p-SCHATTEN NORM INEQUALITIES FOR OPERATORS IN HILBERT SPACES VIA A KITTANEH RESULT

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Abstract. Let H be a complex Hilbert space and $r \geq 1/2$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$. If S, V, D, $E \in \mathcal{B}(H)$ with $S^*D \in \mathcal{B}_{2pr}(H)$ and $VE \in \mathcal{B}_{2qr}(H)$, then $D^*SVE \in \mathcal{B}_{2r}(H)$ and

$$\|D^*SVE\|_{2r} \leq \|S^*D\|_{2pr} \|VE\|_{2qr},$$

where $\|\cdot\|_s$ is the s-Schatten norm for $s \geq 1$. Some examples concerning the generalized *Aluthge transform* are also provided.

1. Introduction

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{1.2}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.3}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.4}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

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An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [16, p. 60-64]

$$\|A\|_p:=[\operatorname{tr}(|A|^p)]^{1/p}=\left(\sum_{i\in I}\langle |A|^p e_i,e_i\rangle\right)^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.5}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{1.6}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [16, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.7}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.8}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \|BA\|_p\le\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.9}$$

This implies that

$$\|CAB\|_p\le\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.10}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\le)\ \|AB\|_1\le\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{1.11}$$

For the theory of trace functionals and their applications the reader is referred to [14] and [16].

For $\mathcal{E}:=\{e_i\}_{i\in I}$ an orthonormal basis of H we define for $A\in\mathcal{B}_p(H)$, $p\ge1$

$$\|A\|_{\mathcal{E},p}:=\left(\sum_{i\in I}|\langle Ae_i,e_i\rangle|^p\right)^{1/p}.$$

We observe that $\|\cdot\|_{\mathcal{E},p}$ is a norm on $\mathcal{B}_p(H)$ and

$$\|A\|_{\mathcal{E},p}\le\|A\|_p \text{ for } A\in\mathcal{B}_p(H).$$

It is known that, if $\mathcal{E}=\{e_i\}_{i\in I}$ and $\mathcal{F}=\{f_i\}_{i\in I}$ are orthonormal basis, then [15]

$$\sup_{\mathcal{E},\mathcal{F}}\sum_{i\in I}|\langle Te_i,f_i\rangle|^s=\|T\|_s^s \text{ for } s\ge1. \tag{1.12}$$

The following result for operator matrices was obtained by F. Kittaneh in [10]:

Lemma 1. *Let $A,\ B,\ C\in\mathcal{B}(H)$ with $A,\ B\ge0$. Then the operator matrix*

$$\begin{bmatrix} A & C^* \\ C & B\end{bmatrix}\in\mathcal{B}(H\oplus H)$$

is positive, if and only if

$$|\langle Cx,y\rangle|^2\le\langle Ax,x\rangle\langle By,y\rangle \tag{1.13}$$

for all x, $y \in H$.

In [10] the author obtained among others that, if A, B, C satisfy the assumptions in Lemma 1 and $A \in \mathcal{B}_p(H)$, $B \in \mathcal{B}_q(H)$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ with p, $q \geq 1$, then $C \in \mathcal{B}_{2r}(H)$ and

$$\|C\|_{2r}^2 \leq \|A\|_p \|B\|_q .$$

In particular, if A, $B \in \mathcal{B}_p(H)$, then $C \in \mathcal{B}_p(H)$ and

$$\|C\|_p^2 \leq \|A\|_p \|B\|_p .$$

Motivated by the above results, in this paper we show among others that, if $r \geq 1/2$, p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr \geq 1$ and if S, V, D, $E \in \mathcal{B}(H)$ with $S^*D \in \mathcal{B}_{2pr}(H)$ and $VE \in \mathcal{B}_{2qr}(H)$, then $D^*SVE \in \mathcal{B}_{2r}(H)$ and

$$\|D^*SVE\|_{2r} \leq \|S^*D\|_{2pr} \|VE\|_{2qr} .$$

Some examples concerning the generalized Aluthge transform are also provided.

2. Main Results

The following vector inequality for positive operators $A \geq 0$, obtained by C. A. McCarthy in [13] is well known,

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle, \ p \geq 1$$

for $x \in H$, $\|x\| = 1$.

