\right) \geq \left|(x,z)\|y\|^2 - (x,y)(y,z)\right|^2 \tag{1.6}$$

and

$$\left(\|x+z\|^2 \|y\|^2 - |(x+z,y)|^2\right)^{\frac{1}{2}} \leq \left(\|x\|^2 \|y\|^2 - |(x,y)|^2\right)^{\frac{1}{2}} + \left(\|y\|^2 \|z\|^2 - |(y,z)|^2\right)^{\frac{1}{2}} \tag{1.7}$$

for any $x, y, z \in X$.

Remark 1. *The case when* $(\cdot,\cdot)$ *is an inner product in Lemma 1 was obtained in 1985 by S. S. Dragomir,* [2].

Remark 2. *Putting* $z = \lambda y$ *in (1.7), we get:*

$$0 \le \|x + \lambda y\|^2 \|y\|^2 - |(x + \lambda y, y)|^2 \le \|x\|^2 \|y\|^2 - |(x, y)|^2 \tag{1.8}$$

and, in particular,

$$0 \le \|x \pm y\|^2 \|y\|^2 - |(x \pm y, y)|^2 \le \|x\|^2 \|y\|^2 - |(x, y)|^2 \tag{1.9}$$

for every $x, y \in H$.

We note here that the inequality (1.8) is in fact equivalent to the following statement

$$\sup_{\lambda\in\mathbb{K}} \left[\|x + \lambda y\|^2 \|y\|^2 - |(x + \lambda y, y)|^2\right] = \|x\|^2 \|y\|^2 - |(x, y)|^2 \tag{1.10}$$

for each $x, y \in H$.

The following result holds (see [5, p. 38] for the case of inner product):

Theorem 2. *Let* X *be a linear space over the real or complex number field* $\mathbb{K}$ *and* $(\cdot,\cdot)$ *a nonnegative Hermitian form on* X. *For any* $x, y, z \in X$, *the following refinement of the Schwarz inequality holds:*

$$\begin{aligned} \|x\| \|z\| \|y\|^2 &\ge \left|(x, z) \|y\|^2 - (x, y)(y, z)\right| + |(x, y)(y, z)| \\ &\ge |(x, z)| \|y\|^2 . \end{aligned} \tag{1.11}$$

Corollary 1. *For any* $x, y, z \in X$ *we have*

$$\frac{1}{2}\left[\|x\| \|z\| + |(x, z)|\right] \|y\|^2 \ge |(x, y)(y, z)| . \tag{1.12}$$

The inequality (1.12) follows from the first inequality in (1.11) and the triangle inequality for modulus

$$\left|(x, z) \|y\|^2 - (x, y)(y, z)\right| \ge |(x, y)(y, z)| - \|y\|^2 |(x, z)|$$

for any $x, y, z \in X$.

Remark 3. *We observe that if* $(\cdot,\cdot)$ *is an inner product, then (1.12) reduces to Buzano's inequality obtained in 1974* [1] *in a different way.*

For some inequalities in inner product spaces and operators on Hilbert spaces see [3]-[13] and the references therein.

The *numerical radius* $w(T)$ of an operator T on H is given by [14, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\} . \tag{1.13}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators $T : H \to H$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [14, p. 9]:

Theorem 3 (Equivalent norm)**.** *For any* $T \in \mathcal{B}(H)$ *one has*

$$w(T) \le \|T\| \le 2w(T) . \tag{1.14}$$

Utilizing Buzano's inequality we obtained the following inequality for the numerical radius [6] or [7]:

Theorem 4. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space and $T:H\to H$ a bounded linear operator on H. Then*

$$w^2(T)\le\frac{1}{2}\left[w\left(T^2\right)+\|T\|^2\right]. \tag{1.15}$$

The constant $\frac{1}{2}$ is best possible in (1.15).

The following general result for the product of two operators holds [14, p. 37]:

Theorem 5. *If U,V are two bounded linear operators on the Hilbert space $(H,\langle\cdot,\cdot\rangle)$, then $w(UV)\le 4w(U)\,w(V)$. In the case that $UV=VU$, then $w(UV)\le 2w(U)\,w(V)$. The constant 2 is best possible here.*

The following results are also well known [14, p. 38].

