VECTOR INEQUALITIES IN TERMS OF SPECTRAL RADIUS OF OPERATORS IN HILBERT SPACES WITH APPLICATIONS TO NUMERICAL RADIUS AND p-SCHATTEN NORMS

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Abstract. Let H be a complex Hilbert space. Assume that f and g are nonnegative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. In this paper we show among others that, if T and V are operators in $\mathcal{B}(H)$ such that $|T|V=V^{*}|T|$, then for A, $B\in\mathcal{B}(H)$ we have

$$|\langle BTVAx,x\rangle|^{2r}\leq\frac{1}{2}r^{2r}(V)\left(\left\langle\left(\frac{1}{p}|f(|T|)A|^{2pr}+\frac{1}{q}|g(|T^{*}|)B^{*}|^{2qr}\right)x,x\right\rangle+\left\langle|g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}x,x\right\rangle^{r}\right)$$

provided that $r\geq1$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\geq2$, for $x\in H$ with $\|x\|=1$. Also, we have the numerical radius inequality

$$|\omega(BTVA)|^{2r}\leq\frac{1}{2}r^{2r}(V)\left(\left\|\frac{1}{p}|f(|T|)A|^{2pr}+\frac{1}{q}|g(|T^{*}|)B^{*}|^{2qr}\right\|+\omega^{r}\left(|g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}\right)\right)$$

and the trace inequality

$$\|BTVA\|_{\mathcal{E},2r}^{2r}\leq\frac{1}{2}r^{2r}(V)\left[\operatorname{tr}\left(\frac{1}{p}|f(|T|)A|^{2pr}+\frac{1}{q}|g(|T^{*}|)B^{*}|^{2qr}\right)+\left\||g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}\right\|_{r}^{r}\right].$$

provided that

$$|f(|T|)A|^{2pr},\ \ |g(|T^{*}|)B^{*}|^{2qr},\ \ |g(|T^{*}|)B^{*}|^{2}|f(|T|)A|^{2}\in\mathcal{B}_{1}(H)$$

and $\|A\|_{\mathcal{E},p}:=\left(\sum_{i\in I}|\langle Ae_i,e_i\rangle|^{p}\right)^{1/p}$ for $A\in\mathcal{B}_p(H)$, $p\geq1$.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T)=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x\in H$ one has

$$|\langle Tx,x\rangle|\leq w(T)\|x\|^{2}. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T:H\to H$, i.e.,

(i) $\omega(T)\geq0$ for any $T\in B(H)$ and $\omega(T)=0$ if and only if $T=0$;
(ii) $\omega(\lambda T)=|\lambda|\omega(T)$ for any $\lambda\in\mathbb{C}$ and $T\in B(H)$;

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(iii) $\omega(T+V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{1.3}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [11], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [12] improved the inequality (1.3) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{1.5}$$

for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [8]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{1.6}$$

and

$$\omega^{2r}(T) \leq \left\|\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|, \tag{1.7}$$

where $\alpha \in (0,1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (1.6) that

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{1.8}$$

and from (1.7) that

$$\omega^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [7] and [4].

Let $T = U|T|$ be the *polar decomposition* of the bounded linear operator T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T} := |T|^{1/2} U |T|^{1/2}$, see [1].

The following properties of $\widetilde{T}$ are as follows:

(i) $\left\|\widetilde{T}\right\| \leq \|T\|$,

(ii) $w\left(\widetilde{T}\right) \leq \omega(T)$,

(iii) $r\left(\widetilde{T}\right) = \omega(T)$,

(iv) $\omega\left(\widetilde{T}\right) \leq \left\|T^2\right\|^{1/2} (\leq \|T\|)$, [15].

Utilizing this transform T. Yamazaki, [15] obtained in 2007 the following refinement of Kittaneh's inequality (1.4):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \omega\left(\widetilde{T}\right)\right) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right) \tag{1.10}$$

for any operator $T \in B(H)$.

We remark that if $\widetilde{T} = 0$, then obviously $w(T) = \frac{1}{2}\|T\|$.

Abu-Omar and Kittaneh [2] improved on inequality (1.10) using generalized Aluthge transform to prove that

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \min_{t\in[0,1]} \omega(\Delta_t(T))\right).$$

For $t = 1$ this also gives the following result for the *Dougal transform*

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \omega\left(\widehat{T}\right)\right). \tag{1.11}$$

In [3] Bunia et al. also proved that

$$\omega(T) \leq \min_{t\in[0,1]} \left\{\frac{1}{2}\omega(\Delta_t(T)) + \frac{1}{4}\left(\|T\|^{2t} + \|T\|^{2(1-t)}\right)\right\},$$

which for $t = 1/2$ gives (1.10) as well.

In 1988, F. Kittaneh obtained the following generalization of Schwarz inequality [10]:

Theorem 1. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Let T, V be operators in $V(H)$ such that $|T|V = V^*|T|$, then*

$$|\langle TVx, y\rangle| \leq r(V)\|f(|T|)x\|\|g(|T^*|)y\| \tag{1.12}$$

for all x, $y \in H$, where $r(V)$ denotes the spectral radius of V.

If we take $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ for $\alpha \in [0,1]$ and $t > 0$,

$$|\langle TVx, y\rangle| \leq r(V)\||T|^\alpha x\|\left\||T^*|^{1-\alpha} y\right\| \tag{1.13}$$

for all x, $y \in H$.

2. Main Results

We recall the following vector inequality for positive operators $A \geq 0$, obtained by C. A. McCarthy in [13]

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle, \ p \geq 1 \tag{2.1}$$

for $x \in H$, $\|x\| = 1$ and *Buzano's inequality* [5],

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}[\|x\|\|y\| + |\langle x, y\rangle|] \tag{2.2}$$

that holds for any x, y, $e \in H$, $\|e\| = 1$.

Our first main result is as follows:

Theorem 2. *Assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Let T, V be operators in $\mathcal{B}(H)$ such that $|T|V = V^*|T|$, then for A, $B \in \mathcal{B}(H)$ we have that*

$$\begin{aligned} &|\langle BTVAx, x\rangle|^{2r} \\ &\leq r^{2r}(V)\left\langle\left[\frac{1}{p}|f(|T|)A|^{2pr} + \frac{1}{q}|g(|T^*|)B^*|^{2qr}\right]x, x\right\rangle \end{aligned} \tag{2.3}$$

for $x \in H$ with $\|x\| = 1$, provided that $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$.

If $r \geq 1$,

$$|\langle BTVAx, x\rangle|^{2r} \leq \frac{1}{2} r^{2r}(V) \times \left(\left\| |f(|T|)A|^2 x \right\|^r \left\| |g(|T^*|)B^*|^2 x \right\|^r + \left\langle |g(|T^*|)B^*|^2 |f(|T|)A|^2 x, x \right\rangle^r \right) \tag{2.4}$$

for $x \in H$ *with* $\|x\| = 1$.

If $r \geq 1$, p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* pr, $qr \geq 2$, *then also*

$$|\langle BTVAx, x\rangle|^{2r} \leq \frac{1}{2} r^{2r}(V) \left(\left\langle \left(\frac{1}{p} |f(|T|)A|^{2pr} + \frac{1}{q} |g(|T^*|)B^*|^{2qr} \right) x, x \right\rangle + \left\langle |g(|T^*|)B^*|^2 |f(|T|)A|^2 x, x \right\rangle^r \right) \tag{2.5}$$

for $x \in H$ *with* $\|x\| = 1$.

Proof. Observe that by (1.12) we have

$$\begin{aligned} |\langle TVx, y\rangle|^2 &\leq r^2(V) \|f(|T|)x\|^2 \|g(|T^*|)y\|^2 \\ &= r^2(V) \langle f(|T|)x, f(|T|)x\rangle \langle g(|T^*|)y, g(|T^*|)y\rangle \\ &= r^2(V) \left\langle f^2(|T|)x, x\right\rangle \left\langle g^2(|T^*|)y, y\right\rangle \end{aligned}$$

for all x, $y \in H$.