Buzano's inequality [5],

$$\frac{1}{2}\left[\|x\| \|y\| + |\langle x, y\rangle|\right] \geq |\langle x, e\rangle \langle e, y\rangle| \tag{2.1}$$

that holds for any x, y, $e \in H$ with $\|e\| = 1$ will also be used in the sequel.

Our first main result is as follows:

Theorem 2. *Let* $r \geq 1/2$, p, $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr \geq 1$. *If* A, B, C *satisfy the assumptions of Lemma 1,* D, $E \in \mathcal{B}(H)$ *with* $D^*AD \in \mathcal{B}_{pr}(H)$ *and* $E^*BE \in \mathcal{B}_{qr}(H)$, *then* $D^*CD \in \mathcal{B}_{2r}(H)$ *and*

$$\|E^*CD\|_{2r}^2 \leq \|D^*AD\|_{pr} \|E^*BE\|_{qr} . \tag{2.2}$$

Proof. From (1.13), by taking instead of x, Dx and instead of y, Ey, then we get

$$|\langle CDx, Ey\rangle|^2 \leq \langle ADx, Dx\rangle \langle BEy, Ey\rangle$$

for all x, $y \in H$.

This is equivalent to

$$|\langle E^*CDx, y\rangle|^2 \leq \langle D^*ADx, x\rangle \langle E^*BEy, y\rangle \tag{2.3}$$

for all x, $y \in H$.

If we take the power $r > 0$ and $x = e_i$, $y = f_i$ where $\{e_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ are orthonormal basis and sum, then we get

$$\sum_{i\in I} |\langle E^*CDe_i, f_i\rangle|^{2r} \leq \sum_{i\in I} \langle D^*ADe_i, e_i\rangle^r \langle E^*BEf_i, f_i\rangle^r . \tag{2.4}$$

If we use the Hölder's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we get

$$\begin{aligned}
(2.5) \qquad & \sum_{i \in I} \langle D^* A D e_i, e_i \rangle^r \langle E^* B E f_i, f_i \rangle^r \\
& \leq \left(\sum_{i \in I} \langle D^* A D e_i, e_i \rangle^{pr} \right)^{1/p} \left(\sum_{i \in I} \langle E^* B E f_i, f_i \rangle^{qr} \right)^{1/q} .
\end{aligned}$$

By the McCarthy inequality for pr, $qr \geq 1$, we have

$$\sum_{i \in I} \langle D^* A D e_i, e_i \rangle^{pr} \leq \sum_{i \in I} \langle (D^* A D)^{pr} e_i, e_i \rangle$$

and

$$\sum_{i \in I} \langle E^* B E f_i, f_i \rangle^{qr} \leq \sum_{i \in I} \langle (E^* B E)^{qr} f_i, f_i \rangle ,$$

therefore

$$\begin{aligned}
& \left(\sum_{i \in I} \langle D^* A D e_i, e_i \rangle^{pr} \right)^{1/p} \left(\sum_{i \in I} \langle E^* B E f_i, f_i \rangle^{qr} \right)^{1/q} \\
& \leq \left(\sum_{i \in I} \langle (D^* A D)^{pr} e_i, e_i \rangle \right)^{1/p} \left(\sum_{i \in I} \langle (E^* B E)^{qr} f_i, f_i \rangle \right)^{1/q} \\
& = \left(\| D^* A D \|_{pr}^{pr} \right)^{1/p} \left(\| E^* B E \|_{qr}^{qr} \right)^{1/q} = \| D^* A D \|_{pr}^{r} \| E^* B E \|_{qr}^{r} .
\end{aligned}$$

By (2.4) and (2.5) we derive

$$(2.6) \qquad \sum_{i \in I} | \langle E^* C D e_i, f_i \rangle |^{2r} \leq \| D^* A D \|_{pr}^{r} \| E^* B E \|_{qr}^{r} .$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (2.6), then by (1.12) we get

$$\| E^* C D \|_{2r}^{2r} \leq \| D^* A D \|_{pr}^{r} \| E^* B E \|_{qr}^{r}$$

and the inequality (2.2) is thus proved. □

Remark 1. *If we take $r = 1/2$ and $p = q = 2$, then by (2.2) we get*

$$(2.7) \qquad \| E^* C D \|_1^2 \leq \| D^* A D \|_1 \| E^* B E \|_1$$

provided that $D^* A D$, $E^* B E \in \mathcal{B}_1 (H)$.