Theorem 6. *If U is a unitary operator that commutes with another operator V, then*

$$w(UV)\le w(V). \tag{1.16}$$

If U is an isometry and $UV=VU$, then (1.16) also holds true.

We say that U and V *double commute* if $UV=VU$ and $UV^*=V^*U$. The following result holds [14, p. 38].

Theorem 7. *If the operators U and V double commute, then*

$$w(UV)\le w(V)\|U\|. \tag{1.17}$$

As a consequence of the above, we have [14, p. 39]:

Corollary 2. *Let U be a normal operator commuting with V. Then*

$$w(UV)\le w(U)\,w(V). \tag{1.18}$$

For a recent survey of inequalities for numerical radius, see [12] and the references therein.

Motivated by the above facts we establish in this paper some new numerical radius inequalities concerning four operators B,C,D and A on a Hilbert space with A nonnegative in the operator order. Some particular cases of interest that generalize and improve an earlier result are also provided. Applications for the trace of operators are also given.

2. Main Results

The following result holds for $(H,\langle.,.\rangle)$ a Hilbert space over the real or complex numbers field $\mathbb{K}$.

Theorem 8. *Let A be a nonnegative operator on H and D,B,C three bounded operators on H. Then for any $e\in H$ we have the inequalities*

$$\begin{aligned}|\langle C^*ABD^*ACe,e\rangle| &\le \|D^*ACe\|\,\|B^*ACe\| \\ &\le \frac{1}{2}\left\|A^{1/2}Ce\right\|^2\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\|+\|B^*AD\|\right].\end{aligned} \tag{2.1}$$

Moreover, we have

$$w(C^*ADB^*AC)\le\frac{1}{2}\left\|A^{1/2}C\right\|^2\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\|+\|B^*AD\|\right]. \tag{2.2}$$

Proof. We observe that if $A \geq 0$, then the mapping $(.,.) : H \times H \to \mathbb{K}$ defined by

$$(x,y)_A := \langle Ax, y\rangle$$

is a Hermitian form on H and by (1.12) we have the inequality

$$\frac{1}{2}\left[\|x\|_A \|y\|_A + |(x,y)_A|\right] \|e\|_A^2 \geq |(x,e)_A (y,e)_A| \tag{2.3}$$

for any $x, y, e \in H$.

This can be written as

$$\frac{1}{2}\left[\langle Ax, x\rangle^{1/2} \langle Ay, y\rangle^{1/2} + |\langle Ax, y\rangle|\right] \langle Ae, e\rangle \geq |\langle Ax, e\rangle \langle Ay, e\rangle| \tag{2.4}$$

for any $x, y, e \in H$.

Now if we replace x by Dx, y by By and e by Ce we get

$$\begin{aligned} &\frac{1}{2}\left[\langle ADx, Dx\rangle^{1/2} \langle ABy, By\rangle^{1/2} + |\langle ADx, By\rangle|\right] \langle ACe, Ce\rangle \\ &\geq |\langle ADx, Ce\rangle \langle ABy, Ce\rangle| \end{aligned} \tag{2.5}$$

for any $x, y, e \in H$, which is equivalent to

$$\begin{aligned} &\frac{1}{2}\left[\langle D^*ADx, x\rangle^{1/2} \langle B^*ABy, y\rangle^{1/2} + |\langle B^*ADx, y\rangle|\right] \langle C^*ACe, e\rangle \\ &\geq |\langle x, D^*ACe\rangle \langle y, B^*ACe\rangle| \end{aligned} \tag{2.6}$$

for any $x, y, e \in H$.

