General sigmoid function based complex valued trigonometric and hyperbolic neural network high order approximations

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Abstract

Here we research the univariate quantitative approximation of complex valued continuous functions on a compact interval by complex valued neural network operators. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the used function's high order derivatives. The kind of our approximations are trigonometric and hyperbolic. Our operators are defined by using a density function generated by a general sigmoid function. The approximations are pointwise and of the uniform norm. The related complex valued feed-forward neural networks are with one hidden layer.

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1 Introduction

The author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support.

Again the author inspired by [12], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation oper-

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ators of sigmoidal and hyperbolic tangent type which resulted into [3]-[7], by treating both the univariate and multivariate cases.

Here the author performs general sigmoid function activated high order neural network approximations to continuous functions over compact intervals of the real line with complex values. All convergences are with rates expressed via the modulus of continuity of the involved functions high order derivatives, deriving by very tight Jackson type inequalities.

The basis of our higher order approximations here are some newly discovered by the author trigonometric and hyperbolic type Taylor's formulae.

Our compact intervals are not necessarily symmetric to the origin. In preparation to prove our results we describe important properties of the basic density function defining our operators which is induced by a general sigmoid function, which is the activation function.

Feed-forward $\mathbb{C}$-valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, \ \ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{C}$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. About neural networks in general read [13], [14], [15].

2 Basics

Here we follow [9].

Let $h : \mathbb{R} \to [-1, 1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0$, $h(-x) = -h(x)$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also h is strictly convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.

We consider the activation function

$$\psi(x) := \frac{1}{4}\left(h(x+1) - h(x-1)\right), \ \ x \in \mathbb{R}, \tag{1}$$

As in [6], p. 34, we get that $\psi(-x) = \psi(x)$, thus ψ is an even function. Since $x + 1 > x - 1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi(0) = \frac{h(1)}{2}. \tag{2}$$

Let $x > 1$, we have that

$$\psi'(x) = \frac{1}{4}\left(h'(x+1) - h'(x-1)\right) < 0,$$

by h' being strictly decreasing over $[0,+\infty)$.

Let now $0<x<1$, then $1-x>0$ and $0<1-x<1+x$. It holds $h'(x-1)=h'(1-x)>h'(x+1)$, so that again $\psi'(x)<0$. Consequently ψ is stritly decreasing on $(0,+\infty)$.

Clearly, ψ is strictly increasing on $(-\infty,0)$, and $\psi'(0)=0$.

See that

$$\lim_{x\to+\infty}\psi(x)=\frac{1}{4}\left(h(+\infty)-h(+\infty)\right)=0, \tag{3}$$

and

$$\lim_{x\to-\infty}\psi(x)=\frac{1}{4}\left(h(-\infty)-h(-\infty)\right)=0. \tag{4}$$

That is the x-axis is the horizontal asymptote on ψ.

Conclusion, ψ is a bell symmetric function with maximum

$$\psi(0)=\frac{h(1)}{2}.$$

We need

Theorem 1 *([10], Ch. 21) We have that*

$$\sum_{i=-\infty}^{\infty}\psi(x-i)=1,\ \ \forall\ x\in\mathbb{R}. \tag{5}$$

Theorem 2 *([10], Ch. 21) It holds*

$$\int_{-\infty}^{\infty}\psi(x)\,dx=1. \tag{6}$$

Thus $\psi(x)$ is a density function on $\mathbb{R}$.

We give

Theorem 3 *([10], Ch. 21) Let $0<\alpha<1$, and $n\in\mathbb{N}$ with $n^{1-\alpha}>2$. It holds*

$$\sum_{\begin{cases}k=-\infty\\ :|nx-k|\geq n^{1-\alpha}\end{cases}}^{\infty}\psi(nx-k)<\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}. \tag{7}$$

Notice that

$$\lim_{n\to+\infty}\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}=0.$$

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number.

We further give

Theorem 4 *([10], Ch. 21) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} < \frac{1}{\psi(1)}, \quad \forall\, x \in [a,b]. \tag{8}$$

Remark 5 *([10], Ch. 21) We have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \neq 1, \tag{9}$$

for at least some $x \in [a,b]$.

See [6], p. 39, same reasoning.

Note 6 *([10], Ch. 21) For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds (by (5))*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \leq 1. \tag{10}$$

Let $(\mathbb{C}, \|\cdot\|)$ be the complex numbers Banach space.

Definition 7 *Let $f \in C([a,b], \mathbb{C})$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the $\mathbb{C}$-valued linear neural network operators*

$$A_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}, \quad x \in [a,b]. \tag{11}$$

Clearly here $A_n(f,x) \in C([a,b], \mathbb{C})$. For convenience we use the same A_n for real valued function when needed. We study here the pointwise and uniform convergence of $A_n(f,x)$ to $f(x)$ with rates.

For convenience also we call

$$A_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi(nx-k), \tag{12}$$

(similarly A_n^* can be defined for real valued function) that is

$$A_n(f,x) = \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}. \tag{13}$$

So that

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} - f(x)$$

$$= \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}. \tag{14}$$

Consequently we derive

$$|A_n(f,x) - f(x)| \leq \frac{1}{\psi(1)}\left|A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\right)\right|. \tag{15}$$

That is

$$|A_n(f,x) - f(x)| \leq \frac{1}{\psi(1)}\left|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right) - f(x)\right)\psi(nx-k)\right|. \tag{16}$$

We will estimate the right hand side of (19).

For that we need, for $f \in C([a,b],\mathbb{C})$ the first modulus of continuity

$$\omega_1(f,\delta)_{[a,b]} := \omega_1(f,\delta) := \sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}}|f(x)-f(y)|, \quad \delta > 0. \tag{17}$$

The fact $f \in C([a,b],\mathbb{C})$, is equivalent to $\lim_{\delta\to0}\omega_1(f,\delta) = 0$, see [8].

We denote by $\|f\|_\infty := \sup_{x\in[a,b]}\|f(x)\|$, for $f \in C([a,b],\mathbb{C})$.

3 Main Results

We present $\mathbb{C}$-valued neural network high order approximations to a function given with rates. We start with a trigonometric approximation.

Theorem 8 *Let $f \in C^2([a,b],\mathbb{C})$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

1)

$$|A_n(f,x) - f(x)| \leq \frac{1}{\psi(1)}\left[|f'(x)|\left(\frac{1}{n^\alpha} + (b-a)\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right)\right. \tag{18}$$

$$+\frac{|f''(x)|}{2}\left(\frac{1}{n^{2\alpha}} + (b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right) +$$

$$\left.\left(\frac{\omega_1\left(f''+f,\frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f''+f\|_\infty(b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right)\right],$$

2) if $f'(x) = f''(x) = 0$, we obtain

$$|A_n(f,x) - f(x)| \leq \frac{1}{\psi(1)} \left[\frac{\omega_1\left(f''+f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}} + \|f''+f\|_{\infty} (b-a)^2 \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2} \right], \tag{19}$$

notice here the high rate of convergence at $n^{-3\alpha}$,

3) furthermore we get

$$\|A_n f - f\|_{\infty} \leq \frac{1}{\psi(1)} \left[\|f'\|_{\infty} \left(\frac{1}{n^{\alpha}} + (b-a) \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2} \right) \right.$$

$$+ \frac{\|f''\|_{\infty}}{2} \left(\frac{1}{n^{2\alpha}} + (b-a)^2 \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2} \right) +$$

$$\left. \left(\frac{\omega_1\left(f''+f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}} + \|f''+f\|_{\infty} (b-a)^2 \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2} \right) \right], \tag{20}$$

i.e. $\lim_{n\to+\infty} A_n(f) = f$, *pointwise and uniformly,*

4) and finally, it holds

$$\left| A_n(f,x) - f'(x) A_n(\sin(\cdot - x), x) - 2f''(x) A_n\left(\sin^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x) \right| \leq \tag{21}$$

$$\frac{1}{\psi(1)} \left[\frac{\omega_1\left(f''+f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}} + \|f''+f\|_{\infty} (b-a)^2 \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2} \right],$$

again here we achieve high speed of convergence at $n^{-3\alpha}$.