Also, if $r = 1$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (2.2) we get

$$(2.8) \qquad \| E^* C D \|_2^2 \leq \| D^* A D \|_p \| E^* B E \|_q$$

provided that $D^* A D \in \mathcal{B}_p (H)$, $E^* B E \in \mathcal{B}_q (H)$.

Corollary 1. *Let $r \geq 1/2$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$. If S, V, D, $E \in \mathcal{B}(H)$ with $S^* D \in \mathcal{B}_{2pr} (H)$ and $V E \in \mathcal{B}_{2qr} (H)$, then $D^* S V E \in \mathcal{B}_{2r} (H)$ and*

$$(2.9) \qquad \| D^* S V E \|_{2r} \leq \| S^* D \|_{2pr} \| V E \|_{2qr} .$$

Proof. Observe that the operator matrix

$$\begin{bmatrix} S S^* & S V \\ V^* S^* & V^* V \end{bmatrix} \in \mathcal{B} (H \oplus H)$$

is positive. Then by (2.2) for $A = |S^*|^2$, $B = |V|^2$ and $C = V^*S^*$ we get

$$\|E^*V^*S^*D\|_{2r}^2 \le \|D^*SS^*D\|_{pr} \|E^*V^*VE\|_{qr} . \tag{2.10}$$

Now, observe that

$$\|E^*V^*S^*D\|_{2r}^2 = \left\|(E^*V^*S^*D)^*\right\|_{2r}^2 = \|D^*SVE\|_{2r}^2 ,$$
$$\|D^*SS^*D\|_{pr} = \left\||S^*D|^2\right\|_{pr} = \|S^*D\|_{2pr}^2$$

and

$$\|E^*V^*VE\|_{qr} = \left\||VE|^2\right\|_{qr} = \|VE\|_{2qr}^2$$

and by (2.10) we get (2.9). □

Remark 2. *If $r = 1$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (2.2) we get*

$$\|D^*SVE\|_2 \le \|S^*D\|_{2p} \|VE\|_{2q} \tag{2.11}$$

provided that $S^*D \in \mathcal{B}_{2p}(H)$, $VE \in \mathcal{B}_{2q}(H)$.

We also have:

Theorem 3. *Let $r \ge 1/2$, p, $q \ge 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If A, B, C satisfy the assumptions of Lemma 1, D, $E \in \mathcal{B}(H)$ with $D^*AD \in \mathcal{B}_p(H)$ and $E^*BE \in \mathcal{B}_q(H)$, then $D^*CD \in \mathcal{B}_{2r}(H)$ and*

$$\|E^*CD\|_{2r}^2 \le \|D^*AD\|_p \|E^*BE\|_q . \tag{2.12}$$

Proof. Observe that we have $\frac{1}{\frac{p}{r}} + \frac{1}{\frac{q}{r}} = 1$ and by Hölder's inequality for $\frac{p}{r}$ and $\frac{q}{r}$ we have

$$\begin{aligned} &\sum_{i\in I} \langle D^*ADe_i, e_i\rangle^r \langle E^*BEf_i, f_i\rangle^r \\ &= \sum_{i\in I} \left[\langle D^*ADe_i, e_i\rangle^p\right]^{\frac{r}{p}} \left[\langle E^*BEf_i, f_i\rangle^q\right]^{\frac{r}{q}} \\ &\le \left(\sum_{i\in I} \langle D^*ADe_i, e_i\rangle^p\right)^{r/p} \left(\sum_{i\in I} \langle E^*BEf_i, f_i\rangle^q\right)^{r/q} . \end{aligned} \tag{2.13}$$