If we take Ax instead of x and B^*y instead of y, then we get

$$\begin{aligned} |\langle TVAx, B^*y\rangle|^2 &\leq r^2(V) \left\langle f^2(|T|)Ax, Ax\right\rangle \left\langle g^2(|T^*|)B^*y, B^*y\right\rangle \\ &= r^2(V) \left\langle A^* f^2(|T|)Ax, x\right\rangle \left\langle Bg^2(|T^*|)B^*y, y\right\rangle \\ &= r^2(V) \left\langle (f(|T|)Ax)^* f(|T|)Ax, x\right\rangle \\ &\times \left\langle (g(|T^*|)B^*)^* g(|T^*|)B^*y, y\right\rangle \\ &= r^2(V) \left\langle |f(|T|)A|^2 x, x\right\rangle \left\langle |g(|T^*|)B^*|^2 y, y\right\rangle, \end{aligned}$$

namely

$$|\langle BTVAx, y\rangle|^2 \leq r^2(V) \left\langle |f(|T|)A|^2 x, x\right\rangle \left\langle |g(|T^*|)B^*|^2 y, y\right\rangle \tag{2.6}$$

for all x, $y \in H$ and, in particular

$$|\langle BTVAx, x\rangle|^2 \leq r^2(V) \left\langle |f(|T|)A|^2 x, x\right\rangle \left\langle |g(|T^*|)B^*|^2 x, x\right\rangle \tag{2.7}$$

for all $x \in H$.

If we take the power $r>0$ in (2.7), then we get, by Young and McCarthy inequalities, that

$$\begin{aligned}
&\left|\langle BTVAx,x\rangle\right|^{2r}\\
&\leq r^{2r}(V)\left\langle \left|f(|T|)A\right|^{2}x,x\right\rangle^{r}\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}x,x\right\rangle^{r}\\
&\leq r^{2r}(V)\left[\frac{1}{p}\left\langle \left|f(|T|)A\right|^{2}x,x\right\rangle^{pr}+\frac{1}{q}\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}x,x\right\rangle^{qr}\right]\\
&\leq r^{2r}(V)\left[\frac{1}{p}\left\langle \left|f(|T|)A\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle \left|g(|T^{*}|)B^{*}\right|^{2qr}x,x\right\rangle\right]\\
&=r^{2r}(V)\left\langle\left[\frac{1}{p}\left|f(|T|)A\right|^{2pr}+\frac{1}{q}\left|g(|T^{*}|)B^{*}\right|^{2qr}\right]x,x\right\rangle
\end{aligned}$$

for $x\in H$ with $\|x\|=1$, which proves (2.3).

Let $x\in H$, $\|x\|=1$, then by Buzano's inequality we derive

$$\begin{aligned}
&\left\langle \left|f(|T|)A\right|^{2}x,x\right\rangle\left\langle x,\left|g(|T^{*}|)B^{*}\right|^{2}x\right\rangle\\
&\leq\frac{1}{2}\left[\left\|\left|f(|T|)A\right|^{2}x\right\|\left\|\left|g(|T^{*}|)B^{*}\right|^{2}x\right\|+\left\langle \left|f(|T|)A\right|^{2}x,\left|g(|T^{*}|)B^{*}\right|^{2}x\right\rangle\right]\\
&=\frac{1}{2}\left[\left\|\left|f(|T|)A\right|^{2}x\right\|\left\|\left|g(|T^{*}|)B^{*}\right|^{2}x\right\|+\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}\left|f(|T|)A\right|^{2}x,x\right\rangle\right].
\end{aligned}$$

By making use of (2.7) we get

$$\begin{aligned}
&\left|\langle BTVAx,x\rangle\right|^{2}\\
&\leq\frac{1}{2}r^{2}(V)\\
&\times\left[\left\|\left|f(|T|)A\right|^{2}x\right\|\left\|\left|g(|T^{*}|)B^{*}\right|^{2}x\right\|+\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}\left|f(|T|)A\right|^{2}x,x\right\rangle\right]
\end{aligned}\tag{2.8}$$

for $x\in H$, $\|x\|=1$.

By taking the power $r\geq 1$ in (2.8) and using the convexity of the power function, we get

$$\begin{aligned}
&\left|\langle BTVAx,x\rangle\right|^{2r}\\
&\leq r^{2r}(V)\\
&\times\left[\frac{\left\|\left|f(|T|)A\right|^{2}x\right\|\left\|\left|g(|T^{*}|)B^{*}\right|^{2}x\right\|+\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}\left|f(|T|)A\right|^{2}x,x\right\rangle}{2}\right]^{r}\\
&\leq r^{2r}(V)\\
&\times\frac{\left\|\left|f(|T|)A\right|^{2}x\right\|^{r}\left\|\left|g(|T^{*}|)B^{*}\right|^{2}x\right\|^{r}+\left\langle \left|g(|T^{*}|)B^{*}\right|^{2}\left|f(|T|)A\right|^{2}x,x\right\rangle^{r}}{2}
\end{aligned}\tag{2.9}$$

for $x\in H$ with $\|x\|=1$, which proves (2.4).

Also, observe that

$$\begin{aligned}
&\left\| |f(|T|)A|^2 x \right\|^r \left\| |g(|T^*|)B^*|^2 x \right\|^r \\
&\le \frac{1}{p}\left\| |f(|T|)A|^2 x \right\|^{pr} + \frac{1}{q}\left\| |g(|T^*|)B^*|^2 x \right\|^{qr} \\
&= \frac{1}{p}\left\| |f(|T|)A|^2 x \right\|^{2\frac{pr}{2}} + \frac{1}{q}\left\| |g(|T^*|)B^*|^2 x \right\|^{2\frac{qr}{2}} \\
&= \frac{1}{p}\left\langle |f(|T|)A|^4 x, x \right\rangle^{\frac{pr}{2}} + \frac{1}{q}\left\langle |g(|T^*|)B^*|^4 x, x \right\rangle^{\frac{qr}{2}} \\
&\le \frac{1}{p}\left\langle |f(|T|)A|^{2pr} x, x \right\rangle + \frac{1}{q}\left\langle |g(|T^*|)B^*|^{2qr} x, x \right\rangle \\
&= \left\langle \left(\frac{1}{p}|f(|T|)A|^{2pr} + \frac{1}{q}|g(|T^*|)B^*|^{2qr} \right) x, x \right\rangle,
\end{aligned}$$

then

$$\begin{aligned}
&\frac{\left\| |f(|T|)A|^2 x \right\|^r \left\| |g(|T^*|)B^*|^2 x \right\|^r + \left\langle |g(|T^*|)B^*|^2 |f(|T|)A|^2 x, x \right\rangle^r}{2} \\
&\le \frac{1}{2}\left\langle \left(\frac{1}{p}|f(|T|)A|^{2pr} + \frac{1}{q}|g(|T^*|)B^*|^{2qr} \right) x, x \right\rangle \\
&+ \frac{1}{2}\left\langle |g(|T^*|)B^*|^2 |f(|T|)A|^2 x, x \right\rangle^r
\end{aligned}$$

and by (2.9) we get

$$\begin{aligned}
|\langle BTVAx, x\rangle|^{2r} \le{}& \frac{1}{2}r^{2r}(V)\left\langle \left(\frac{1}{p}|f(|T|)A|^{2pr} + \frac{1}{q}|g(|T^*|)B^*|^{2qr} \right) x, x \right\rangle \\
&+ \frac{1}{2}r^{2r}(V)\left\langle |g(|T^*|)B^*|^2 |f(|T|)A|^2 x, x \right\rangle^r
\end{aligned}$$

or $x \in H$ with $\|x\| = 1$. This proves (2.5). □

Remark 1. *If we take $r = 1/2$ and $p = q = 2$ in (2.3), then we get*

$$|\langle BTVAx, x\rangle| \le \frac{1}{2}r(V)\left\langle \left[|f(|T|)A|^2 + |g(|T^*|)B^*|^2 \right] x, x \right\rangle \tag{2.10}$$

for $x \in H$ with $\|x\| = 1$.