Proof. Here $f \in C^2([a,b], \mathbb{C})$, and we apply the trigonometric Taylor's formula for $f \in C^2([a,b], \mathbb{C})$, see Theorem 6 of [11].

Let $\frac{k}{n}, x \in [a,b]$, then

$$f\left(\frac{k}{n}\right) = f(x) + f'(x)\sin\left(\frac{k}{n} - x\right) + 2f''(x)\sin^2\left(\frac{\frac{k}{n} - x}{2}\right) +$$

$$\int_x^{\frac{k}{n}} \left[(f''(t) + f(t)) - (f''(x) + f(x)) \right] \sin\left(\frac{k}{n} - t\right) dt. \tag{22}$$

Hence it holds

$$f\left(\frac{k}{n}\right)\psi(nx-k) = f(x)\psi(nx-k) +$$

$$f'(x)\sin\left(\frac{k}{n} - x\right)\psi(nx-k) + 2f''(x)\sin^2\left(\frac{\frac{k}{n} - x}{2}\right)\psi(nx-k) +$$

$$\psi\left(nx-k\right)\left(\int_{x}^{\frac{k}{n}}\left[\left(f''\left(t\right)+f\left(t\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right]\sin\left(\frac{k}{n}-t\right)dt\right). \tag{23}$$

So that we have

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\psi\left(nx-k\right)-f\left(x\right)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)=$$

$$f'\left(x\right)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\sin\left(\frac{k}{n}-x\right)+2f''\left(x\right)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\sin^{2}\left(\frac{\frac{k}{n}-x}{2}\right)+$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\left(\int_{x}^{\frac{k}{n}}\left[\left(f''\left(t\right)+f\left(t\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right]\sin\left(\frac{k}{n}-t\right)dt\right). \tag{24}$$

Thus, we obtain

$$A_{n}^{*}\left(f,x\right)-f\left(x\right)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)=$$

$$f'\left(x\right)A_{n}^{*}\left(\sin\left(\cdot-x\right),x\right)+2f''\left(x\right)A_{n}^{*}\left(\sin^{2}\left(\frac{\cdot-x}{2}\right),x\right)+\Lambda_{n}\left(x\right), \tag{25}$$

where

$$\Lambda_{n}\left(x\right):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\left(\int_{x}^{\frac{k}{n}}\left[\left(f''\left(t\right)+f\left(t\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right]\sin\left(\frac{k}{n}-t\right)dt\right).$$

We call

$$R_{2}\left(n\right):=\int_{x}^{\frac{k}{n}}\left[\left(f''\left(t\right)+f\left(t\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right]\sin\left(\frac{k}{n}-t\right)dt. \tag{26}$$

We assume that $b-a>\frac{1}{n^{\alpha}}$, which is always the case for large enough $n\in\mathbb{N}$, that is when $n>\left\lceil\left(b-a\right)^{-\frac{1}{\alpha}}\right\rceil$.

Thus $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}$ or $\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}$.

In case of $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}$, we have the following cases:

i) if $\frac{k}{n}\geq x$, then

$$\left|R_{2}\left(n\right)\right|=\left|\int_{x}^{\frac{k}{n}}\left[\left(f''\left(t\right)+f\left(t\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right]\sin\left(\frac{k}{n}-t\right)dt\right|\leq$$

$$\int_{x}^{\frac{k}{n}}\omega_{1}\left(f''+f,t-x\right)\left|\sin\left(\frac{k}{n}-t\right)\right|dt\leq \tag{27}$$

(by $|\sin x| \leq |x|$, $\forall\, x \in \mathbb{R}$)

$$\int_x^{\frac{k}{n}} \omega_1 \left(f'' + f, t - x\right) \left(\frac{k}{n} - t\right) dt \leq \omega_1 \left(f'' + f, \frac{k}{n} - x\right) \frac{\left(\frac{k}{n} - x\right)^2}{2}$$

$$\leq \frac{\omega_1 \left(f'' + f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}},$$

that is

$$|R_2(n)| \leq \frac{\omega_1 \left(f'' + f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}}. \tag{28}$$

ii) if $\frac{k}{n} < x$, then

$$|R_2(n)| = \left| \int_x^{\frac{k}{n}} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right) \right] \sin \left(\frac{k}{n} - t\right) dt \right| =$$

$$\left| \int_{\frac{k}{n}}^{x} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right) \right] \sin \left(\frac{k}{n} - t\right) dt \right| \leq$$

$$\int_{\frac{k}{n}}^{x} \left| \left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right) \right| \left| \sin \left(\frac{k}{n} - t\right) \right| dt \leq \tag{29}$$

$$\int_{\frac{k}{n}}^{x} \omega_1 \left(f'' + f, x - \frac{k}{n}\right) \left(t - \frac{k}{n}\right) dt \leq \omega_1 \left(f'' + f, x - \frac{k}{n}\right) \frac{\left(x - \frac{k}{n}\right)^2}{2}$$

$$\leq \frac{\omega_1 \left(f'' + f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}}.$$

That is

$$|R_2(n)| \leq \frac{\omega_1 \left(f'' + f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}}. \tag{30}$$

So, we have proved when $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^\alpha}$, always it holds

$$|R_2(n)| \leq \frac{\omega_1 \left(f'' + f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}}. \tag{31}$$

Next assume again $\frac{k}{n} \geq x$, then

$$|R_2(n)| = \left| \int_x^{\frac{k}{n}} \left[\left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right) \right] \sin \left(\frac{k}{n} - t\right) dt \right| \leq$$

$$\int_x^{\frac{k}{n}} \left| \left(f''(t) + f(t)\right) - \left(f''(x) + f(x)\right) \right| \left| \sin \left(\frac{k}{n} - t\right) \right| dt \leq$$

(by $|\sin x| \leq |x|$, $\forall\, x \in \mathbb{R}$)

$$2 \left\| f'' + f \right\|_\infty \left(\int_x^{\frac{k}{n}} \left| \sin \left(\frac{k}{n} - t\right) \right| dt \right) \leq$$

$$2\left\|f''+f\right\|_{\infty}\left(\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-t\right)dt\right)=$$

$$2\left\|f''+f\right\|_{\infty}\frac{\left(\frac{k}{n}-x\right)^{2}}{2}\leq\left\|f''+f\right\|_{\infty}(b-a)^{2}. \tag{32}$$