By McCarthy inequality for p, $q > 1$ we get

$$\sum_{i\in I} \langle D^*ADe_i, e_i\rangle^p \le \sum_{i\in I} \langle (D^*AD)^p e_i, e_i\rangle$$

and

$$\sum_{i\in I} \langle E^*BEf_i, f_i\rangle^q \le \sum_{i\in I} \langle (E^*BE)^q f_i, f_i\rangle$$

and by (2.13) we obtain

$$\begin{aligned} &\sum_{i\in I} \langle D^*ADe_i, e_i\rangle^r \langle E^*BEf_i, f_i\rangle^r \\ &\le \left(\sum_{i\in I} \langle (D^*AD)^p e_i, e_i\rangle\right)^{r/p} \left(\sum_{i\in I} \langle (E^*BE)^q f_i, f_i\rangle\right)^{r/q} \\ &= \|D^*AD\|_p^r \|E^*BE\|_q^r . \end{aligned} \tag{2.14}$$

By (2.4) and (2.14) we derive

$$\sum_{i\in I}\left|\langle E^{*}CDe_{i},f_{i}\rangle\right|^{2r}\leq\left\|D^{*}AD\right\|_{p}^{r}\left\|E^{*}BE\right\|_{q}^{r}. \tag{2.15}$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (2.15) we get

$$\left\|E^{*}CD\right\|_{2r}^{2r}\leq\left\|D^{*}AD\right\|_{p}^{r}\left\|E^{*}BE\right\|_{q}^{r}$$

and the inequality (2.12) is thus proved. □

Remark 3. *If we take $p=q=2r=s\geq 1$, then by (2.12) we get*

$$\left\|E^{*}CD\right\|_{s}^{2}\leq\left\|D^{*}AD\right\|_{s}\left\|E^{*}BE\right\|_{s} \tag{2.16}$$

provided that $D^{*}AD,\ E^{*}BE\in\mathcal{B}_{s}(H)$.

If $r=2$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, then

$$\left\|E^{*}CD\right\|_{4}^{2}\leq\left\|D^{*}AD\right\|_{p}\left\|E^{*}BE\right\|_{q} \tag{2.17}$$

provided that $D^{*}AD\in\mathcal{B}_{p}(H)$ and $E^{*}BE\in\mathcal{B}_{q}(H)$.

We also notice that if we take $D=E=I$ in Theorem 3, then we recapture the result from Corollary 4 in [10].

Also, we have:

Corollary 2. *Let $r\geq 1/2,\ p,\ q\geq 1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If $S,\ V,\ D,\ E\in\mathcal{B}(H)$ with $S^{*}D\in\mathcal{B}_{2p}(H)$ and $VE\in\mathcal{B}_{2q}(H)$, then $D^{*}SVE\in\mathcal{B}_{2r}(H)$ and*

$$\left\|D^{*}SVE\right\|_{2r}\leq\left\|S^{*}D\right\|_{2p}\left\|VE\right\|_{2q}. \tag{2.18}$$

The proof follows by Theorem 3 by taking $A=\left|S^{*}\right|^{2},\ B=\left|V\right|^{2}$ and $C=V^{*}S^{*}$.

Theorem 4. *Assume that $A,\ B,\ C$ satisfy the assumptions of Lemma 1 and $\mathcal{E}=\{e_{i}\}_{i\in I}$ is an orthonormal basis. If $r\geq 1/2,\ p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1,\ pr,\ qr\geq 1$ and $(D^{*}AD)^{pr},\ (E^{*}BE)^{qr}\in\mathcal{B}_{1}(H)$, then $E^{*}CD\in\mathcal{B}_{2r}(H)$ and*

$$\left\|E^{*}CD\right\|_{\mathcal{E},2r}^{2r}\leq\operatorname{tr}\left(\frac{1}{p}(D^{*}AD)^{pr}+\frac{1}{q}(E^{*}BE)^{qr}\right). \tag{2.19}$$