The choice $r = 1$ in the same inequality produces

$$|\langle BTVAx, x\rangle|^2 \le r^2(V)\left\langle \left[\frac{1}{p}|f(|T|)A|^{2p} + \frac{1}{q}|g(|T^*|)B^*|^{2q} \right] x, x \right\rangle \tag{2.11}$$

for $x \in H$ with $\|x\| = 1$, provided that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

For $r = 1$ in (2.4) we obtain

$$\begin{aligned}
|\langle BTVAx, x\rangle|^2 \le{}& \frac{1}{2}r^2(V)\Big(\left\| |f(|T|)A|^2 x \right\| \left\| |g(|T^*|)B^*|^2 x \right\| \\
&+ \left\langle |g(|T^*|)B^*|^2 |f(|T|)A|^2 x, x \right\rangle \Big)
\end{aligned} \tag{2.12}$$

for $x \in H$ with $\|x\| = 1$.

If we take $r=1$ in (2.5), then we get

$$\begin{aligned}(2.13)\qquad & |\langle BTVAx,x\rangle|^2 \\ & \leq \frac{1}{2}r^2(V)\left(\left\langle\left(\frac{1}{p}|f(|T|)A|^{2p}+\frac{1}{q}|g(|T^*|)B^*|^{2q}\right)x,x\right\rangle\right. \\ & \left.+\left\langle|g(|T^*|)B^*|^2|f(|T|)A|^2x,x\right\rangle\right)\end{aligned}$$

for $x\in H$ with $\|x\|=1$, provided that p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 1. *Assume that f and g are non-negative functions on $[0,\infty)$ that are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Let X, A, $B\in\mathcal{B}(H)$, then for all α, $\beta\geq 0$ with $\alpha+\beta\geq 1$, we have*

$$\begin{aligned}(2.14)\qquad & \left|\left\langle BX|X|^{\alpha+\beta-1}Ax,x\right\rangle\right|^{2r} \\ & \leq \|X\|^{2\beta r}\left\langle\left[\frac{1}{p}|f(|X|^{\alpha})A|^{2pr}+\frac{1}{q}|g(|X^*|^{\alpha})B^*|^{2qr}\right]x,x\right\rangle\end{aligned}$$

for $x\in H$ with $\|x\|=1$, provided that $r>0$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\geq 1$.

If $r\geq 1$,

$$\begin{aligned}(2.15)\qquad & \left|\left\langle BX|X|^{\alpha+\beta-1}Ax,x\right\rangle\right|^{2r} \\ & \leq \frac{1}{2}\|X\|^{2\beta r}\left(\left\||f(|X|^{\alpha})A|^2x\right\|^r\left\||g(|X^*|^{\alpha})B^*|^2x\right\|^r\right. \\ & \left.+\left\langle|g(|X^*|^{\alpha})B^*|^2|f(|X|^{\alpha})A|^2x,x\right\rangle^r\right)\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

If $r\geq 1$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\geq 2$, then also

$$\begin{aligned}(2.16)\qquad & \left|\left\langle BX|X|^{\alpha+\beta-1}Ax,x\right\rangle\right|^{2r} \\ & \leq \frac{1}{2}\|X\|^{2\beta r}\left(\left\langle\left(\frac{1}{p}|f(|X|^{\alpha})A|^{2pr}+\frac{1}{q}|g(|X^*|^{\alpha})B^*|^{2qr}\right)x,x\right\rangle\right. \\ & \left.+\left\langle|g(|X^*|^{\alpha})B^*|^2|f(|X|^{\alpha})A|^2x,x\right\rangle^r\right)\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

Proof. Let $X=U|X|$ be the *polar decomposition* of the bounded linear operator X, with U a partial isometry. If we take $T=U|X|^{\alpha}$ and $V=|X|^{\beta}$, then we have

$$TV=U|X|^{\alpha+\beta}=X|X|^{\alpha+\beta-1},\ \ |T|=|X|^{\alpha}\ \text{ and }\ |T^*|=|X^*|^{\alpha}$$

and since

$$|T|V=|X|^{\alpha+\beta}=V^*|T|$$

and

$$r(V)=r\left(|X|^{\beta}\right)=\left\||X|^{\beta}\right\|=\|X\|^{\beta},$$

hence by Theorem 2 we derive the desired inequalities. □

Remark 2. *Let $\alpha \in [0,1]$. If we take $\beta = 1-\alpha$ in Corollary 1, then we get*

$$
\begin{aligned}
(2.17) \qquad & |\langle BXAx, x\rangle|^{2r} \\
& \leq \|X\|^{2(1-\alpha)r} \left\langle \left[\frac{1}{p} |f(|X|^{\alpha}) A|^{2pr} + \frac{1}{q} |g(|X^*|^{\alpha}) B^*|^{2qr} \right] x, x \right\rangle
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$, provided that $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$.

If $r \geq 1$,

$$
\begin{aligned}
(2.18) \qquad & |\langle BXAx, x\rangle|^{2r} \\
& \leq \frac{1}{2} \|X\|^{2(1-\alpha)r} \left(\left\| |f(|X|^{\alpha}) A|^2 x \right\|^r \left\| |g(|X^*|^{\alpha}) B^*|^2 x \right\|^r \right. \\
& \left. + \left\langle |g(|X^*|^{\alpha}) B^*|^2 |f(|X|^{\alpha}) A|^2 x, x \right\rangle^r \right)
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$.

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then also

$$
\begin{aligned}
(2.19) \qquad & |\langle BXAx, x\rangle|^{2r} \\
& \leq \frac{1}{2} \|X\|^{2(1-\alpha)r} \left(\left\langle \left(\frac{1}{p} |f(|X|^{\alpha}) A|^{2pr} + \frac{1}{q} |g(|X^*|^{\alpha}) B^*|^{2qr} \right) x, x \right\rangle \right. \\
& \left. + \left\langle |g(|X^*|^{\alpha}) B^*|^2 |f(|X|^{\alpha}) A|^2 x, x \right\rangle^r \right)
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$.

We also have:

Theorem 3. *With the assumptions of Theorem 2, we have for $r \geq 1$ and $\delta \in [0,1]$ that*

$$
\begin{aligned}
(2.20) \quad |\langle BTVAx, x\rangle|^{2r} & \leq r^{2r}(V) \left\langle \left[(1-\delta) |f(|T|) A|^{2r} + \delta |g(|T^*|) B^*|^{2r} \right] x, x \right\rangle \\
& \times \left\langle |f(|T|) A|^2 x, x \right\rangle^{r\delta} \left\langle |g(|T^*|) B^*|^2 x, x \right\rangle^{r(1-\delta)}
\end{aligned}
$$

for all $x \in H$, $\|x\| = 1$.

Also, we have

$$
\begin{aligned}
(2.21) \quad |\langle BTVAx, x\rangle|^{2} & \leq r^{2}(V) \left\langle \left[(1-\delta) |f(|T|) A|^{2r} + \delta |g(|T^*|) B^*|^{2r} \right] x, x \right\rangle^{1/r} \\
& \times \left\langle \left[\delta |f(|T|) A|^{2r} + (1-\delta) |g(|T^*|) B^*|^{2r} \right] x, x \right\rangle^{1/r}
\end{aligned}
$$

for all $x \in H$, $\|x\| = 1$.

Proof. From (2.7) we have for all $\delta \in [0,1]$ that

$$\begin{aligned} \left|\langle BTVAx,x\rangle\right|^{2} &\leq r^{2}\left(V\right)\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle \left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle \\ &= r^{2}\left(V\right)\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle^{1-\delta}\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle^{\delta} \\ &\quad\times\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle^{\delta}\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle^{1-\delta} \\ &\leq r^{2}\left(V\right)\left[\left(1-\delta\right)\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle +\delta\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle\right] \\ &\quad\times\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle^{\delta}\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle^{1-\delta} \end{aligned}$$

for all $x\in H$, $\|x\|=1$.

If we take the power $r\geq 1$, then we get by the convexity of power r that

$$\begin{aligned} &\left|\langle BTVAx,x\rangle\right|^{2r} \\ &\leq r^{2r}\left(V\right)\left[\left(1-\delta\right)\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle +\delta\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle\right]^{r} \\ &\times\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle^{r\delta}\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle^{r(1-\delta)} \\ &\leq r^{2r}\left(V\right)\left[\left(1-\delta\right)\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle^{r} +\delta\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle^{r}\right] \\ &\times\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle^{r\delta}\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle^{r(1-\delta)} \end{aligned} \tag{2.22}$$

for all $x\in H$, $\|x\|=1$.