Hence it is

$$\left|R_{2}(n)\right|\leq\left\|f''+f\right\|_{\infty}(b-a)^{2}. \tag{33}$$

When $\frac{k}{n}<x$, we have

$$\left|R_{2}(n)\right|=\left|\int_{x}^{\frac{k}{n}}\left[\left(f''(t)+f(t)\right)-\left(f''(x)+f(x)\right)\right]\sin\left(\frac{k}{n}-t\right)dt\right|=$$

$$\left|\int_{\frac{k}{n}}^{x}\left[\left(f''(x)+f(x)\right)-\left(f''(t)+f(t)\right)\right]\sin\left(\frac{k}{n}-t\right)dt\right|\leq$$

$$\int_{\frac{k}{n}}^{x}\left|\left(f''(x)+f(x)\right)-\left(f''(t)+f(t)\right)\right|\left|\sin\left(\frac{k}{n}-t\right)\right|dt\leq$$

$$2\left\|f''+f\right\|_{\infty}\int_{\frac{k}{n}}^{x}\left|\sin\left(\frac{k}{n}-t\right)\right|dt\leq$$

$$2\left\|f''+f\right\|_{\infty}\int_{\frac{k}{n}}^{x}\left(t-\frac{k}{n}\right)dt=\left\|f''+f\right\|_{\infty}\left(x-\frac{k}{n}\right)^{2}\leq$$

$$\left\|f''+f\right\|_{\infty}(b-a)^{2}. \tag{34}$$

Therefore, always it holds

$$\left|R_{2}(n)\right|\leq\left\|f''+f\right\|_{\infty}(b-a)^{2}. \tag{35}$$

And we have

$$\Lambda_{n}(x)=\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor}\psi(nx-k)\,R_{2}(n)+$$

$$\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor}\psi(nx-k)\,R_{2}(n). \tag{36}$$

Hence it holds

$$\left|\Lambda_{n}(x)\right|\leq\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor}\psi(nx-k)\left|R_{2}(n)\right|+$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} \psi(nx-k) \left| R_2(n) \right| \leq \tag{37}$$

$$\left(\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} \psi(nx-k) \right) \frac{\omega_1\left(f''+f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}} +$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} \psi(nx-k) \left\| f''+f \right\|_{\infty} (b-a)^2 \overset{\text{(by (10))}}{\leq}$$

$$\frac{\omega_1\left(f''+f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}} + \left\| f''+f \right\|_{\infty} (b-a)^2$$

$$\left(\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} \psi(nx-k) \right) \overset{\text{(by Theorem 3)}}{\leq}$$

$$\frac{\omega_1\left(f''+f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}} + \left\| f''+f \right\|_{\infty} (b-a)^2 \frac{\left(1 - h\left(n^{1-\alpha}-2\right)\right)}{2}. \tag{38}$$

Consequently, we have derived that

$$\left| \Lambda_n(x) \right| \leq \frac{\omega_1\left(f''+f, \frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}} + \left\| f''+f \right\|_{\infty} (b-a)^2 \frac{\left(1 - h\left(n^{1-\alpha}-2\right)\right)}{2}. \tag{39}$$

Next we use again $|\sin x| \leq |x|$, $\forall\, x \in \mathbb{R}$.

We have that

$$A_n^*\left(\sin(\cdot - x), x\right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \sin\left(\frac{k}{n} - x\right), \tag{40}$$

and

$$\left| A_n^*\left(\sin(\cdot - x), x\right) \right| \leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \left| \sin\left(\frac{k}{n} - x\right) \right| =$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb \rfloor} \psi(nx-k) \left| \sin\left(\frac{k}{n} - x\right) \right| +$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \psi (nx - k) \left| \sin \left(\frac{k}{n} - x \right) \right| \leq \tag{41}$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \psi (nx - k) \left| \frac{k}{n} - x \right| +$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \psi (nx - k) \left| \frac{k}{n} - x \right| \leq$$

$$\frac{1}{n^\alpha} + (b - a) \left(\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \psi (nx - k) \right) \overset{\text{(by (7))}}{\leq} \tag{42}$$

$$\frac{1}{n^\alpha} + (b - a) \frac{\left(1 - h \left(n^{1-\alpha} - 2\right)\right)}{2}.$$

We found that

$$\left| A_n^* \left(\sin \left(\cdot - x \right), x \right) \right| \leq \frac{1}{n^\alpha} + (b - a) \frac{\left(1 - h \left(n^{1-\alpha} - 2\right)\right)}{2}. \tag{43}$$

Next we estimate

$$A_n^* \left(\sin^2 \left(\frac{\cdot - x}{2} \right), x \right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi (nx - k) \sin^2 \left(\frac{\frac{k}{n} - x}{2} \right), \tag{44}$$

We have that (by $|\sin x| \leq |x|$, $\forall\ x \in \mathbb{R}$)

$$A_n^* \left(\sin^2 \left(\frac{\cdot - x}{2} \right), x \right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi (nx - k) \left| \sin \left(\frac{\frac{k}{n} - x}{2} \right) \right|^2 \leq$$

$$\frac{1}{4} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi (nx - k) \left| \frac{k}{n} - x \right|^2 =$$

$$\frac{1}{4} \left[\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \psi (nx - k) \left| \frac{k}{n} - x \right|^2 + \right.$$

$$\left. \sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor} \psi(nx-k)\left|\frac{k}{n}-x\right|^{2}\right] \leq \tag{45}$$

$$\frac{1}{4}\left[\frac{1}{n^{2\alpha}}+(b-a)^{2}\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right].$$

That is

$$A_{n}^{*}\left(\sin^{2}\left(\frac{\cdot-x}{2}\right),x\right)\leq\frac{1}{4}\left[\frac{1}{n^{2\alpha}}+(b-a)^{2}\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right]. \tag{46}$$

Consequently we have derived:

1)

$$|A_{n}(f,x)-f(x)|\leq\frac{1}{\psi(1)}\left[|f'(x)|\left(\frac{1}{n^{\alpha}}+(b-a)\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right)\right. \tag{47}$$

$$+\frac{|f''(x)|}{2}\left(\frac{1}{n^{2\alpha}}+(b-a)^{2}\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right)+$$

$$\left.\left(\frac{\omega_{1}\left(f''+f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''+f\|_{\infty}(b-a)^{2}\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right)\right].$$

2) if $f'(x)=f''(x)=0$, by (47), we obtain

$$|A_{n}(f,x)-f(x)|\leq\frac{1}{\psi(1)}\left[\frac{\omega_{1}\left(f''+f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''+f\|_{\infty}(b-a)^{2}\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right], \tag{48}$$

notice here the high rate of convergence at $n^{-3\alpha}$.

3) Furthermore, by (47), we get

$$\|A_{n}f-f\|_{\infty}\leq\frac{1}{\psi(1)}\left[\|f'\|_{\infty}\left(\frac{1}{n^{\alpha}}+(b-a)\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right)+\right.$$

$$\frac{\|f''\|_{\infty}}{2}\left(\frac{1}{n^{2\alpha}}+(b-a)^{2}\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right)+$$

$$\left.\left(\frac{\omega_{1}\left(f''+f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''+f\|_{\infty}(b-a)^{2}\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right)\right]. \tag{49}$$

We derive that $\lim_{n\to+\infty}A_{n}(f)=f$, pointwise and uniformly.

We observe that

$$A_n(f,x) - f'(x) A_n(\sin(\cdot - x), x) - 2f''(x) A_n\left(\sin^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x) =$$

$$\frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} - f'(x) \frac{A_n^*(\sin(\cdot - x), x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} -$$

$$2f''(x) \frac{A_n^*\left(\sin^2\left(\frac{\cdot - x}{2}\right), x\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} - f(x) \left(\frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}\right) = \tag{50}$$

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} \left[A_n^*(f,x) - f'(x) A_n^*(\sin(\cdot - x), x) -\right.$$

$$\left. 2f''(x) A_n^*\left(\sin^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\right] =$$

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} (\Lambda_n(x)). \tag{51}$$

Finally, we obtain ($\forall\ x \in [a,b]$, $n \in \mathbb{N}$):

4)

$$\left|A_n(f,x) - f'(x) A_n(\sin(\cdot - x), x) - 2f''(x) A_n\left(\sin^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x)\right| \overset{(8)}{\leq}$$

$$\frac{1}{\psi(1)} |\Lambda_n(x)| \overset{(39)}{\leq}$$

$$\frac{1}{\psi(1)} \left[\frac{\omega_1\left(f'' + f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f'' + f\|_\infty (b-a)^2 \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2}\right]. \tag{52}$$

The theorem is proved. ■

We continue with a hyperbolic high order neural network approximation.