If $r\geq 1$ and $D^{*}AD,\ E^{*}BE\in\mathcal{B}_{2r}(H),\ E^{*}BED^{*}AD\in\mathcal{B}_{r}(H)$, then $E^{*}CD\in\mathcal{B}_{2r}(H)$ and

$$\begin{aligned}\left\|E^{*}CD\right\|_{\mathcal{E},2r}^{2r}&\leq\frac{1}{2}\left(\left\|D^{*}AD\right\|_{2r}^{r}\left\|E^{*}BE\right\|_{2r}^{r}+\left\|E^{*}BED^{*}AD\right\|_{\mathcal{E},r}^{r}\right)\\&\leq\frac{1}{2}\left(\left\|D^{*}AD\right\|_{2r}^{r}\left\|E^{*}BE\right\|_{2r}^{r}+\left\|E^{*}BED^{*}AD\right\|_{r}^{r}\right).\end{aligned} \tag{2.20}$$

If $r\geq 1,\ p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1,\ pr,qr\geq 2$ and $\left|D^{*}AD\right|^{p},\ \left|E^{*}BE\right|^{q}\in\mathcal{B}_{r}(H)$, $E^{*}BED^{*}AD\in\mathcal{B}_{r}(H)$ then $E^{*}CD\in\mathcal{B}_{2r}(H)$ and

$$\begin{aligned}&\left\|E^{*}CD\right\|_{\mathcal{E},2r}^{2r}\\&\leq\frac{1}{2}\left[\operatorname{tr}\left(\frac{1}{p}\left|D^{*}AD\right|^{pr}+\frac{1}{q}\left|E^{*}BE\right|^{qr}\right)+\left\|E^{*}BED^{*}AD\right\|_{\mathcal{E},r}^{r}\right]\\&\leq\frac{1}{2}\left[\operatorname{tr}\left(\frac{1}{p}\left|D^{*}AD\right|^{pr}+\frac{1}{q}\left|E^{*}BE\right|^{qr}\right)+\left\|E^{*}BED^{*}AD\right\|_{r}^{r}\right].\end{aligned} \tag{2.21}$$

Proof. Let $x \in H$ with $\|x\| = 1$. Then by Lemma 1 we get

$$|\langle E^*CDx, x\rangle|^2 \le \langle D^*ADx, x\rangle \langle E^*BEx, x\rangle . \tag{2.22}$$

If we take the power $r > 0$, we get, by Young and McCarthy inequalities, that

$$\begin{aligned} |\langle E^*CDx, x\rangle|^{2r} &\le \langle D^*ADx, x\rangle^r \langle E^*BEx, x\rangle^r \\ &\le \frac{1}{p} \langle D^*ADx, x\rangle^{pr} + \frac{1}{q} \langle E^*BEx, x\rangle^{qr} \\ &\le \frac{1}{p} \langle (D^*AD)^{pr} x, x\rangle + \frac{1}{q} \langle (E^*BE)^{qr} x, x\rangle \\ &= \left\langle \left(\frac{1}{p} (D^*AD)^{pr} + \frac{1}{q} (E^*BE)^{qr}\right) x, x \right\rangle \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

If $\mathcal{E} = \{e_i\}_{i\in I}$ is an orthonormal basis, then by taking $x = e_i$ and summing over $i \in I$ we get

$$\begin{aligned} \|E^*CD\|_{\mathcal{E},2r}^{2r} &= \sum_{i\in I} |\langle E^*CDe_i, e_i\rangle|^{2r} \\ &\le \sum_{i\in I} \left\langle \left(\frac{1}{p} (D^*AD)^{pr} + \frac{1}{q} (E^*BE)^{qr}\right) e_i, e_i \right\rangle \\ &= \operatorname{tr}\left(\frac{1}{p} (D^*AD)^{pr} + \frac{1}{q} (E^*BE)^{qr}\right), \end{aligned}$$

which proves (2.19).