If we use McCarthy inequality for power $r\geq 1$, then we get

$$\begin{aligned} &\left(1-\delta\right)\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle^{r} +\delta\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle^{r} \\ &\leq \left(1-\delta\right)\left\langle \left|f\left(\left|T\right|\right)A\right|^{2r}x,x\right\rangle +\delta\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2r}x,x\right\rangle \\ &= \left\langle \left[\left(1-\delta\right)\left|f\left(\left|T\right|\right)A\right|^{2r}+\delta\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2r}\right]x,x\right\rangle \end{aligned}$$

and by (2.22) we get (2.20).

We also have

$$\begin{aligned} &\left|\langle BTVAx,x\rangle\right|^{2} \\ &\leq r^{2}\left(V\right)\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle^{1-\delta}\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle^{\delta} \\ &\times\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle^{\delta}\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle^{1-\delta} \\ &\leq r^{2}\left(V\right)\left[\left(1-\delta\right)\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle +\delta\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle\right] \\ &\times\left[\delta\left\langle \left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle +\left(1-\delta\right)\left\langle \left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x,x\right\rangle\right], \end{aligned}$$

for all $x\in H$, $\|x\|=1$.

This implies in the same way that

$$\begin{aligned} \left|\langle BTVAx,x\rangle\right|^{2r} &\leq r^{2r}\left(V\right)\left\langle \left[\left(1-\delta\right)\left|f\left(\left|T\right|\right)A\right|^{2r}+\delta\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2r}\right]x,x\right\rangle \\ &\quad\times\left\langle \left[\delta\left|f\left(\left|T\right|\right)A\right|^{2r}+\left(1-\delta\right)\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2r}\right]x,x\right\rangle \end{aligned}$$

for all $x\in H$, $\|x\|=1$, which proves (2.21). □

Corollary 2. *With the assumptions of Corollary 1, we have that*

$$\left|\left\langle BX\left|X\right|^{\alpha+\beta-1}Ax,x\right\rangle\right|^{2r} \leq \left\|X\right\|^{2\beta r}\left\langle\left[\left(1-\delta\right)\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+\delta\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right]x,x\right\rangle \times\left\langle\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2}x,x\right\rangle^{r\delta}\left\langle\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2}x,x\right\rangle^{r(1-\delta)} \tag{2.23}$$

for all $x\in H$, $\|x\|=1$.

Also, we have

$$\left|\left\langle BX\left|X\right|^{\alpha+\beta-1}Ax,x\right\rangle\right|^{2} \leq \left\|X\right\|^{2\beta}\left\langle\left[\left(1-\delta\right)\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+\delta\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right]x,x\right\rangle^{1/r} \times\left\langle\left[\delta\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+\left(1-\delta\right)\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right]x,x\right\rangle^{1/r} \tag{2.24}$$

for all $x\in H$, $\|x\|=1$.

For $r\geq1$ and $\alpha\in[0,1]$ we derive

$$\left|\left\langle BXAx,x\right\rangle\right|^{2r} \leq \left\|X\right\|^{2(1-\alpha)r}\left\langle\left[\left(1-\delta\right)\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+\delta\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right]x,x\right\rangle \times\left\langle\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2}x,x\right\rangle^{r\delta}\left\langle\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2}x,x\right\rangle^{r(1-\delta)} \tag{2.25}$$

for all $x\in H$, $\|x\|=1$.

Also, we have

$$\left|\left\langle BXAx,x\right\rangle\right|^{2} \leq \left\|X\right\|^{2(1-\alpha)}\left\langle\left[\left(1-\delta\right)\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+\delta\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right]x,x\right\rangle^{1/r} \times\left\langle\left[\delta\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+\left(1-\delta\right)\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right]x,x\right\rangle^{1/r} \tag{2.26}$$

for all $x\in H$, $\|x\|=1$.

Consider $f(u)=u^{\lambda}$ and $g(u)=u^{1-\lambda}$ with $\lambda\in[0,1]$. Let T, V be operators in $\mathcal{B}(H)$ such that $|T|V=V^{*}|T|$, then for A, $B\in\mathcal{B}(H)$ we have from Theorem 2 that

$$\left|\left\langle BTVAx,x\right\rangle\right|^{2r}\leq r^{2r}\left(V\right)\left\langle\left[\frac{1}{p}\left|\left|T\right|^{\lambda}A\right|^{2pr}+\frac{1}{q}\left|\left|T^{*}\right|^{1-\lambda}B^{*}\right|^{2qr}\right]x,x\right\rangle \tag{2.27}$$

for $x\in H$ with $\|x\|=1$, provided that $r>0$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\geq1$.

If $r\geq1$,

$$\left|\left\langle BTVAx,x\right\rangle\right|^{2r}\leq\frac{1}{2}r^{2r}\left(V\right)\left(\left\|\left|\left|T\right|^{\lambda}A\right|^{2}x\right\|^{r}\left\|\left|\left|T^{*}\right|^{1-\lambda}B^{*}\right|^{2}x\right\|^{r} +\left\langle\left|\left|T^{*}\right|^{1-\lambda}B^{*}\right|^{2}\left|\left|T\right|^{\lambda}A\right|^{2}x,x\right\rangle^{r}\right) \tag{2.28}$$

for $x \in H$ with $\|x\| = 1$.

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then also

$$|\langle BTVAx, x\rangle|^{2r} \leq \frac{1}{2} r^{2r}(V) \left(\left\langle \left(\frac{1}{p} \left| |T|^{\lambda} A \right|^{2pr} + \frac{1}{q} \left| |T^*|^{1-\lambda} B^* \right|^{2qr} \right) x, x \right\rangle + \left\langle \left| |T^*|^{1-\lambda} B^* \right|^2 \left| |T|^{\lambda} A \right|^2 x, x \right\rangle^r \right) \tag{2.29}$$

for $x \in H$ with $\|x\| = 1$.

From Corollary 1 we get for X, A, $B \in \mathcal{B}(H)$ and α, $\beta \geq 0$ with $\alpha + \beta \geq 1$, that

$$\left| \left\langle BX |X|^{\alpha+\beta-1} Ax, x \right\rangle \right|^{2r} \leq \|X\|^{2\beta r} \left\langle \left[\frac{1}{p} \left| |X|^{\lambda\alpha} A \right|^{2pr} + \frac{1}{q} \left| |X^*|^{(1-\lambda)\alpha} B^* \right|^{2qr} \right] x, x \right\rangle \tag{2.30}$$

for $x \in H$ with $\|x\| = 1$, provided that $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$.

If $r \geq 1$, then

$$\left| \left\langle BX |X|^{\alpha+\beta-1} Ax, x \right\rangle \right|^{2r} \leq \frac{1}{2} \|X\|^{2\beta r} \left(\left\| \left| |X|^{\lambda\alpha} A \right|^2 x \right\|^r \left\| \left| |X^*|^{(1-\lambda)\alpha} B^* \right|^2 x \right\|^r + \left\langle \left| |X^*|^{(1-\lambda)\alpha} B^* \right|^2 \left| |X|^{\lambda\alpha} A \right|^2 x, x \right\rangle^r \right) \tag{2.31}$$

for $x \in H$ with $\|x\| = 1$.

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then also

$$\left| \left\langle BX |X|^{\alpha+\beta-1} Ax, x \right\rangle \right|^{2r} \leq \frac{1}{2} \|X\|^{2\beta r} \left(\left\langle \left(\frac{1}{p} \left| |X|^{\lambda\alpha} A \right|^{2pr} + \frac{1}{q} \left| |X^*|^{(1-\lambda)\alpha} B^* \right|^{2qr} \right) x, x \right\rangle + \left\langle \left| |X^*|^{(1-\lambda)\alpha} B^* \right|^2 \left| |X|^{\lambda\alpha} A \right|^2 x, x \right\rangle^r \right) \tag{2.32}$$

for $x \in H$ with $\|x\| = 1$.