Theorem 9 *Let* $f \in C^2([a,b], \mathbb{C})$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. *Then*

1)

$$|A_n(f,x) - f(x)| \leq \frac{1}{\psi(1)} \cosh(b-a) \left[|f'(x)| \left(\frac{1}{n^\alpha} + (b-a) \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2}\right) + \right.$$

$$\frac{|f''(x)|}{2} \left(\frac{1}{n^{2\alpha}} + (b-a)^2 \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2}\right) +$$

$$\left(\frac{\omega_1 \left(f'' - f, \frac{1}{n^\alpha} \right)}{2n^{2\alpha}} + \|f'' - f\|_\infty (b-a)^2 \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} \right) \Bigg], \qquad (53)$$

2) if $f'(x) = f''(x) = 0$, we obtain

$$|A_n(f,x) - f(x)| \leq \frac{1}{\psi(1)} \cosh(b-a)$$

$$\left[\frac{\omega_1 \left(f'' - f, \frac{1}{n^\alpha} \right)}{2n^{2\alpha}} + \|f'' - f\|_\infty (b-a)^2 \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} \right], \qquad (54)$$

notice here the high rate of convergence at $n^{-3\alpha}$,

3) furthermore, we get

$$\|A_n f - f\|_\infty \leq \frac{1}{\psi(1)} \cosh(b-a) \left[\|f'\|_\infty \left(\frac{1}{n^\alpha} + (b-a) \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} \right) + \right.$$

$$\frac{\|f''\|_\infty}{2} \left(\frac{1}{n^{2\alpha}} + (b-a)^2 \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} \right) +$$

$$\left. \left(\frac{\omega_1 \left(f'' - f, \frac{1}{n^\alpha} \right)}{2n^{2\alpha}} + \|f'' - f\|_\infty (b-a)^2 \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} \right) \right], \qquad (55)$$

it follows that $\lim_{n \to +\infty} A_n(f) = f$, pointwise and uniformly,

and

4)

$$\left| A_n(f,x) - f'(x) A_n(\sinh(\cdot - x), x) - 2f''(x) A_n\left(\sinh^2\left(\frac{\cdot - x}{2} \right), x \right) - f(x) \right| \leq$$

$$\frac{\cosh(b-a)}{\psi(1)} \left[\frac{\omega_1 \left(f'' - f, \frac{1}{n^\alpha} \right)}{2n^{2\alpha}} + \|f'' - f\|_\infty (b-a)^2 \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} \right], \qquad (56)$$

again here we achieve high speed of convergence at $n^{-3\alpha}$.

Proof. By the mean value theorem we have that

$$\sinh x = \sinh x - \sinh 0 = (\cosh \xi)(x - 0),$$

for some ξ between $\{0, x\}$, for any $x \in \mathbb{R}$.

Hence

$$|\sinh x| \leq \|\cosh\|_{\infty, [-(b-a), b-a]} |x|, \ \ \forall\, x \in [-(b-a), b-a]. \qquad (57)$$

That is, there exists $M \geq 1$ such that

$$|\sinh x| \leq M |x|, \ \ \forall\, x \in [-(b-a), b-a], \qquad (58)$$

where $M := \|\cosh\|_{\infty,[-(b-a),b-a]} = \cosh(b-a)$.

Here $f \in C^2([a,b],\mathbb{C})$, and we apply the hyperbolic Taylor's formula for $f \in C^2([a,b],\mathbb{C})$, see Theorem 7 of [11].

Let $\frac{k}{n}, x \in [a,b]$, then

$$f\left(\frac{k}{n}\right) = f(x) + f'(x)\sinh\left(\frac{k}{n}-x\right) + 2f''(x)\sinh^2\left(\frac{\frac{k}{n}-x}{2}\right) +$$

$$\int_x^{\frac{k}{n}} \left[(f''(t)-f(t)) - (f''(x)-f(x))\right]\sinh\left(\frac{k}{n}-t\right)dt. \tag{59}$$

Hence it holds

$$f\left(\frac{k}{n}\right)\psi(nx-k) = f(x)\psi(nx-k) +$$

$$f'(x)\sinh\left(\frac{k}{n}-x\right)\psi(nx-k) + 2f''(x)\sinh^2\left(\frac{\frac{k}{n}-x}{2}\right)\psi(nx-k) +$$

$$\psi(nx-k)\left(\int_x^{\frac{k}{n}} \left[(f''(t)-f(t)) - (f''(x)-f(x))\right]\sinh\left(\frac{k}{n}-t\right)dt\right). \tag{60}$$

So that we have

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\psi(nx-k) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k) =$$

$$f'(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\sinh\left(\frac{k}{n}-x\right) + 2f''(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\sinh^2\left(\frac{\frac{k}{n}-x}{2}\right) +$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\left(\int_x^{\frac{k}{n}} \left[(f''(t)-f(t)) - (f''(x)-f(x))\right]\sinh\left(\frac{k}{n}-t\right)dt\right). \tag{61}$$

Thus, we obtain

$$A_n^*(f,x) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k) =$$

$$f'(x)A_n^*(\sinh(\cdot - x), x) + 2f''(x)A_n^*\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) + \Lambda_n(x), \tag{62}$$

where

$$\Lambda_n(x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\left(\int_x^{\frac{k}{n}} \left[(f''(t)-f(t)) - (f''(x)-f(x))\right]\sinh\left(\frac{k}{n}-t\right)dt\right). \tag{63}$$

We call

$$R_2(n) := \int_x^{\frac{k}{n}} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh\left(\frac{k}{n} - t \right) dt. \tag{64}$$

We assume that $b - a > \frac{1}{n^{\alpha}}$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\alpha}} \right\rceil$.

Thus $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\alpha}}$ or $\left| \frac{k}{n} - x \right| > \frac{1}{n^{\alpha}}$.

In case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\alpha}}$, we have the following cases:

i) if $\frac{k}{n} \geq x$, then

$$|R_2(n)| = \left| \int_x^{\frac{k}{n}} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh\left(\frac{k}{n} - t \right) dt \right| \leq$$

$$\int_x^{\frac{k}{n}} \omega_1\left(f'' - f, t - x \right) \left| \sinh\left(\frac{k}{n} - t \right) \right| dt \overset{(58)}{\leq}$$

$$\int_x^{\frac{k}{n}} \omega_1\left(f'' - f, t - x \right) M \left(\frac{k}{n} - t \right) dt \leq M\omega_1\left(f'' - f, \frac{k}{n} - x \right) \int_x^{\frac{k}{n}} \left(\frac{k}{n} - t \right) dt = \tag{65}$$

$$M\omega_1\left(f'' - f, \frac{k}{n} - x \right) \frac{\left(\frac{k}{n} - x \right)^2}{2} \leq \frac{M\omega_1\left(f'' - f, \frac{1}{n^{\alpha}} \right)}{2n^{2\alpha}},$$

that is

$$|R_2(n)| \leq \frac{M\omega_1\left(f'' - f, \frac{1}{n^{\alpha}} \right)}{2n^{2\alpha}}. \tag{66}$$