By Buzano's inequality we have

$$\begin{aligned} &\langle D^*ADx, x\rangle \langle E^*BEx, x\rangle \\ &\le \frac{1}{2} [\|D^*ADx\| \|E^*BEx\| + |\langle D^*ADx, E^*BEx\rangle|] \\ &= \frac{1}{2} [\|D^*ADx\| \|E^*BEx\| + |\langle E^*BED^*ADx, x\rangle|] \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

If we take the power $r \ge 1$ and use the convexity of power function, then we get

$$\begin{aligned} &\langle D^*ADx, x\rangle^r \langle E^*BEx, x\rangle^r \\ &\le \left(\frac{\|D^*ADx\| \|E^*BEx\| + |\langle E^*BED^*ADx, x\rangle|}{2}\right)^r \\ &\le \frac{\|D^*ADx\|^r \|E^*BEx\|^r + |\langle E^*BED^*ADx, x\rangle|^r}{2} \\ &= \frac{\|D^*ADx\|^{2\frac{r}{2}} \|E^*BEx\|^{2\frac{r}{2}} + |\langle E^*BED^*ADx, x\rangle|^r}{2} \\ &= \frac{\left\langle |D^*AD|^2 x, x\right\rangle^{\frac{r}{2}} \left\langle |E^*BE|^2 x, x\right\rangle^{\frac{r}{2}} + |\langle E^*BED^*ADx, x\rangle|^r}{2} \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

Therefore

$$\begin{aligned}(2.23)\quad & \|E^*CD\|_{\mathcal{E},2r}^{2r} \\ & = \sum_{i\in I} |\langle E^*CDe_i, e_i\rangle|^{2r} \le \sum_{i\in I} \langle D^*ADe_i, e_i\rangle^r \langle E^*BEe_i, e_i\rangle^r \\ & \le \frac{1}{2}\sum_{i\in I}\left[\left\langle |D^*AD|^2 e_i, e_i\right\rangle^{\frac{r}{2}} \left\langle |E^*BE|^2 e_i, e_i\right\rangle^{\frac{r}{2}} + |\langle E^*BED^*ADe_i, e_i\rangle|^r\right].\end{aligned}$$

Using Cauchy-Schwarz inequality we have

$$\begin{aligned}& \sum_{i\in I}\left\langle |D^*AD|^2 e_i, e_i\right\rangle^{\frac{r}{2}} \left\langle |E^*BE|^2 e_i, e_i\right\rangle^{\frac{r}{2}} \\ & \le \left(\sum_{i\in I}\left\langle |D^*AD|^2 e_i, e_i\right\rangle^{r}\right)^{1/2} \left(\sum_{i\in I}\left\langle |E^*BE|^2 e_i, e_i\right\rangle^{r}\right)^{1/2} \\ & \le \left(\sum_{i\in I}\left\langle |D^*AD|^{2r} e_i, e_i\right\rangle\right)^{1/2} \left(\sum_{i\in I}\left\langle |E^*BE|^{2r} e_i, e_i\right\rangle\right)^{1/2} \\ & = \|D^*AD\|_{2r}^{r} \|E^*BE\|_{2r}^{r},\end{aligned}$$

where for the last inequality we used McCarthy's result for $r \ge 1$.

Therefore by (2.23) we derive the first part of (2.20). The second part is obvious.

Further, if we use Young's inequality

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q,\ \ a, b \ge 0,$$

then we get

$$\begin{aligned}\|D^*ADx\|^r \|E^*BEx\|^r & \le \frac{1}{p}\|D^*ADx\|^{rp} + \frac{1}{q}\|E^*BEx\|^{rq} \\ & = \frac{1}{p}\|D^*ADx\|^{2\frac{rp}{2}} + \frac{1}{q}\|E^*BEx\|^{2\frac{rq}{2}} \\ & = \frac{1}{p}\left\langle |D^*AD|^2 x, x\right\rangle^{\frac{rp}{2}} + \frac{1}{q}\left\langle |E^*BE|^2 x, x\right\rangle^{\frac{rq}{2}} \\ & \le \frac{1}{p}\langle |D^*AD|^{rp} x, x\rangle + \frac{1}{q}\langle |E^*BE|^{rq} x, x\rangle \\ & = \left\langle \left(\frac{1}{p}|D^*AD|^{rp} + \frac{1}{q}|E^*BE|^{rq}\right) x, x\right\rangle\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