Taking the supremum over $x, y \in H$ with $\|x\| = \|y\| = 1$ we have

$$\begin{aligned} &\|D^*ACe\| \|B^*ACe\| \\ &= \sup_{\|x\|=1} |\langle x, D^*ACe\rangle| \sup_{\|y\|=1} |\langle y, B^*ACe\rangle| \\ &= \sup_{\|x\|=\|y\|=1} \{|\langle x, D^*ACe\rangle \langle y, B^*ACe\rangle|\} \\ &\leq \frac{1}{2}\langle C^*ACe, e\rangle \\ &\times \sup_{\|x\|=\|y\|=1} \left[\langle D^*ADx, x\rangle^{1/2} \langle B^*ABy, y\rangle^{1/2} + |\langle B^*ADx, y\rangle|\right] \\ &\leq \frac{1}{2}\langle C^*ACe, e\rangle \\ &\times \left[\sup_{\|x\|=1} \langle D^*ADx, x\rangle^{1/2} \sup_{\|y\|=1} \langle B^*ABy, y\rangle^{1/2} + \sup_{\|x\|=\|y\|=1} |\langle B^*ADx, y\rangle|\right] \\ &= \frac{1}{2}\langle C^*ACe, e\rangle \left[\|D^*AD\|^{1/2} \|B^*AB\|^{1/2} + \|B^*AD\|\right] \end{aligned} \tag{2.7}$$

for any $e \in H$.

Since

$$D^*AD = \left|A^{1/2}D\right|^2, \; B^*AB = \left|A^{1/2}B\right|^2$$

and

$$C^*AC = \left|A^{1/2}C\right|^2$$

then by (2.7) we get the desired inequality in (2.1).

By Schwarz inequality we have

$$|\langle C^*ABD^*ACe, e\rangle| \leq \|D^*ACe\| \|B^*ACe\| \tag{2.8}$$

for any $e \in H$.

Using inequality (2.1) we then have

$$|\langle C^*ABD^*ACe, e\rangle| \leq \frac{1}{2}\left\|A^{1/2}Ce\right\|^2\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\| + \|B^*AD\|\right] \tag{2.9}$$

for any $e \in H$.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (2.9) we get

$$w(C^*ABD^*AC) \leq \frac{1}{2}\left\|A^{1/2}C\right\|^2\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\| + \|B^*AD\|\right] \tag{2.10}$$

and since

$$w(C^*ABD^*AC) = w(C^*ADB^*AC)$$

then by (2.10) we get the desired result (2.2). □

Remark 4. *If P is a projection and if we take $A = P^2 = P$ in Theorem 8 we obtain*

$$\begin{aligned} |\langle C^*PBD^*PCe, e\rangle| &\leq \|D^*PCe\|\|B^*PCe\| \\ &\leq \frac{1}{2}\|PCe\|^2[\|PD\|\|PB\| + \|B^*PD\|]. \end{aligned} \tag{2.11}$$

for any $e \in H$ and

$$w(C^*PDB^*PC) \leq \frac{1}{2}\|PC\|^2[\|PD\|\|PB\| + \|B^*PD\|]. \tag{2.12}$$

For $P = I$ we get

$$\begin{aligned} |\langle C^*BD^*Ce, e\rangle| &\leq \|D^*Ce\|\|B^*Ce\| \\ &\leq \frac{1}{2}\|Ce\|^2[\|D\|\|B\| + \|B^*D\|]. \end{aligned} \tag{2.13}$$

for any $e \in H$ and

$$w(C^*DB^*C) \leq \frac{1}{2}\|C\|^2[\|D\|\|B\| + \|B^*D\|]. \tag{2.14}$$

The following result also holds.

Theorem 9. *Let A be a nonnegative operator on H and D, B, C three bounded operators on H such that $B^*AC = C^*AD$, then*

$$w^2(C^*AD) \leq \frac{1}{2}\left\|A^{1/2}C\right\|^2\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\| + w(B^*AD)\right] \tag{2.15}$$

and

$$w^2(C^*AD) \leq \frac{1}{2}\left\|A^{1/2}C\right\|^2\left[\left\|\frac{\left|A^{1/2}D\right|^2 + \left|A^{1/2}B\right|^2}{2}\right\| + w(B^*AD)\right]. \tag{2.16}$$

Proof. From the inequality (2.6) we have

$$\begin{aligned} &\frac{1}{2}\left[\langle D^*ADe, e\rangle^{1/2}\langle B^*ABe, e\rangle^{1/2} + |\langle B^*ADe, e\rangle|\right]\langle C^*ACe, e\rangle \\ &\geq |\langle e, D^*ACe\rangle\langle e, B^*ACe\rangle| \end{aligned} \tag{2.17}$$

for any $e \in H$.