ii) if $\frac{k}{n} < x$, then

$$|R_2(n)| = \left| \int_x^{\frac{k}{n}} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh\left(\frac{k}{n} - t \right) dt \right| =$$

$$\left| \int_{\frac{k}{n}}^{x} \left[\left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right] \sinh\left(\frac{k}{n} - t \right) dt \right| \leq$$

$$\int_{\frac{k}{n}}^{x} \left| \left(f''(t) - f(t) \right) - \left(f''(x) - f(x) \right) \right| \left| \sinh\left(\frac{k}{n} - t \right) \right| dt \leq \tag{67}$$

$$M\omega_1\left(f'' - f, x - \frac{k}{n} \right) \int_{\frac{k}{n}}^{x} \left(t - \frac{k}{n} \right) dt = M\omega_1\left(f'' - f, x - \frac{k}{n} \right) \frac{\left(x - \frac{k}{n} \right)^2}{2}$$

$$\leq \frac{M\omega_1\left(f'' - f, \frac{1}{n^{\alpha}} \right)}{2n^{2\alpha}},$$

that is

$$|R_2(n)| \leq \frac{M\omega_1\left(f'' - f, \frac{1}{n^{\alpha}} \right)}{2n^{2\alpha}}. \tag{68}$$

So, we have proved when $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}$, always it holds

$$\left|R_2(n)\right| \leq \frac{M\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}. \tag{69}$$

Next assume again $\frac{k}{n} \geq x$, then

$$\left|R_2(n)\right| = \left|\int_x^{\frac{k}{n}} \left[\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right]\sinh\left(\frac{k}{n}-t\right)dt\right| \leq$$

$$\int_x^{\frac{k}{n}} \left|\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right|\left|\sinh\left(\frac{k}{n}-t\right)\right|dt \leq$$

$$2M\left\|f''-f\right\|_{\infty}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-t\right)dt = \tag{70}$$

$$2M\left\|f''-f\right\|_{\infty}\frac{\left(\frac{k}{n}-x\right)^2}{2} \leq M\left\|f''-f\right\|_{\infty}(b-a)^2.$$

Hence

$$\left|R_2(n)\right| \leq M\left\|f''-f\right\|_{\infty}(b-a)^2. \tag{71}$$

When $\frac{k}{n} < x$, we have

$$\left|R_2(n)\right| = \left|\int_{\frac{k}{n}}^{x} \left[\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right]\sinh\left(\frac{k}{n}-t\right)dt\right| \leq$$

$$\int_{\frac{k}{n}}^{x} \left|\left(f''(t)-f(t)\right)-\left(f''(x)-f(x)\right)\right|\left|\sinh\left(\frac{k}{n}-t\right)\right|dt \leq$$

$$2M\left\|f''-f\right\|_{\infty}\int_{\frac{k}{n}}^{x}\left(t-\frac{k}{n}\right)dt = \tag{72}$$

$$2M\left\|f''-f\right\|_{\infty}\frac{\left(x-\frac{k}{n}\right)^2}{2} \leq M\left\|f''-f\right\|_{\infty}(b-a)^2.$$

Therefore, always it holds

$$\left|R_2(n)\right| \leq M\left\|f''-f\right\|_{\infty}(b-a)^2. \tag{73}$$

And we have

$$\Lambda_n(x) = \sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb\rfloor} \psi(nx-k)\,R_2(n) + \tag{74}$$

$$\sum_{\left\{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}\right.}^{\lfloor nb\rfloor} \psi\left(nx-k\right)R_2\left(n\right).$$

Hence it holds

$$\left|\Lambda_n\left(x\right)\right| \leq \sum_{\left\{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}\right.}^{\lfloor nb\rfloor} \psi\left(nx-k\right)\left|R_2\left(n\right)\right| + \tag{75}$$

$$\sum_{\left\{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}\right.}^{\lfloor nb\rfloor} \psi\left(nx-k\right)\left|R_2\left(n\right)\right| \leq$$

$$\left(\sum_{\left\{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}\right.}^{\lfloor nb\rfloor} \psi\left(nx-k\right)\right)\frac{\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)M}{2n^{2\alpha}}+$$

$$\left(\sum_{\left\{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}\right.}^{\lfloor nb\rfloor} \psi\left(nx-k\right)\right)M\left\|f''-f\right\|_{\infty}\left(b-a\right)^2 \overset{((10))}{\leq}$$

$$\frac{M\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+M\left\|f''-f\right\|_{\infty}\left(b-a\right)^2$$

$$\left(\sum_{\left\{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}\right.}^{\lfloor nb\rfloor} \psi\left(nx-k\right)\right) \overset{\text{(by Theorem 3)}}{\leq} \tag{76}$$

$$M\frac{\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+M\left\|f''-f\right\|_{\infty}\left(b-a\right)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}.$$

Consequently, we have derived that

$$\left|\Lambda_n\left(x\right)\right| \leq M\left[\frac{\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f''-f\right\|_{\infty}\left(b-a\right)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right]. \tag{77}$$

We have that

$$A_n^*\left(\sinh\left(\cdot-x\right),x\right)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\sinh\left(\frac{k}{n}-x\right), \tag{78}$$

and

$$\left|A_n^*\left(\sinh\left(\cdot-x\right),x\right)\right|\leq\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\left|\sinh\left(\frac{k}{n}-x\right)\right|=$$

$$\sum_{\substack{k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\left|\sinh\left(\frac{k}{n}-x\right)\right|+$$

$$\sum_{\substack{k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\left|\sinh\left(\frac{k}{n}-x\right)\right|\leq \tag{79}$$

$$M\left[\sum_{\substack{k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\left|\frac{k}{n}-x\right|+\right.$$

$$\left.\sum_{\substack{k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\left|\frac{k}{n}-x\right|\right]\leq \tag{80}$$

$$M\left[\frac{1}{n^{\alpha}}+\left(b-a\right)\left(\sum_{\substack{k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\right)\right]\leq$$

$$M\left[\frac{1}{n^{\alpha}}+\left(b-a\right)\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right].$$

We found that

$$\left|A_n^*\left(\sinh\left(\cdot-x\right),x\right)\right|\leq M\left[\frac{1}{n^{\alpha}}+\left(b-a\right)\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right]. \tag{81}$$

Next we estimate

$$A_n^*\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\sinh^2\left(\frac{\frac{k}{n}-x}{2}\right), \tag{82}$$

We have that

$$A_n^*\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\left(\sinh\left(\frac{\frac{k}{n}-x}{2}\right)\right)^2 \leq$$

$$\frac{M}{4}\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\left(\frac{k}{n}-x\right)^2 = \tag{83}$$

$$\frac{M}{4}\left[\sum_{\substack{k=\lceil na \rceil \\ :\left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \psi(nx-k)\left(\frac{k}{n}-x\right)^2 + \right.$$

$$\left.\sum_{\substack{k=\lceil na \rceil \\ :\left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \psi(nx-k)\left(\frac{k}{n}-x\right)^2\right] \leq$$

$$\frac{M}{4}\left[\frac{1}{n^{2\alpha}} + (b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right].$$

That is

$$A_n^*\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) \leq \frac{M}{4}\left[\frac{1}{n^{2\alpha}} + (b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right]. \tag{84}$$

By (16) and putting together (62), (77), (81) and (84) we derive
1)

$$|A_n(f,x) - f(x)| \leq \frac{1}{\psi(1)}M\left[|f'(x)|\left(\frac{1}{n^\alpha} + (b-a)\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right) + \right.$$

$$\frac{|f''(x)|}{2}\left(\frac{1}{n^{2\alpha}} + (b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right) +$$

$$\left.\left(\frac{\omega_1\left(f''-f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f''-f\|_\infty (b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right)\right]. \tag{85}$$

2) If $f'(x) = f''(x) = 0$, by (85), we obtain

$$|A_n(f,x) - f(x)| \leq$$

$$\frac{M}{\psi(1)} \left[\frac{\omega_1\left(f''-f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f''-f\|_\infty (b-a)^2 \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2} \right], \quad (86)$$

notice here the high rate of convergence at $n^{-3\alpha}$.