Therefore

$$\begin{aligned}
&\|E^*CD\|_{\mathcal{E},2r}^{2r} \\
&= \sum_{i\in I} |\langle E^*CDe_i, e_i\rangle|^{2r} \le \sum_{i\in I} \langle D^*ADe_i, e_i\rangle^r \langle E^*BEe_i, e_i\rangle^r \\
&\le \frac{1}{2}\left[\sum_{i\in I} \|D^*ADe_i\|^r \|E^*BEe_i\|^r + \sum_{i\in I} |\langle E^*BED^*ADe_i, e_i\rangle|^r\right] \\
&\le \frac{1}{2}\sum_{i\in I} \left\langle \left(\frac{1}{p}|D^*AD|^{rp} + \frac{1}{q}|E^*BE|^{rq}\right) e_i, e_i\right\rangle \\
&+ \frac{1}{2}\sum_{i\in I} |\langle E^*BED^*ADe_i, e_i\rangle|^r \\
&= \frac{1}{2}\left[\operatorname{tr}\left(\frac{1}{p}|D^*AD|^{rp} + \frac{1}{q}|E^*BE|^{rq}\right) + \|E^*BED^*AD\|_{\mathcal{E},r}^r\right],
\end{aligned}$$

which proves the first part of (2.21). The second part is obvious. □

The following corollary is a natural consequence to consider and follows by Theorem 4 for $A = SS^*$, $B = V^*V$ and $C = V^*S^*$.

Corollary 3. *Let $S, V, D, E \in \mathcal{B}(H)$ and $\mathcal{E} = \{e_i\}_{i\in I}$ is an orthonormal basis. If $r \ge 1/2$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $pr, qr \ge 1$ and $|S^*D|^{2pr}, |VE|^{2qr} \in \mathcal{B}_1(H)$, then $D^*SVE \in \mathcal{B}_{2r}(H)$ and*

$$\|D^*SVE\|_{\mathcal{E},2r}^{2r} \le \operatorname{tr}\left(\frac{1}{p}|S^*D|^{2pr} + \frac{1}{q}|VE|^{2qr}\right). \tag{2.24}$$

If $r \ge 1$ and $S^*D, VE \in \mathcal{B}_{2r}(H)$, $|VE|^2 |S^*D|^2 \in \mathcal{B}_r(H)$, then $D^*SVE \in \mathcal{B}_{2r}(H)$ and

$$\begin{aligned}
\|D^*SVE\|_{\mathcal{E},2r}^{2r} &\le \frac{1}{2}\left(\|S^*D\|_{2r}^{2r}\|VE\|_{2r}^{2r} + \left\||VE|^2|S^*D|^2\right\|_{\mathcal{E},r}^r\right) \\
&\le \frac{1}{2}\left(\|S^*D\|_{2r}^{2r}\|VE\|_{2r}^{2r} + \left\||VE|^2|S^*D|^2\right\|_{r}^r\right).
\end{aligned} \tag{2.25}$$

If $r \ge 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $pr, qr \ge 2$ and $|S^*D|^{2pr}, |VE|^{2qr} \in \mathcal{B}_1(H)$, $|VE|^2 |S^*D|^2 \in \mathcal{B}_r(H)$ then $D^*SVE \in \mathcal{B}_{2r}(H)$ and

$$\begin{aligned}
&\|D^*SVE\|_{\mathcal{E},2r}^{2r} \\
&\le \frac{1}{2}\left[\operatorname{tr}\left(\frac{1}{p}|S^*D|^{2pr} + \frac{1}{q}|VE|^{2qr}\right) + \left\||VE|^2|S^*D|^2\right\|_{\mathcal{E},r}^r\right] \\
&\le \frac{1}{2}\left[\operatorname{tr}\left(\frac{1}{p}|S^*D|^{2pr} + \frac{1}{q}|VE|^{2qr}\right) + \left\||VE|^2|S^*D|^2\right\|_{r}^r\right].
\end{aligned} \tag{2.26}$$

3. Inequalities Via Polar Decomposition

If the operator T has the *polar decomposition* $T = U|T|$ with U *a partial isometry*, we define the transform

$$\Delta_{p,q}(T) := |T|^p U |T|^q,$$

for $p, q \ge 0$. Here we assume that $|T|^0 = I$.