Since

$$B^*AC = C^*AD = (D^*AC)^*$$

then

$$|\langle e, D^*ACe\rangle \langle e, B^*ACe\rangle| = \left|\langle e, D^*ACe\rangle \langle e, (D^*AC)^* e\rangle\right| = |\langle D^*ACe, e\rangle|^2 = |\langle C^*ADe, e\rangle|^2 \tag{2.18}$$

for any $e \in H$.

By (2.17) and (2.18) we then have

$$|\langle C^*ADe, e\rangle|^2 \le \frac{1}{2}\left[\langle D^*ADe, e\rangle^{1/2} \langle B^*ABe, e\rangle^{1/2} + |\langle B^*ADe, e\rangle|\right] \langle C^*ACe, e\rangle \tag{2.19}$$

for any $e \in H$. This inequality is of interest in itself.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (2.19) we have

$$\begin{aligned} & w^2 (C^*AD) \\ & = \sup_{\|e\|=1} |\langle C^*ADe, e\rangle|^2 \\ & \le \frac{1}{2} \sup_{\|e\|=1} \left\{\left[\langle D^*ADe, e\rangle^{1/2} \langle B^*ABe, e\rangle^{1/2} + |\langle B^*ADe, e\rangle|\right] \langle C^*ACe, e\rangle\right\} \\ & \le \frac{1}{2} \sup_{\|e\|=1} \left[\langle D^*ADe, e\rangle^{1/2} \langle B^*ABe, e\rangle^{1/2} + |\langle B^*ADe, e\rangle|\right] \sup_{\|e\|=1} \langle C^*ACe, e\rangle \\ & \le \frac{1}{2} \left[\sup_{\|e\|=1} \langle D^*ADe, e\rangle^{1/2} \sup_{\|e\|=1} \langle B^*ABe, e\rangle^{1/2} + \sup_{\|e\|=1} |\langle B^*ADe, e\rangle|\right] \\ & \times \sup_{\|e\|=1} \langle C^*ACe, e\rangle \\ & = \frac{1}{2}\left[\|D^*AD\|^{1/2} \|B^*AB\|^{1/2} + w(B^*AD)\right] \|C^*AC\|, \end{aligned}$$

which proves the inequality (2.15).

Using the *arithmetic mean - geometric mean* inequality we also have

$$\langle D^*ADe, e\rangle^{1/2} \langle B^*ABe, e\rangle^{1/2} \le \frac{1}{2}\left[\langle D^*ADe, e\rangle + \langle B^*ABe, e\rangle\right] = \left\langle \frac{D^*AD + B^*AB}{2} e, e\right\rangle$$

for any $e \in H$.

By (2.19) we then have

$$|\langle C^*ADe, e\rangle|^2 \le \frac{1}{2}\left[\left\langle \frac{D^*AD + B^*AB}{2} e, e\right\rangle + |\langle B^*ADe, e\rangle|\right] \langle C^*ACe, e\rangle \tag{2.20}$$

for any $e \in H$.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (2.20) we obtain the desired result (2.16). □

Remark 5. *If P is a projection and $B^*PC = C^*PD$, then by Theorem 9 for $A = P$ we get*

$$w^2 (C^*PD) \le \frac{1}{2} \|PC\|^2 \left[\|PD\| \|PB\| + w(B^*PD)\right] \tag{2.21}$$

and

$$w^{2}\left(C^{*}PD\right)\leq\frac{1}{2}\left\|PC\right\|^{2}\left[\left\|\frac{\left|PD\right|^{2}+\left|PB\right|^{2}}{2}\right\|+w\left(B^{*}PD\right)\right]. \tag{2.22}$$

If $B^{*}C=C^{*}D$, *then*

$$w^{2}\left(C^{*}D\right)\leq\frac{1}{2}\left\|C\right\|^{2}\left[\left\|D\right\|\left\|B\right\|+w\left(B^{*}D\right)\right] \tag{2.23}$$

and

$$w^{2}\left(C^{*}D\right)\leq\frac{1}{2}\left\|C\right\|^{2}\left[\left\|\frac{\left|D\right|^{2}+\left|B\right|^{2}}{2}\right\|+w\left(B^{*}PD\right)\right]. \tag{2.24}$$