3) Furthermore, by (85), we get

$$\|A_n f - f\|_\infty \leq \frac{M}{\psi(1)} \left[\|f'\|_\infty \left(\frac{1}{n^\alpha} + (b-a) \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2} \right) + \right.$$

$$\frac{\|f''\|_\infty}{2} \left(\frac{1}{n^{2\alpha}} + (b-a)^2 \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2} \right) +$$

$$\left. \left(\frac{\omega_1\left(f''-f, \frac{1}{n^\alpha}\right)}{2n^{2\alpha}} + \|f''-f\|_\infty (b-a)^2 \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2} \right) \right]. \quad (87)$$

It follows that $\lim_{n\to+\infty} A_n(f) = f$, pointwise and uniformly.

We observe that

$$A_n(f,x) - f'(x) A_n(\sinh(\cdot - x), x) - 2f''(x) A_n\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x) =$$

$$\frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)} - f'(x) \frac{A_n^*(\sinh(\cdot - x), x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)} -$$

$$2f''(x) \frac{A_n^*\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)} - f(x) \left(\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)} \right) = \quad (88)$$

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)} \left[A_n^*(f,x) - f'(x) A_n^*(\sinh(\cdot - x), x) - \right.$$

$$\left. 2f''(x) A_n^*\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x) \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k) \right] = \quad (89)$$

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi(nx-k)} \left(\Lambda_n(x)\right).$$

Finally, we obtain ($\forall\, x \in [a,b]$, $n \in \mathbb{N}$):

4)

$$\left| A_n(f,x) - f'(x) A_n(\sinh(\cdot - x), x) - 2f''(x) A_n\left(\sinh^2\left(\frac{\cdot - x}{2}\right), x\right) - f(x) \right| \overset{(8)}{\leq}$$

$$|\Lambda_n(x)|/\psi(1) \overset{(77)}{\leq}$$

$$\frac{M}{\psi(1)}\left[\frac{\omega_1\left(f''-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\|f''-f\|_{\infty}(b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right]. \tag{90}$$

The theorem is established. ■

Next follows a mixed hyperbolic-trigonometric high order neural network approximation.

Theorem 10 *Let $f \in C^4([a,b],\mathbb{C})$, $0<\alpha<1$, $n \in \mathbb{N} : n^{1-\alpha}>2$, $x \in [a,b]$. Then*

1)

$$\Bigg|A_n(f,x)-f(x)-\frac{f'(x)}{2}A_n((\sinh(\cdot-x)+\sin(\cdot-x)),x)$$

$$-\frac{f''(x)}{2}A_n((\cosh(\cdot-x)-\cos(\cdot-x)),x) \tag{91}$$

$$-\frac{f^{(3)}(x)}{2}A_n((\sinh(\cdot-x)-\sin(\cdot-x)),x)$$

$$-f^{(4)}(x)A_n\left(\left(\sinh^2\left(\frac{\cdot-x}{2}\right)-\sin^2\left(\frac{\cdot-x}{2}\right)\right),x\right)\Bigg| \leq$$

$$\frac{(\cosh(b-a)+1)}{2\psi(1)}$$

$$\left[\frac{\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f^{(4)}-f\right\|_{\infty}(b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right], \tag{92}$$

2) if $f^{(i)}(x)=0$, $i=1,2,3,4$, we get

$$|A_n(f,x)-f(x)| \leq \frac{(\cosh(b-a)+1)}{2\psi(1)}$$

$$\left[\frac{\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f^{(4)}-f\right\|_{\infty}(b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right], \tag{93}$$

in the last (93) observe the high speed of convergence at $n^{-3\alpha}$.

Proof. Here $f \in C^4([a,b],\mathbb{C})$, and we apply the hyperbolic-trigonometric Taylor's formula for $f \in C^4([a,b],\mathbb{C})$, see Theorem 8 of [11].

Let $\frac{k}{n}, x \in [a,b]$, then

$$f\left(\frac{k}{n}\right)-f(x)-f'(x)\left(\frac{\sinh\left(\frac{k}{n}-x\right)+\sin\left(\frac{k}{n}-x\right)}{2}\right)$$

$$-f''(x)\left(\frac{\cosh\left(\frac{k}{n}-x\right)-\cos\left(\frac{k}{n}-x\right)}{2}\right)$$

$$-f^{(3)}(x)\left(\frac{\sinh\left(\frac{k}{n}-x\right)-\sin\left(\frac{k}{n}-x\right)}{2}\right)$$

$$-f^{(4)}(x)\left(\sinh^2\left(\frac{\frac{k}{n}-x}{2}\right)-\sin^2\left(\frac{\frac{k}{n}-x}{2}\right)\right)= \tag{94}$$

$$\int_x^{\frac{k}{n}}\left[\left(f^{(4)}(t)-f(t)\right)-\left(f^{(4)}(x)-f(x)\right)\right]\left(\frac{\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)}{2}\right)dt$$

$$=: R_4\left(\frac{k}{n},x\right).$$

As in Theorems 8, 9 we derive

$$A_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)-$$

$$\frac{f'(x)}{2}A_n^*\left(\left(\sinh(\cdot-x)+\sin(\cdot-x)\right),x\right)-$$

$$\frac{f''(x)}{2}A_n^*\left(\left(\cosh(\cdot-x)-\cos(\cdot-x)\right),x\right)-$$

$$\frac{f^{(3)}(x)}{2}A_n^*\left(\left(\sinh(\cdot-x)-\sin(\cdot-x)\right),x\right)-$$

$$\frac{f^{(4)}(x)}{2}A_n^*\left(\left(\sinh^2\left(\frac{\cdot-x}{2}\right)-\sin^2\left(\frac{\cdot-x}{2}\right)\right),x\right)=\Phi_n(x), \tag{95}$$

where

$$\Phi_n(x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\,R_4\left(\frac{k}{n},x\right). \tag{96}$$

Without loss of generality we can assume that $n>\left\lceil (b-a)^{-\frac{1}{\alpha}}\right\rceil$.

Thus $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}$ or $\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}$.