Let $\alpha \in [0, 1]$. From Remark 2 we obtain

$$|\langle BXAx, x\rangle|^{2r} \leq \|X\|^{2(1-\alpha)r} \left\langle \left[\frac{1}{p} \left| |X|^{\lambda\alpha} A \right|^{2pr} + \frac{1}{q} \left| |X^*|^{(1-\lambda)\alpha} B^* \right|^{2qr} \right] x, x \right\rangle \tag{2.33}$$

for $x \in H$ with $\|x\| = 1$, provided that $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$.

If $r \geq 1$, then

$$\begin{aligned}(2.34) \quad & |\langle BXAx, x\rangle|^{2r} \\ & \leq \frac{1}{2}\|X\|^{2(1-\alpha)r}\left(\left\|\left||X|^{\lambda\alpha}A\right|^2 x\right\|^r \left\|\left||X^*|^{(1-\lambda)\alpha}B^*\right|^2 x\right\|^r \right. \\ & \left. + \left\langle \left||X^*|^{(1-\lambda)\alpha}B^*\right|^2 \left||X|^{\lambda\alpha}A\right|^2 x, x\right\rangle^r\right)\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then also

$$\begin{aligned}(2.35) \quad & |\langle BXAx, x\rangle|^{2r} \\ & \leq \frac{1}{2}\|X\|^{2(1-\alpha)r}\left(\left\langle\left(\frac{1}{p}\left||X|^{\lambda\alpha}A\right|^{2pr} + \frac{1}{q}\left||X^*|^{(1-\lambda)\alpha}B^*\right|^{2qr}\right)x, x\right\rangle \right. \\ & \left. + \left\langle \left||X^*|^{(1-\lambda)\alpha}B^*\right|^2 \left||X|^{\lambda\alpha}A\right|^2 x, x\right\rangle^r\right)\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

3. Numerical Radius Inequalities

We can state the following result:

Proposition 1. *Assume that f and g are non-negative functions on $[0, \infty)$ that are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Let T, V be operators in $\mathcal{B}(H)$ such that $|T|V = V^*|T|$, then for A, $B \in \mathcal{B}(H)$ we have the norm inequality*

$$(3.1) \quad |\omega(BTVA)|^{2r} \leq r^{2r}(V)\left\|\frac{1}{p}|f(|T|)A|^{2pr} + \frac{1}{q}|g(|T^*|)B^*|^{2qr}\right\|,$$

provided that $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$.

If $r \geq 1$,

$$(3.2) \quad |\omega(BTVA)|^{2r} \leq \frac{1}{2}r^{2r}(V)\left(\|f(|T|)A\|^{2r}\|g(|T^*|)B^*\|^{2r}\right.$$

$$(3.3) \quad \left. + \omega^r\left(|g(|T^*|)B^*|^2 |f(|T|)A|^2\right)\right).$$

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then also

$$\begin{aligned}(3.4) \quad & |\omega(BTVA)|^{2r} \\ & \leq \frac{1}{2}r^{2r}(V)\left(\left\|\frac{1}{p}|f(|T|)A|^{2pr} + \frac{1}{q}|g(|T^*|)B^*|^{2qr}\right\| \right. \\ & \left. + \omega^r\left(|g(|T^*|)B^*|^2 |f(|T|)A|^2\right)\right).\end{aligned}$$

Proof. If we take the supremum over $\|x\|=1$ in (2.3), then we get

$$\begin{aligned}
&\left|\omega\left(BTVA\right)\right|^{2r} \\
&= \sup_{\|x\|=1}\left|\left\langle BTVAx,x\right\rangle\right|^{2r} \\
&\leq r^{2r}\left(V\right)\sup_{\|x\|=1}\left\langle \left[\frac{1}{p}\left|f\left(\left|T\right|\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2qr}\right]x,x\right\rangle \\
&= r^{2r}\left(V\right)\left\|\frac{1}{p}\left|f\left(\left|T\right|\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2qr}\right\|,
\end{aligned}$$

which proves (3.1).

By taking the supremum over $\|x\|=1$ in (2.4) we obtain

$$\begin{aligned}
&\left|\omega\left(BTVA\right)\right|^{2r}=\sup_{\|x\|=1}\left|\left\langle BTVAx,x\right\rangle\right|^{2r} \\
&\leq \frac{1}{2}r^{2r}\left(V\right)\sup_{\|x\|=1}\left(\left\|\left|f\left(\left|T\right|\right)A\right|^{2}x\right\|^{r}\left\|\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x\right\|^{r}\right. \\
&\left.+\left\langle\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}\left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle^{r}\right) \\
&\leq \frac{1}{2}r^{2r}\left(V\right)\left[\sup_{\|x\|=1}\left\|\left|f\left(\left|T\right|\right)A\right|^{2}x\right\|^{r}\sup_{\|x\|=1}\left\|\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}x\right\|^{r}\right. \\
&\left.+\sup_{\|x\|=1}\left\langle\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}\left|f\left(\left|T\right|\right)A\right|^{2}x,x\right\rangle^{r}\right] \\
&= \frac{1}{2}r^{2r}\left(V\right)\left[\left\|f\left(\left|T\right|\right)A\right\|^{2r}\left\|g\left(\left|T^{*}\right|\right)B^{*}\right\|^{2r}+\omega^{r}\left(\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2}\left|f\left(\left|T\right|\right)A\right|^{2}\right)\right],
\end{aligned}$$

which proves (3.2).

The inequality (3.4) follows in a similar way from (2.5). □

We also have:

Corollary 3. *Assume that f and g are non-negative functions on $[0,\infty)$ that are continuous and satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Let X, A, $B\in\mathcal{B}\left(H\right)$, then for all α, $\beta\geq 0$ with $\alpha+\beta\geq 1$, we have*

$$\begin{aligned}
&\omega^{2r}\left(BX\left|X\right|^{\alpha+\beta-1}A\right) \\
&\leq \left\|X\right\|^{2\beta r}\left\|\frac{1}{p}\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2qr}\right\|,
\end{aligned}\tag{3.5}$$

provided that $r>0$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr\geq 1$.

If $r\geq 1$,

$$\begin{aligned}
&\omega^{2r}\left(BX\left|X\right|^{\alpha+\beta-1}A\right) \\
&\leq \frac{1}{2}\left\|X\right\|^{2\beta r}\left(\left\|f\left(\left|X\right|^{\alpha}\right)A\right\|^{2r}\left\|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right\|^{2r}\right. \\
&\left.+\omega^{r}\left(\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2}\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2}\right)\right).
\end{aligned}\tag{3.6}$$

If $r\geq 1$, p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr\geq 2$, *then also*

$$\begin{aligned}(3.7)\qquad &\omega^{2r}\left(BX\left|X\right|^{\alpha+\beta-1}A\right)\\ &\leq\frac{1}{2}\left\|X\right\|^{2\beta r}\left(\left\|\frac{1}{p}\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2qr}\right\|\right.\\ &\left.+\omega^{r}\left(\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2}\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2}\right)\right).\end{aligned}$$

Remark 3. *Let* $\alpha\in[0,1]$. *Then we get*

$$\begin{aligned}(3.8)\qquad &\omega^{2r}\left(BXA\right)\\ &\leq\left\|X\right\|^{2(1-\alpha)r}\left\|\frac{1}{p}\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2qr}\right\|,\end{aligned}$$

provided that $r>0$, p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr\geq 1$.