3. Trace Inequalities

Let $(H;\langle.,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\left\|A\right\|_{1}:=\sum_{i\in I}\left\langle\left|A\right|e_{i},e_{i}\right\rangle<\infty. \tag{3.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\left\langle Ae_{i},e_{i}\right\rangle, \tag{3.2}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (3.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 10. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^{*}\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^{*})=\overline{\operatorname{tr}(A)}; \tag{3.3}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ \left|\operatorname{tr}(AT)\right|\leq\left\|A\right\|_{1}\left\|T\right\|; \tag{3.4}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

Proposition 1. *Let* A *be a nonnegative operator on* H *and* D,B,C *three bounded operators on* H. *If* $C^{*}AC\in\mathcal{B}_1(H)$, *then* $C^{*}ABD^{*}AC\in\mathcal{B}_1(H)$ *and*

$$\left|\operatorname{tr}\left(C^{*}ABD^{*}AC\right)\right|\leq\frac{1}{2}\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\|+\left\|B^{*}AD\right\|\right]\operatorname{tr}\left(C^{*}AC\right). \tag{3.5}$$

Proof. Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H, then by (2.1) we obtain

$$\begin{aligned}
|\operatorname{tr}(C^*ABD^*AC)| &= \left|\sum_{i\in I}\langle C^*ABD^*ACe_i, e_i\rangle\right| \\
&\le \sum_{i\in I}|\langle C^*ABD^*ACe_i, e_i\rangle| \le \\
&\le \frac{1}{2}\sum_{i\in I}\left\|A^{1/2}Ce_i\right\|^2\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\| + \|B^*AD\|\right] \\
&= \frac{1}{2}\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\| + \|B^*AD\|\right]\sum_{i\in I}\left\langle A^{1/2}Ce_i, A^{1/2}Ce_i\right\rangle \\
&= \frac{1}{2}\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\| + \|B^*AD\|\right]\sum_{i\in I}\langle C^*ACe_i, e_i\rangle \\
&= \frac{1}{2}\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\| + \|B^*AD\|\right]\operatorname{tr}(C^*AC),
\end{aligned}$$

which proves (3.5). □

Corollary 3. *Let P be a projection. If $C^*PC \in \mathcal{B}_1(H)$, then $C^*PBD^*PC \in \mathcal{B}_1(H)$ and*

$$|\operatorname{tr}(C^*PBD^*PC)| \le \frac{1}{2}\left[\|PD\|\|PB\| + \|B^*PD\|\right]\operatorname{tr}(C^*PC). \tag{3.6}$$

If $C^*C \in \mathcal{B}_1(H)$, then $C^*BD^*C \in \mathcal{B}_1(H)$ and

$$|\operatorname{tr}(C^*BD^*C)| \le \frac{1}{2}\left[\|D\|\|B\| + \|B^*D\|\right]\operatorname{tr}(C^*C). \tag{3.7}$$

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite [17, p. 60-64]

$$\|A\|_p := \left[\operatorname{tr}(|A|^p)\right]^{1/p} = \left(\sum_{i\in I}\langle |A|^p e_i, e_i\rangle\right)^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{3.8}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{3.9}$$

For $p \ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [17, p. 60-64],

$$\|A\|_p = \|A^*\|_p,\ A \in \mathcal{B}_p(H) \tag{3.10}$$

$$\|AB\|_p \le \|A\|_p\|B\|_p,\ A, B \in \mathcal{B}_p(H) \tag{3.11}$$

and

$$\|AB\|_p \le \|A\|_p\|B\|,\ \|BA\|_p \le \|B\|\|A\|_p,\ A \in \mathcal{B}_p(H),\ B \in \mathcal{B}(H). \tag{3.12}$$

This implies that

$$\|CAB\|_p \le \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{3.13}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \le) \|AB\|_1 \le \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{3.14}$$

For the theory of trace functionals and their applications the reader is referred to [16] and [17].