In case of $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}$, we have the following cases:

i) if $\frac{k}{n}\geq x$, then

$$\left|R_4\left(\frac{k}{n},x\right)\right|=$$

$$\left|\frac{1}{2}\int_x^{\frac{k}{n}}\left[\left(f^{(4)}(t)-f(t)\right)-\left(f^{(4)}(x)-f(x)\right)\right]\left(\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)\right)dt\right|\leq$$

$$\frac{1}{2}\int_{x}^{\frac{k}{n}}\left|\left(f^{(4)}(t)-f(t)\right)-\left(f^{(4)}(x)-f(x)\right)\right|\left|\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)\right|dt\leq \tag{97}$$

$$\frac{1}{2}\int_{x}^{\frac{k}{n}}\omega_1\left(f^{(4)}-f,t-x\right)\left(\left|\sinh\left(\frac{k}{n}-t\right)\right|+\left|\sin\left(\frac{k}{n}-t\right)\right|\right)dt\leq$$

$$\frac{\omega_1\left(f^{(4)}-f,\frac{k}{n}-x\right)}{2}\int_{x}^{\frac{k}{n}}\left(\cosh(b-a)\left(\frac{k}{n}-t\right)+\left(\frac{k}{n}-t\right)\right)dt=$$

$$\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,\frac{k}{n}-x\right)}{2}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-t\right)dt=$$

$$\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,\frac{k}{n}-x\right)}{4}\left(\frac{k}{n}-x\right)^2\leq$$

$$\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}. \tag{98}$$

That is , when $\frac{k}{n}\geq x$, then

$$\left|R_4\left(\frac{k}{n},x\right)\right|\leq\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}. \tag{99}$$

ii) if $\frac{k}{n}<x$, then

$$\left|R_4\left(\frac{k}{n},x\right)\right|=$$

$$\left|\frac{1}{2}\int_{\frac{k}{n}}^{x}\left[\left(f^{(4)}(t)-f(t)\right)-\left(f^{(4)}(x)-f(x)\right)\right]\left(\sinh\left(\frac{k}{n}-t\right)-\sin\left(\frac{k}{n}-t\right)\right)dt\right|\leq$$

$$\frac{\omega_1\left(f^{(4)}-f,x-\frac{k}{n}\right)}{2}\int_{\frac{k}{n}}^{x}\left(\cosh(b-a)\left(t-\frac{k}{n}\right)+\left(t-\frac{k}{n}\right)\right)dt=$$

$$\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,x-\frac{k}{n}\right)}{2}\int_{\frac{k}{n}}^{x}\left(t-\frac{k}{n}\right)dt=$$

$$\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,x-\frac{k}{n}\right)}{4}\left(x-\frac{k}{n}\right)^2\leq \tag{100}$$

$$\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}.$$

Consequently, when $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}$, we always obtain that

$$\left|R_4\left(\frac{k}{n},x\right)\right|\leq\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}. \tag{101}$$

Next assume again $\frac{k}{n} \geq x$, then

$$\left|R_4\left(\frac{k}{n}, x\right)\right| \leq$$

$$\frac{1}{2}\int_x^{\frac{k}{n}} \left|\left(f^{(4)}(t) - f(t)\right) - \left(f^{(4)}(x) - f(x)\right)\right| \left|\sinh\left(\frac{k}{n} - t\right) - \sin\left(\frac{k}{n} - t\right)\right| dt \leq$$

$$\left\|f^{(4)} - f\right\|_\infty \int_x^{\frac{k}{n}} \left[\cosh(b-a)\left(\frac{k}{n} - t\right) + \left(\frac{k}{n} - t\right)\right] dt =$$

$$\left\|f^{(4)} - f\right\|_\infty (\cosh(b-a) + 1) \int_x^{\frac{k}{n}} \left(\frac{k}{n} - t\right) dt =$$

$$\frac{\left\|f^{(4)} - f\right\|_\infty (\cosh(b-a) + 1)}{2}\left(\frac{k}{n} - x\right)^2 \leq \tag{102}$$

$$\leq \frac{\left\|f^{(4)} - f\right\|_\infty (\cosh(b-a) + 1)(b-a)^2}{2}.$$

Hence

$$\left|R_4\left(\frac{k}{n}, x\right)\right| \leq \frac{\left\|f^{(4)} - f\right\|_\infty (\cosh(b-a) + 1)(b-a)^2}{2}. \tag{103}$$

When $\frac{k}{n} < x$, we have

$$\left|R_4\left(\frac{k}{n}, x\right)\right| \leq$$

$$\frac{1}{2}\int_{\frac{k}{n}}^{x} \left|\left(f^{(4)}(t) - f(t)\right) - \left(f^{(4)}(x) - f(x)\right)\right| \left|\sinh\left(\frac{k}{n} - t\right) - \sin\left(\frac{k}{n} - t\right)\right| dt \leq$$

$$\left\|f^{(4)} - f\right\|_\infty \int_{\frac{k}{n}}^{x} \left[\cosh(b-a)\left(t - \frac{k}{n}\right) + \left(t - \frac{k}{n}\right)\right] dt =$$

$$\left\|f^{(4)} - f\right\|_\infty (\cosh(b-a) + 1) \int_{\frac{k}{n}}^{x} \left(t - \frac{k}{n}\right) dt =$$

$$\frac{\left\|f^{(4)} - f\right\|_\infty (\cosh(b-a) + 1)}{2}\left(x - \frac{k}{n}\right)^2 \leq \tag{104}$$

$$\frac{\left\|f^{(4)} - f\right\|_\infty (\cosh(b-a) + 1)(b-a)^2}{2}.$$

So, it is always true that

$$\left|R_4\left(\frac{k}{n}, x\right)\right| \leq \frac{\left\|f^{(4)} - f\right\|_\infty (\cosh(b-a) + 1)(b-a)^2}{2}. \tag{105}$$

Thus

$$|\Phi_n(x)| \leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx - k)\left|R_4\left(\frac{k}{n}, x\right)\right| =$$

$$\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}\end{array}\right.}^{\lfloor nb\rfloor} \psi(nx-k)\left|R_4\left(\frac{k}{n},x\right)\right| + \tag{106}$$

$$\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}\end{array}\right.}^{\lfloor nb\rfloor} \psi(nx-k)\left|R_4\left(\frac{k}{n},x\right)\right| \leq$$

$$\left(\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}\end{array}\right.}^{\lfloor nb\rfloor} \psi(nx-k)\right)\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}+ \tag{107}$$

$$\left(\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}\end{array}\right.}^{\lfloor nb\rfloor} \psi(nx-k)\right)$$

$$\left(\frac{\left\|f^{(4)}-f\right\|_{\infty}(\cosh(b-a)+1)(b-a)^2}{2}\right)\overset{\text{(by Theorem 3)}}{\leq}$$

$$\frac{(\cosh(b-a)+1)\,\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{4n^{2\alpha}}+ \tag{108}$$

$$\frac{\left\|f^{(4)}-f\right\|_{\infty}(\cosh(b-a)+1)(b-a)^2}{2}\,\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}.$$

We have proved that

$$|\Phi_n(x)| \leq \frac{(\cosh(b-a)+1)}{2}$$

$$\left[\frac{\omega_1\left(f^{(4)}-f,\frac{1}{n^{\alpha}}\right)}{2n^{2\alpha}}+\left\|f^{(4)}-f\right\|_{\infty}(b-a)^2\,\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right]. \tag{109}$$

We observe that

$$A_n(f,x)-f(x)-\frac{f'(x)}{2}A_n\left((\sinh(\cdot-x)+\sin(\cdot-x)),x\right)$$

$$-\frac{f''(x)}{2}A_n\left((\cosh(\cdot-x)-\cos(\cdot-x)),x\right) \tag{110}$$

$$-\frac{f^{(3)}(x)}{2}A_n\left(\left(\sinh(\cdot-x)-\sin(\cdot-x)\right),x\right)$$

$$-f^{(4)}(x)A_n\left(\left(\sinh^2\left(\frac{\cdot-x}{2}\right)-\sin^2\left(\frac{\cdot-x}{2}\right)\right),x\right)=$$

$$\Bigg[A_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)-\frac{f'(x)}{2}A_n^*\left(\left(\sinh(\cdot-x)+\sin(\cdot-x)\right),x\right)$$

$$-\frac{f''(x)}{2}A_n^*\left(\left(\cosh(\cdot-x)-\cos(\cdot-x)\right),x\right) \tag{111}$$

$$-\frac{f^{(3)}(x)}{2}A_n^*\left(\left(\sinh(\cdot-x)-\sin(\cdot-x)\right),x\right)$$