If $r\geq 1$,

$$\begin{aligned}(3.9)\qquad &\omega^{2r}\left(BXA\right)\\ &\leq\frac{1}{2}\left\|X\right\|^{2(1-\alpha)r}\left(\left\|f\left(\left|X\right|^{\alpha}\right)A\right\|^{2r}\left\|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right\|^{2r}\right.\\ &\left.+\omega^{r}\left(\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2}\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2}\right)\right).\end{aligned}$$

If $r\geq 1$, p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr\geq 2$, *then also*

$$\begin{aligned}(3.10)\qquad &\omega^{2r}\left(BXA\right)\\ &\leq\frac{1}{2}\left\|X\right\|^{2(1-\alpha)r}\left(\left(\left\|\frac{1}{p}\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2qr}\right\|\right)\right.\\ &\left.+\omega^{r}\left(\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2}\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2}\right)\right).\end{aligned}$$

We also have:

Proposition 2. *With the assumptions of Theorem 2, we have for* $r\geq 1$ *and* $\delta\in[0,1]$ *that*

$$\begin{aligned}(3.11)\qquad \left|\omega\left(BTVA\right)\right|^{2r}&\leq r^{2r}\left(V\right)\left\|\left(1-\delta\right)\left|f\left(\left|T\right|\right)A\right|^{2r}+\delta\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2r}\right\|\\ &\times\left\|f\left(\left|T\right|\right)A\right\|^{2r\delta}\left\|g\left(\left|T^{*}\right|\right)B^{*}\right\|^{2r(1-\delta)}.\end{aligned}$$

Also, we have

$$\begin{aligned}(3.12)\qquad \left|\omega\left(BTVA\right)\right|^{2}&\leq r^{2}\left(V\right)\left\|\left(1-\delta\right)\left|f\left(\left|T\right|\right)A\right|^{2r}+\delta\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2r}\right\|^{1/r}\\ &\times\left\|\delta\left|f\left(\left|T\right|\right)A\right|^{2r}+\left(1-\delta\right)\left|g\left(\left|T^{*}\right|\right)B^{*}\right|^{2r}\right\|^{1/r}.\end{aligned}$$

The proof follows by Theorem 3 on taking the supremum over $\|x\|=1$.

Corollary 4. *With the assumptions of Corollary 1, we have for* $\delta\in[0,1]$, *that*

$$\begin{aligned}(3.13)\qquad &\omega^{2r}\left(BX\left|X\right|^{\alpha+\beta-1}A\right)\\ &\leq\left\|X\right\|^{2\beta r}\left\|\left(1-\delta\right)\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+\delta\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right\|\\ &\times\left\|f\left(\left|X\right|^{\alpha}\right)A\right\|^{2r\delta}\left\|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right\|^{r(1-\delta)}.\end{aligned}$$

Also, we have

$$\omega^{2}\left(BX\left|X\right|^{\alpha+\beta-1}A\right) \le \left\|X\right\|^{2\beta}\left\|(1-\delta)\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+\delta\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right\|^{1/r} \times\left\|\delta\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+(1-\delta)\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right\|^{1/r}. \tag{3.14}$$

Now, if we take $\beta=1-\alpha$, $\alpha\in[0,1]$ in Corollary 4, then we get

$$\omega^{2r}\left(BXA\right) \le \left\|X\right\|^{2(1-\alpha)r}\left\|(1-\delta)\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+\delta\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right\| \times\left\|f\left(\left|X\right|^{\alpha}\right)A\right\|^{2r\delta}\left\|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right\|^{r(1-\delta)}. \tag{3.15}$$

and

$$\omega^{2}\left(BXA\right) \le \left\|X\right\|^{2(1-\alpha)}\left\|(1-\delta)\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+\delta\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right\|^{1/r} \times\left\|\delta\left|f\left(\left|X\right|^{\alpha}\right)A\right|^{2r}+(1-\delta)\left|g\left(\left|X^{*}\right|^{\alpha}\right)B^{*}\right|^{2r}\right\|^{1/r} \tag{3.16}$$

for $\delta\in[0,1]$.

If the operator T has the polar decomposition $T=U\left|T\right|$ with U a partial isometry, we define the transform

$$\Delta_{p,q}\left(T\right):=\left|T\right|^{p}U\left|T\right|^{q},$$

for p, $q\ge 0$. Here we assume that $\left|T\right|^{0}=I$.

The *p-generalized Dougal transform* is defined by

$$\widehat{T}_{p}:=\left|T\right|^{p}U,$$

the usual *Dougal transform* is then

$$\widehat{T}:=\left|T\right|U,$$

and the *p-generalized Aluthge transform*

$$\widetilde{T}_{p}:=\left|T\right|^{p}U\left|T\right|^{p},$$

which for $p=1/2$ gives the usual Aluthge transform

$$\widetilde{T}:=\left|T\right|^{1/2}U\left|T\right|^{1/2},$$

Also

$$T_{q}:=U\left|T\right|^{q},$$

which gives for $q=1$ the usual polar decomposition $T=U\left|T\right|$.

For $p=t$, $q=1-t$, where $t\in[0,1]$ we have

$$\Delta_{t}\left(T\right):=\Delta_{t,1-t}\left(T\right)=\left|T\right|^{t}V\left|T\right|^{1-t}.$$

The transform $\Delta_{t}\left(T\right)$ was introduced and studied in [6].

Now, if we use Remark 3 for $X = \Delta_{p,q}(T)$ and $A = |T|^m$ and $B = |T|^n$ for $p, q, m, n \geq 0$, then for $\alpha \in [0,1]$ we get

$$\begin{aligned}
(3.17)\qquad & \omega^{2r}\left(\Delta_{p+n,q+m}(T)\right) \\
& \leq \left\|\Delta_{p,q}(T)\right\|^{2(1-\alpha)r} \\
& \times \left\| \frac{1}{p}\left|f\left(\left|\Delta_{p,q}(T)\right|^{\alpha}\right)|T|^m\right|^{2pr} + \frac{1}{q}\left|g\left(\left|\left(\Delta_{p,q}(T)\right)^*\right|^{\alpha}\right)|T|^n\right|^{2qr}\right\|,
\end{aligned}$$

provided that $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$.

If $r \geq 1$,

$$\begin{aligned}
(3.18)\qquad & \omega^{2r}\left(\Delta_{p+n,q+m}(T)\right) \\
& \leq \frac{1}{2}\left\|\Delta_{p,q}(T)\right\|^{2(1-\alpha)r} \\
& \times \left(\left\|f\left(\left|\Delta_{p,q}(T)\right|^{\alpha}\right)|T|^m\right\|^{2r} \left\|g\left(\left|\left(\Delta_{p,q}(T)\right)^*\right|^{\alpha}\right)|T|^n\right\|^{2r} \right. \\
& \left. +\omega^r\left(\left|g\left(\left|\left(\Delta_{p,q}(T)\right)^*\right|^{\alpha}\right)|T|^n\right|^2 \left|f\left(\left|\Delta_{p,q}(T)\right|^{\alpha}\right)|T|^m\right|^2 \right)\right).
\end{aligned}$$

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then also

$$\begin{aligned}
(3.19)\qquad & \omega^{2r}\left(\Delta_{p+n,q+m}(T)\right) \\
& \leq \frac{1}{2}\left\|\Delta_{p,q}(T)\right\|^{2(1-\alpha)r} \\
& \times \left(\left(\left\| \frac{1}{p}\left|f\left(\left|\Delta_{p,q}(T)\right|^{\alpha}\right)|T|^m\right|^{2pr} + \frac{1}{q}\left|g\left(\left|\left(\Delta_{p,q}(T)\right)^*\right|^{\alpha}\right)|T|^n\right|^{2qr}\right\| \right) \right. \\
& \left. +\omega^r\left(\left|g\left(\left|\left(\Delta_{p,q}(T)\right)^*\right|^{\alpha}\right)|T|^n\right|^2 \left|f\left(\left|\Delta_{p,q}(T)\right|^{\alpha}\right)|T|^m\right|^2 \right)\right).
\end{aligned}$$

From (3.15) and (3.16) we also get for α, $\delta \in [0,1]$ that

$$\begin{aligned}
(3.20)\qquad & \omega^{2r}\left(\Delta_{p+n,q+m}(T)\right) \\
& \leq \left\|\Delta_{p,q}(T)\right\|^{2(1-\alpha)r} \\
& \times \left\| (1-\delta)\left|f\left(\left|\Delta_{p,q}(T)\right|^{\alpha}\right)|T|^m\right|^{2r} + \delta\left|g\left(\left|\left(\Delta_{p,q}(T)\right)^*\right|^{\alpha}\right)|T|^n\right|^{2r}\right\| \\
& \times \left\|f\left(\left|\Delta_{p,q}(T)\right|^{\alpha}\right)|T|^m\right\|^{2r\delta} \left\|g\left(\left|\left(\Delta_{p,q}(T)\right)^*\right|^{\alpha}\right)|T|^n\right\|^{r(1-\delta)}
\end{aligned}$$

and

$$\begin{aligned}
(3.21)\qquad & \omega^{2}\left(\Delta_{p+n,q+m}(T)\right) \\
& \leq \left\|\Delta_{p,q}(T)\right\|^{2(1-\alpha)} \\
& \times \left\| (1-\delta)\left|f\left(\left|\Delta_{p,q}(T)\right|^{\alpha}\right)|T|^m\right|^{2r} + \delta\left|g\left(\left|\left(\Delta_{p,q}(T)\right)^*\right|^{\alpha}\right)|T|^n\right|^{2r}\right\|^{1/r} \\
& \times \left\| \delta\left|f\left(\left|\Delta_{p,q}(T)\right|^{\alpha}\right)|T|^m\right|^{2r} + (1-\delta)\left|g\left(\left|\left(\Delta_{p,q}(T)\right)^*\right|^{\alpha}\right)|T|^n\right|^{2r}\right\|^{1/r}
\end{aligned}$$

By taking some particular values for $p, q, m, n \geq 0$ we can obtain certain inequalities for the Aluthge and Dougal transforms. The details are omitted.