For $\mathcal{E} := \{e_i\}_{i\in I}$ an orthonormal basis of H we define for $A \in \mathcal{B}_p(H)$, $p \ge 1$

$$\|A\|_{\mathcal{E},p} := \left(\sum_{i\in I} |\langle Ae_i, e_i\rangle|^p\right)^{1/p}.$$

We observe that $\|\cdot\|_{\mathcal{E},p}$ is a norm on $\mathcal{B}_p(H)$ and

$$\|A\|_{\mathcal{E},p} \le \|A\|_p \text{ for } A \in \mathcal{B}_p(H).$$

Proposition 2. *Let A be a nonnegative operator on H and D, B, C three bounded operators on H such that $B^*AC = C^*AD$. If D^*AD, B^*AB and $B^*AD \in \mathcal{B}_1(H)$, then*

$$\|C^*AD\|_{\mathcal{E},2}^2 \le \frac{1}{2} \|C^*AC\| \left([\operatorname{tr}(D^*AD)]^{1/2} [\operatorname{tr}(B^*AB)]^{1/2} + \|C^*AD\|_{\mathcal{E},1}\right), \tag{3.15}$$

where $\mathcal{E} := \{e_i\}_{i\in I}$ an orthonormal basis of H.

Proof. Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. From (2.19) we have

$$\begin{aligned}
&|\langle C^*ADe_i, e_i\rangle|^2 \\
&\le \frac{1}{2}\left[\langle D^*ADe_i, e_i\rangle^{1/2} \langle B^*ABe_i, e_i\rangle^{1/2} + |\langle B^*ADe_i, e_i\rangle|\right] \langle C^*ACe_i, e_i\rangle \\
&\le \frac{1}{2}\|C^*AC\| \left[\langle D^*ADe_i, e_i\rangle^{1/2} \langle B^*ABe_i, e_i\rangle^{1/2} + |\langle B^*ADe_i, e_i\rangle|\right]
\end{aligned}$$

for all $i \in I$.

If we sum over $i \in I$, then we get

$$\begin{aligned}
&\sum_{i\in I} |\langle C^*ADe_i, e_i\rangle|^2 \\
&\le \frac{1}{2}\|C^*AC\| \left[\sum_{i\in I} \langle D^*ADe_i, e_i\rangle^{1/2} \langle B^*ABe_i, e_i\rangle^{1/2} + \sum_{i\in I} |\langle B^*ADe_i, e_i\rangle|\right].
\end{aligned}$$

By Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\sum_{i\in I} \langle D^*ADe_i, e_i\rangle^{1/2} \langle B^*ABe_i, e_i\rangle^{1/2} \\
&\le \left(\sum_{i\in I} \left[\langle D^*ADe_i, e_i\rangle^{1/2}\right]^2\right)^{1/2} \left(\sum_{i\in I} \left[\langle B^*ABe_i, e_i\rangle^{1/2}\right]^2\right)^{1/2} \\
&= \left(\sum_{i\in I} \langle D^*ADe_i, e_i\rangle\right)^{1/2} \left(\sum_{i\in I} \langle B^*ABe_i, e_i\rangle\right)^{1/2} \\
&= [\operatorname{tr}(D^*AD)]^{1/2} [\operatorname{tr}(B^*AB)]^{1/2},
\end{aligned}$$

$$\sum_{i\in I}|\langle C^*ADe_i,e_i\rangle|^2=\|C^*AD\|_{\mathcal{E},2}^2 \text{ and } \sum_{i\in I}|\langle B^*ADe_i,e_i\rangle|=\|C^*AD\|_{\mathcal{E},1}$$

and the inequality (3.15) is proved. □

4. Some Particular Inequalities

In this section we explore some particular inequalities of interest that can be obtained from the main results stated above.

If we take in (2.1) and (2.2) $B=D^*$, then we get

$$\left|\left\langle C^*AD^2ACe,e\right\rangle\right|\le\|DACe\|\,\|D^*ACe\| \le\frac{1}{2}\left\|A^{1/2}Ce\right\|^2\left[\left\|A^{1/2}D\right\|\left\|DA^{1/2}\right\|+\|DAD\|\right] \tag{4.1}$$

for any $e\in H$ and

$$w\left(C^*AD^2AC\right)\le\frac{1}{2}\left\|A^{1/2}C\right\|^2\left[\left\|A^{1/2}D\right\|\left\|DA^{1/2}\right\|+\|DAD\|\right], \tag{4.2}$$

where D,C are bounded operators on H and A is a nonnegative operator on H.