$$-f^{(4)}(x)A_n^*\left(\left(\sinh^2\left(\frac{\cdot-x}{2}\right)-\sin^2\left(\frac{\cdot-x}{2}\right)\right),x\right)\Bigg]\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}$$

$$=\frac{\Phi_n(x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}.$$

Finally, we obtain ($\forall\ x\in[a,b]$, $n\in\mathbb{N}$):

$$\Bigg|A_n(f,x)-f(x)-\frac{f'(x)}{2}A_n\left(\left(\sinh(\cdot-x)+\sin(\cdot-x)\right),x\right)$$

$$-\frac{f''(x)}{2}A_n\left(\left(\cosh(\cdot-x)-\cos(\cdot-x)\right),x\right) \tag{112}$$

$$-\frac{f^{(3)}(x)}{2}A_n\left(\left(\sinh(\cdot-x)-\sin(\cdot-x)\right),x\right)$$

$$-f^{(4)}(x)A_n\left(\left(\sinh^2\left(\frac{\cdot-x}{2}\right)-\sin^2\left(\frac{\cdot-x}{2}\right)\right),x\right)\Bigg|=$$

$$\frac{|\Phi_n(x)|}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}\leq\frac{|\Phi_n(x)|}{\psi(1)}\overset{(\text{by }(109))}{\leq}\frac{(\cosh(b-a)+1)}{2\psi(1)}$$

$$\left[\frac{\omega_1\left(f^{(4)}-f,\frac{1}{n^\alpha}\right)}{2n^{2\alpha}}+\left\|f^{(4)}-f\right\|_\infty(b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right]. \tag{113}$$

The theorem is proved. ■

We continue with a general trigonometric result.

Theorem 11 *Let $f\in C^4([a,b],\mathbb{C})$, $0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in[a,b]$. Let also $\overline{\alpha},\overline{\beta}\in\mathbb{R}$ with $\overline{\alpha}\overline{\beta}\left(\overline{\alpha}^2-\overline{\beta}^2\right)\neq 0$. Then*

1)

$$\Bigg|A_n(f,x)-f(x)-\frac{f'(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)}A_n\left(\left(\overline{\beta}^3\sin\left(\overline{\alpha}(\cdot-x)\right)-\overline{\alpha}^3\sin\left(\overline{\beta}(\cdot-x)\right)\right),x\right)$$

$$-\frac{f''(x)}{\left(\overline{\beta}^2-\overline{\alpha}^2\right)} A_n\left(\left(\cos\left(\overline{\alpha}(\cdot-x)\right)-\cos\left(\overline{\beta}(\cdot-x)\right)\right),x\right)$$

$$-\frac{f'''(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)} A_n\left(\left(\overline{\beta}\sin\left(\overline{\alpha}(\cdot-x)\right)-\overline{\alpha}\sin\left(\overline{\beta}(\cdot-x)\right)\right),x\right)$$

$$-\left(\frac{2f^{(4)}(x)+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''(x)}{\left(\overline{\alpha}\overline{\beta}\right)^2\left(\overline{\beta}^2-\overline{\alpha}^2\right)}\right)$$

$$A_n\left(\left(\overline{\beta}^2\sin^2\left(\frac{\overline{\alpha}(\cdot-x)}{2}\right)-\overline{\alpha}^2\sin^2\left(\frac{\overline{\beta}(\cdot-x)}{2}\right)\right),x\right)\Bigg| \leq \tag{114}$$

$$\frac{1}{\psi(1)\left|\overline{\beta}^2-\overline{\alpha}^2\right|}\left[\frac{\omega_1\left(\left(f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right),\frac{1}{n^{\alpha}}\right)}{n^{2\alpha}}+\right.$$

$$\left.2\left\|f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right\|_{\infty}(b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right],$$

2) if $f^{(i)}(x)=0$, $i=1,2,3,4$, we get

$$\left|A_n(f,x)-f(x)\right| \leq \frac{1}{\left|\overline{\beta}^2-\overline{\alpha}^2\right|\psi(1)}$$

$$\left[\frac{\omega_1\left(\left(f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right),\frac{1}{n^{\alpha}}\right)}{n^{2\alpha}}+\right. \tag{115}$$

$$\left.2\left\|f^{(4)}+\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2 f\right\|_{\infty}(b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right].$$

The high speed of convergence in (1) and (2) is $n^{-3\alpha}$.

Proof. As similar to Theorem 8 is omitted. It is based on Theorem 9 of [11]. ■

We finish with a general hyperbolic result.

Theorem 12 *Let $f\in C^4([a,b],\mathbb{C})$, $0<\alpha<1$, $n\in\mathbb{N}: n^{1-\alpha}>2$, $x\in[a,b]$. Let also $\overline{\alpha},\overline{\beta}\in\mathbb{R}$ with $\overline{\alpha}\overline{\beta}\left(\overline{\alpha}^2-\overline{\beta}^2\right)\neq 0$. Then*

1)

$$\Bigg|A_n(f,x)-f(x)-\frac{f'(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)}A_n\left(\left(\overline{\beta}^3\sinh\left(\overline{\alpha}(\cdot-x)\right)-\overline{\alpha}^3\sinh\left(\overline{\beta}(\cdot-x)\right)\right),x\right)$$

$$-\frac{f''(x)}{\overline{\beta}^2-\overline{\alpha}^2}A_n\left(\left(\cosh\left(\overline{\beta}(\cdot-x)\right)-\cosh\left(\overline{\alpha}(\cdot-x)\right)\right),x\right)$$

$$-\frac{f'''(x)}{\overline{\alpha}\overline{\beta}\left(\overline{\beta}^2-\overline{\alpha}^2\right)}A_n\left(\left(\overline{\alpha}\sinh\left(\overline{\beta}(\cdot-x)\right)-\overline{\beta}\sinh\left(\overline{\alpha}(\cdot-x)\right)\right),x\right)$$

$$-\left(\frac{2(f^{(4)}(x)-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''(x))}{\left(\overline{\alpha}\overline{\beta}\right)^2\left(\overline{\beta}^2-\overline{\alpha}^2\right)}\right)$$

$$A_n\left(\left(\overline{\alpha}^2\sinh^2\left(\frac{\overline{\beta}(\cdot-x)}{2}\right)-\overline{\beta}^2\sinh^2\left(\frac{\overline{\alpha}(\cdot-x)}{2}\right)\right),x\right)\Bigg|\leq \tag{116}$$

$$\frac{\cosh(b-a)}{\psi(1)\left|\overline{\beta}^2-\overline{\alpha}^2\right|}\left[\frac{\omega_1\left(\left(f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2f\right),\frac{1}{n^\alpha}\right)}{n^{2\alpha}}\right.$$

$$\left.+2\left\|f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2f\right\|_\infty(b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right],$$

2) if $f^{(i)}(x)=0$, $i=1,2,3,4$, we get

$$|A_n(f,x)-f(x)|\leq\frac{\cosh(b-a)}{\left|\overline{\beta}^2-\overline{\alpha}^2\right|\psi(1)} \tag{117}$$

$$\left[\frac{\omega_1\left(\left(f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2f\right),\frac{1}{n^\alpha}\right)}{n^{2\alpha}}+\right.$$

$$\left.2\left\|f^{(4)}-\left(\overline{\alpha}^2+\overline{\beta}^2\right)f''+\overline{\alpha}^2\overline{\beta}^2f\right\|_\infty(b-a)^2\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\right].$$

The high speed of convergence in (1) and (2) is $n^{-3\alpha}$.

Proof. As similar to Theorem 8 is omitted. It is based on Theorem 10 of [11]. ∎

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