4. Inequalities for p-Schatten Norms

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty. \tag{4.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{4.2}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (4.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 4. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{4.3}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1\|T\|; \tag{4.4}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [16, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} = \left(\sum_{i\in I}\langle |A|^p e_i, e_i\rangle\right)^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{4.5}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{4.6}$$

For $p\ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [16, p. 60-64],

$$\|A\|_p = \|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{4.7}$$

$$\|AB\|_p \le \|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{4.8}$$

and

$$\|AB\|_p \le \|A\|_p \|B\|, \ \|BA\|_p \le \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{4.9}$$

This implies that

$$\|CAB\|_p \le \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{4.10}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \le) \|AB\|_1 \le \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{4.11}$$

For the theory of trace functionals and their applications the reader is referred to [14] and [16].

For $\mathcal{E} := \{e_i\}_{i \in I}$ an orthonormal basis of H we define for $A \in \mathcal{B}_p(H)$, $p \ge 1$

$$\|A\|_{\mathcal{E},p} := \left(\sum_{i \in I} |\langle Ae_i, e_i \rangle|^p \right)^{1/p}.$$

We observe that $\|\cdot\|_{\mathcal{E},p}$ is a norm on $\mathcal{B}_p(H)$ and $\|A\|_{\mathcal{E},p} \le \|A\|_p$ for $A \in \mathcal{B}_p(H)$.

Proposition 3. *Assume that f and g are non-negative functions on $[0, \infty)$ that are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Let T, V be operators in $\mathcal{B}(H)$ such that $|T| V = V^* |T|$, then for A, $B \in \mathcal{B}(H)$ with $|f(|T|) A|^{2pr}$, $|g(|T^*|) B^*|^{2qr} \in \mathcal{B}_1(H)$ for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$ and pr, $qr \ge 1$, we have*

$$\|BTVA\|_{\mathcal{E},2r}^{2r} \le r^{2r}(V) \operatorname{tr} \left[\frac{1}{p} |f(|T|) A|^{2pr} + \frac{1}{q} |g(|T^*|) B^*|^{2qr} \right]. \tag{4.12}$$

If $r \ge 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, pr, $qr \ge 2$, and $|f(|T|) A|^{2pr}$, $|g(|T^*|) B^*|^{2qr} \in \mathcal{B}_1(H)$ while $|g(|T^*|) B^*|^2 |f(|T|) A|^2 \in \mathcal{B}_1(H)$, then

$$\begin{aligned} \|BTVA\|_{\mathcal{E},2r}^{2r} &\le \frac{1}{2} r^{2r}(V) \left[\operatorname{tr} \left(\frac{1}{p} |f(|T|) A|^{2pr} + \frac{1}{q} |g(|T^*|) B^*|^{2qr} \right) \right. \\ &\quad \left. + \left\| |g(|T^*|) B^*|^2 |f(|T|) A|^2 \right\|_{\mathcal{E},r}^r \right] \\ &\le \frac{1}{2} r^{2r}(V) \left[\operatorname{tr} \left(\frac{1}{p} |f(|T|) A|^{2pr} + \frac{1}{q} |g(|T^*|) B^*|^{2qr} \right) \right. \\ &\quad \left. + \left\| |g(|T^*|) B^*|^2 |f(|T|) A|^2 \right\|_r^r \right]. \end{aligned} \tag{4.13}$$

Proof. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, then by (2.3) for $y = x = e_i$, $i \in I$, we get

$$\begin{aligned} &|\langle BTVAe_i, e_i \rangle|^{2r} \\ &\le r^{2r}(V) \left\langle \left[\frac{1}{p} |f(|T|) A|^{2pr} + \frac{1}{q} |g(|T^*|) B^*|^{2qr} \right] e_i, e_i \right\rangle \end{aligned}$$

for $i \in I$.

If we sum over $i \in I$, then we get

$$\sum_{i\in I} |\langle BTVAe_i, e_i\rangle|^{2r} \\ \le r^{2r}(V) \sum_{i\in I} \left\langle \left[\frac{1}{p}|f(|T|)A|^{2pr} + \frac{1}{q}|g(|T^*|)B^*|^{2qr}\right] e_i, e_i \right\rangle,$$

which proves (4.12).

If $r \ge 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 2$, then by (2.5) we get

$$|\langle BTVAe_i, e_i\rangle|^{2r} \\ \le \frac{1}{2} r^{2r}(V) \left(\left\langle \left(\frac{1}{p}|f(|T|)A|^{2pr} + \frac{1}{q}|g(|T^*|)B^*|^{2qr}\right) e_i, e_i \right\rangle \right. \\ \left. + \left\langle |g(|T^*|)B^*|^2 |f(|T|)A|^2 e_i, e_i \right\rangle^r \right)$$

for $i \in I$.

If we sum over $i \in I$, then we get the first part of (4.13). The second part is obvious. □

Corollary 5. *Assume that f and g are non-negative functions on $[0, \infty)$ that are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Let X, A, $B \in \mathcal{B}(H)$ and α, $\beta \ge 0$ with $\alpha + \beta \ge 1$, if $|f(|X|^\alpha)A|^{2pr}$, $|g(|X^*|^\alpha)B^*|^{2qr} \in \mathcal{B}_1(H)$ with $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$ and pr, $qr \ge 1$, then we have $BX|X|^{\alpha+\beta-1}A \in \mathcal{B}_1(H)$ and*

$$\left\| BX|X|^{\alpha+\beta-1}A \right\|_{\mathcal{E},2r}^{2r} \\ \le \|X\|^{2\beta r} \operatorname{tr}\left[\frac{1}{p}|f(|X|^\alpha)A|^{2pr} + \frac{1}{q}|g(|X^*|^\alpha)B^*|^{2qr}\right]. \tag{4.14}$$

If $r \ge 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 2$, then also

$$\left\| BX|X|^{\alpha+\beta-1}A \right\|_{\mathcal{E},2r}^{2r} \\ \le \frac{1}{2}\|X\|^{2\beta r}\left(\operatorname{tr}\left[\frac{1}{p}|f(|X|^\alpha)A|^{2pr} + \frac{1}{q}|g(|X^*|^\alpha)B^*|^{2qr}\right] \right. \\ \left. + \left\| |g(|X^*|^\alpha)B^*|^2 |f(|X|^\alpha)A|^2 \right\|_{\mathcal{E},r}^r \right) \\ \le \frac{1}{2}\|X\|^{2\beta r}\left(\operatorname{tr}\left[\frac{1}{p}|f(|X|^\alpha)A|^{2pr} + \frac{1}{q}|g(|X^*|^\alpha)B^*|^{2qr}\right] \right. \\ \left. + \left\| |g(|X^*|^\alpha)B^*|^2 |f(|X|^\alpha)A|^2 \right\|_r^r \right). \tag{4.15}$$