If we take in these two inequalities $A=P$, a projector, then we obtain

$$\left|\left\langle C^*PD^2PCe,e\right\rangle\right|\le\|DPCe\|\,\|D^*PCe\| \le\frac{1}{2}\|PCe\|^2\left[\|PD\|\,\|DP\|+\|DPD\|\right] \tag{4.3}$$

for any $e\in H$ and

$$w\left(C^*PD^2PC\right)\le\frac{1}{2}\|PC\|^2\left[\|PD\|\,\|DP\|+\|DPD\|\right], \tag{4.4}$$

Choosing $B=D^*$ in (2.13) and (2.14), we get

$$\left|\left\langle C^*D^2Ce,e\right\rangle\right|\le\|D^*Ce\|\,\|DCe\|\le\frac{1}{2}\|Ce\|^2\left[\|D\|^2+\left\|D^2\right\|\right] \tag{4.5}$$

for any $e\in H$ and

$$w\left(C^*D^2C\right)\le\frac{1}{2}\|C\|^2\left[\|D\|^2+\left\|D^2\right\|\right]. \tag{4.6}$$

If we take in (2.1) and (2.2) $C=1_H$, then we get

$$|\langle ADB^*Ae,e\rangle|\le\|D^*Ae\|\,\|B^*Ae\| \le\frac{1}{2}\left\|A^{1/2}e\right\|^2\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\|+\|B^*AD\|\right] \tag{4.7}$$

for any $e\in H$ and

$$w\left(ADB^*A\right)\le\frac{1}{2}\|A\|\left[\left\|A^{1/2}D\right\|\left\|A^{1/2}B\right\|+\|B^*AD\|\right], \tag{4.8}$$

where D,B are bounded operators on H and A is a nonnegative operator on H.

The choice $A=P$ in (4.7) and (4.8) gives

$$|\langle PDB^*Pe,e\rangle|\le\|D^*Pe\|\,\|B^*Pe\| \le\frac{1}{2}\|Pe\|^2\left[\|PD\|\,\|PB\|+\|B^*PD\|\right] \tag{4.9}$$

for any $e\in H$ and

$$w\left(PDB^*P\right)\le\frac{1}{2}\left[\|PD\|\,\|PB\|+\|B^*PD\|\right], \tag{4.10}$$

Moreover, if in (4.7) and (4.8) we take $B = D^*$, then we get the inequalities

$$\left|\left\langle AD^2Ae, e\right\rangle\right| \leq \|D^*Ae\| \|DAe\| \leq \frac{1}{2}\left\|A^{1/2}e\right\|^2 \left[\left\|A^{1/2}D\right\| \left\|DA^{1/2}\right\| + \|DAD\|\right] \tag{4.11}$$

for any $e \in H$ and

$$w\left(AD^2A\right) \leq \frac{1}{2}\|A\| \left[\left\|A^{1/2}D\right\| \left\|DA^{1/2}\right\| + \|DAD\|\right]. \tag{4.12}$$

For A a projection P we obtain

$$\left|\left\langle PD^2Pe, e\right\rangle\right| \leq \|D^*Pe\| \|DPe\| \leq \frac{1}{2}\|Pe\|^2 \left[\|PD\| \|DP\| + \|DPD\|\right] \tag{4.13}$$

for any $e \in H$ and

$$w\left(PD^2P\right) \leq \frac{1}{2}\left[\|PD\| \|DP\| + \|DPD\|\right]. \tag{4.14}$$

Further, if we assume that $DAC = C^*AD$, then by taking $B = D^*$ in (2.15) and (2.16) we get

$$w^2(DAC) \leq \frac{1}{2}\left\|A^{1/2}C\right\|^2 \left[\left\|A^{1/2}D\right\| \left\|DA^{1/2}\right\| + w(DAD)\right] \tag{4.15}$$

and

$$w^2(DAC) \leq \frac{1}{2}\left\|A^{1/2}C\right\|^2 \left[\left\|\frac{\left|A^{1/2}D\right|^2 + \left|A^{1/2}D^*\right|^2}{2}\right\| + w(DAD)\right]. \tag{4.16}$$

If $DC = C^*D$, then by taking $A = 1_H$ in (4.15) and (4.16) we deduce

$$w^2(DC) \leq \frac{1}{2}\|C\|^2 \left[\|D\|^2 + w\left(D^2\right)\right] \tag{4.17}$$

and

$$w^2(DC) \leq \frac{1}{2}\|C\|^2 \left[\left\|\frac{|D|^2 + |D^*|^2}{2}\right\| + w\left(D^2\right)\right]. \tag{4.18}$$

Since

$$\left\|\frac{|D|^2 + |D^*|^2}{2}\right\| \leq \frac{1}{2}\left[\left\||D|^2\right\| + \left\||D^*|^2\right\|\right] = \|D\|^2,$$

then the inequality (4.18) is better than (4.17).

If $DA = AD$, then by taking $C = 1_H$ in (4.15) and (4.16) we also have

$$w^2(DA) \leq \frac{1}{2}\|A\| \left[\left\|A^{1/2}D\right\| \left\|DA^{1/2}\right\| + w\left(AD^2\right)\right] \tag{4.19}$$

and

$$w^2(DA) \leq \frac{1}{2}\|A\| \left[\left\|\frac{\left|A^{1/2}D\right|^2 + \left|A^{1/2}D^*\right|^2}{2}\right\| + w\left(AD^2\right)\right]. \tag{4.20}$$

Taking into account the above results, we can state the following two inequalities for an operator T, namely

$$w^2(T) \leq \frac{1}{2}\left[\|T\|^2 + w\left(T^2\right)\right], \text{ see (1.15)}, \tag{4.21}$$

and

$$w^2(T) \leq \frac{1}{2}\left[\left\|\frac{|T|^2+|T^*|^2}{2}\right\| + w\left(T^2\right)\right]. \tag{4.22}$$

The inequality (4.22) is better than (4.21).

Let P be a projection on H. If $C^*PC \in \mathcal{B}_1(H)$, then $C^*PBD^*PC \in \mathcal{B}_1(H)$ and by (3.5) we obtain

$$|\mathrm{tr}(C^*PBD^*PC)| \leq \frac{1}{2}\left[\|PD\|\|PB\| + \|B^*PD\|\right]\mathrm{tr}(C^*PC). \tag{4.23}$$

If we take $P = I$ and assume that, if $C^*C \in \mathcal{B}_1(H)$, then $C^*BD^*C \in \mathcal{B}_1(H)$ and by (4.23) we obtain

$$|\mathrm{tr}(C^*BD^*C)| \leq \frac{1}{2}\left[\|D\|\|B\| + \|B^*D\|\right]\mathrm{tr}(C^*C). \tag{4.24}$$

Let P be a projection on H and D, B, C three bounded operators on H such that $B^*PC = C^*PD$. If D^*PD, B^*PB and $B^*PD \in \mathcal{B}_1(H)$, then

$$\|C^*PD\|_{\mathcal{E},2}^2 \leq \frac{1}{2}\|C^*PC\|\left(\left[\mathrm{tr}(D^*PD)\right]^{1/2}\left[\mathrm{tr}(B^*PB)\right]^{1/2} + \|C^*PD\|_{\mathcal{E},1}\right), \tag{4.25}$$

where $\mathcal{E} := \{e_i\}_{i\in I}$ an orthonormal basis of H.

Moreover, if $B^*C = C^*D$, D^*D, B^*B and $B^*D \in \mathcal{B}_1(H)$, then also

$$\|C^*D\|_{\mathcal{E},2}^2 \leq \frac{1}{2}\|C\|^2\left(\left[\mathrm{tr}(D^*D)\right]^{1/2}\left[\mathrm{tr}(B^*B)\right]^{1/2} + \|C^*D\|_{\mathcal{E},1}\right), \tag{4.26}$$

where $\mathcal{E} := \{e_i\}_{i\in I}$ an orthonormal basis of H.

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