We notice that, if we take $\beta = 1 - \alpha$ in Corollary 5, then we get

$$\|BXA\|_{\mathcal{E},2r}^{2r} \le \|X\|^{2(1-\alpha)r} \operatorname{tr}\left[\frac{1}{p}|f(|X|^\alpha)A|^{2pr} + \frac{1}{q}|g(|X^*|^\alpha)B^*|^{2qr}\right] \tag{4.16}$$

and

$$\begin{aligned}
(4.17)\qquad & \|BXA\|_{\mathcal{E},2r}^{2r} \\
& \leq \frac{1}{2}\|X\|^{2(1-\alpha)r}\left(\operatorname{tr}\left[\frac{1}{p}\left|f\left(|X|^{\alpha}\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(|X^{*}|^{\alpha}\right)B^{*}\right|^{2qr}\right]\right. \\
& \left.+\left\|\left|g\left(|X^{*}|^{\alpha}\right)B^{*}\right|^{2}\left|f\left(|X|^{\alpha}\right)A\right|^{2}\right\|_{\mathcal{E},r}^{r}\right) \\
& \leq \frac{1}{2}\|X\|^{2(1-\alpha)r}\left(\operatorname{tr}\left[\frac{1}{p}\left|f\left(|X|^{\alpha}\right)A\right|^{2pr}+\frac{1}{q}\left|g\left(|X^{*}|^{\alpha}\right)B^{*}\right|^{2qr}\right]\right. \\
& \left.+\left\|\left|g\left(|X^{*}|^{\alpha}\right)B^{*}\right|^{2}\left|f\left(|X|^{\alpha}\right)A\right|^{2}\right\|_{r}^{r}\right).
\end{aligned}$$

Corollary 6. *Let T, V be operators in $\mathcal{B}(H)$ such that $|T|V=V^{*}|T|$ and $\lambda\in[0,1]$, then for $A,B\in\mathcal{B}(H)$ with $\left||T|^{\lambda}A\right|^{2pr}$, $\left||T^{*}|^{1-\lambda}B^{*}\right|^{2qr}\in\mathcal{B}_{1}(H)$ for $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>0$ and pr, $qr\geq 1$, we have*

$$(4.18)\qquad \|BTVA\|_{\mathcal{E},2r}^{2r}\leq r^{2r}(V)\operatorname{tr}\left[\frac{1}{p}\left||T|^{\lambda}A\right|^{2pr}+\frac{1}{q}\left||T^{*}|^{1-\lambda}B^{*}\right|^{2qr}\right].$$

If $r\geq 1$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, pr, $qr\geq 2$, and $\left||T|^{\lambda}A\right|^{2pr}$, $\left||T^{*}|^{1-\lambda}B^{*}\right|^{2qr}\in\mathcal{B}_{1}(H)$ while $\left||T^{*}|^{1-\lambda}B^{*}\right|^{2}\left||T|^{\lambda}A\right|^{2}\in\mathcal{B}_{1}(H)$, then

$$\begin{aligned}
(4.19)\qquad \|BTVA\|_{\mathcal{E},2r}^{2r} & \leq \frac{1}{2}r^{2r}(V)\left[\operatorname{tr}\left(\frac{1}{p}\left||T|^{\lambda}A\right|^{2pr}+\frac{1}{q}\left||T^{*}|^{1-\lambda}B^{*}\right|^{2qr}\right)\right. \\
& \left.+\left\|\left||T^{*}|^{1-\lambda}B^{*}\right|^{2}\left||T|^{\lambda}A\right|^{2}\right\|_{\mathcal{E},r}^{r}\right] \\
& \leq \frac{1}{2}r^{2r}(V)\left[\operatorname{tr}\left(\frac{1}{p}\left||T|^{\lambda}A\right|^{2pr}+\frac{1}{q}\left||T^{*}|^{1-\lambda}B^{*}\right|^{2qr}\right)\right. \\
& \left.+\left\|\left||T^{*}|^{1-\lambda}B^{*}\right|^{2}\left|f\left(|T|\right)A\right|^{2}\right\|_{r}^{r}\right].
\end{aligned}$$

Finally, by Corollary 5 we get for X, A, $B\in\mathcal{B}(H)$ and $\lambda\in[0,1]$, α, $\beta\geq 0$ with $\alpha+\beta\geq 1$, and if $\left||X|^{\lambda\alpha}A\right|^{2pr}$, $\left||X^{*}|^{(1-\lambda)\alpha}B^{*}\right|^{2qr}\in\mathcal{B}_{1}(H)$ for $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>0$ and pr, $qr\geq 1$, then we have $BX|X|^{\alpha+\beta-1}A\in\mathcal{B}_{1}(H)$ and

$$\begin{aligned}
(4.20)\qquad & \left\|BX|X|^{\alpha+\beta-1}A\right\|_{\mathcal{E},2r}^{2r} \\
& \leq \|X\|^{2\beta r}\operatorname{tr}\left[\frac{1}{p}\left||X|^{\lambda\alpha}A\right|^{2pr}+\frac{1}{q}\left||X^{*}|^{(1-\lambda)\alpha}B^{*}\right|^{2qr}\right].
\end{aligned}$$

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 2$, then also

$$\begin{aligned}(4.21)\qquad & \left\| BX \left| X \right|^{\alpha+\beta-1} A \right\|_{\mathcal{E},2r}^{2r} \\ & \leq \frac{1}{2} \left\| X \right\|^{2\beta r} \left(\operatorname{tr}\left[\frac{1}{p} \left| \left| X \right|^{\lambda\alpha} A \right|^{2pr} + \frac{1}{q} \left| \left| X^* \right|^{(1-\lambda)} B^* \right|^{2qr} \right] \right. \\ & \left. + \left\| \left| \left| X^* \right|^{(1-\lambda)\alpha} B^* \right|^2 \left| \left| X \right|^{\lambda\alpha} A \right|^2 \right\|_{\mathcal{E},r}^{r} \right) \\ & \leq \frac{1}{2} \left\| X \right\|^{2\beta r} \left(\operatorname{tr}\left[\frac{1}{p} \left| \left| X \right|^{\lambda\alpha} A \right|^{2pr} + \frac{1}{q} \left| \left| X^* \right|^{(1-\lambda)\alpha} B^* \right|^{2qr} \right] \right. \\ & \left. + \left\| \left| \left| X^* \right|^{(1-\lambda)\alpha} B^* \right|^2 \left| \left| X \right|^{\lambda\alpha} A \right|^2 \right\|_{r}^{r} \right).\end{aligned}$$

Moreover, if we take in (4.20) and (4.21) $\beta = 1 - \alpha$, then we get

$$(4.22)\qquad \left\| BXA \right\|_{\mathcal{E},2r}^{2r} \leq \left\| X \right\|^{2(1-\alpha)r} \operatorname{tr}\left[\frac{1}{p} \left| \left| X \right|^{\lambda\alpha} A \right|^{2pr} + \frac{1}{q} \left| \left| X^* \right|^{(1-\lambda)\alpha} B^* \right|^{2qr} \right]$$

and

$$\begin{aligned}(4.23)\qquad & \left\| BXA \right\|_{\mathcal{E},2r}^{2r} \\ & \leq \frac{1}{2} \left\| X \right\|^{2(1-\alpha)r} \left(\operatorname{tr}\left[\frac{1}{p} \left| \left| X \right|^{\lambda\alpha} A \right|^{2pr} + \frac{1}{q} \left| \left| X^* \right|^{(1-\lambda)} B^* \right|^{2qr} \right] \right. \\ & \left. + \left\| \left| \left| X^* \right|^{(1-\lambda)\alpha} B^* \right|^2 \left| \left| X \right|^{\lambda\alpha} A \right|^2 \right\|_{\mathcal{E},r}^{r} \right) \\ & \leq \frac{1}{2} \left\| X \right\|^{2(1-\alpha)r} \left(\operatorname{tr}\left[\frac{1}{p} \left| \left| X \right|^{\lambda\alpha} A \right|^{2pr} + \frac{1}{q} \left| \left| X^* \right|^{(1-\lambda)\alpha} B^* \right|^{2qr} \right] \right. \\ & \left. + \left\| \left| \left| X^* \right|^{(1-\lambda)\alpha} B^* \right|^2 \left| \left| X \right|^{\lambda\alpha} A \right|^2 \right\|_{r}^{r} \right).\end{aligned}$$

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