The *p-generalized Dougal transform* is defined by

$$\widehat{T}_p := |T|^p U,$$

the usual *Dougal transform* is then

$$\widehat{T} := |T| U,$$

and the *p-generalized Aluthge transform*

$$\widetilde{T}_p := |T|^p U |T|^p ,$$

which for $p = 1/2$ gives the usual Aluthge transform [1]

$$\widetilde{T} := |T|^{1/2} U |T|^{1/2} .$$

Also

$$T_q := U |T|^q ,$$

which gives for $q = 1$ the usual polar decomposition $T = U |T|$.

For $p = t$, $q = 1 - t$, where $t \in [0, 1]$ we have

$$\Delta_t (T) := \Delta_{t,1-t} (T) = |T|^t V |T|^{1-t} .$$

The transform $\Delta_t (T)$ was introduced and studied in [6].

For some recent result concerning these transforms, see [2]-[4], [6]-[8] and [10]-[12].

We also have:

Proposition 1. *Let $r \geq 1/2$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$. For $s \geq 0$, $v \geq 1$ and $t \in [0, 1]$, assume that $\Delta_{s,t} (T) \in \mathcal{B}_{2pr} (H)$ and $T \in \mathcal{B}_{2(v-t)qr} (H)$, then $\Delta_{s,v} (T) \in \mathcal{B}_{2r} (H)$ and we have*

$$\|\Delta_{s,v} (T)\|_{2r} \leq \|\Delta_{s,t} (T)\|_{2pr} \|T\|^{v-t}_{2(v-t)qr} , \tag{3.1}$$

Proof. If we take $S = U |T|^t$ and $V = |T|^{1-t}$, $t \in [0, 1]$ and observe that $SV = U |T| = T$, then by (2.9) we get

$$\|D^* T E\|_{2r} \leq \left\| |T|^t U^* D \right\|_{2pr} \left\| |T|^{1-t} E \right\|_{2qr} = \left\| DU |T|^t \right\|_{2pr} \left\| |T|^{1-t} E \right\|_{2qr}$$

for $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$.

If we take $D = |T|^s$ and $E = |T|^{v-1}$ for $s \geq 0$ and $v \geq 1$, then we get

$$\| |T|^s U |T|^v \|_{2r} \leq \left\| |T|^s U |T|^t \right\|_{2pr} \left\| |T|^{v-t} \right\|_{2qr} ,$$

which proves (3.1). □

If we take $r = 1/2$ and $p = q = 2$ in (3.1), then we get

$$(|\operatorname{tr} [\Delta_{s,v} (T)]| \leq) \|\Delta_{s,v} (T)\|_1 \leq \|\Delta_{s,t} (T)\|_2 \|T\|^{v-t}_{2(v-t)} , \tag{3.2}$$

for $s \geq 0$, $v \geq 1$ and $t \in [0, 1]$.

For $r = 1$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we also obtain from (3.1) that

$$\|\Delta_{s,v} (T)\|_2 \leq \|\Delta_{s,t} (T)\|_{2p} \|T\|^{v-t}_{2(v-t)q} , \tag{3.3}$$

for $s \geq 0$, $v \geq 1$ and $t \in [0, 1]$.

Proposition 2. *Let $r \geq 1/2$, p, $q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. For $s \geq 0$, $v \geq 1$ and $t \in [0,1]$, assume that $\Delta_{s,t}(T) \in \mathcal{B}_{2p}(H)$ and $T \in \mathcal{B}_{2(v-t)q}(H)$, then $\Delta_{s,v}(T) \in \mathcal{B}_{2r}(H)$ and we have*

$$\|\Delta_{s,v}(T)\|_{2r} \leq \|\Delta_{s,t}(T)\|_{2p} \|T\|_{2(v-t)q}^{v-t}. \tag{3.4}$$

For $p=q=2r=u \geq 1$, we get from (3.4) that

$$\|\Delta_{s,v}(T)\|_{u} \leq \|\Delta_{s,t}(T)\|_{2u} \|T\|_{2(v-t)u}^{v-t}. \tag{3.5}$$

for $s \geq 0$, $v \geq 1$ and $t \in [0,1]$.

The proof follows in a similar way by Corollary 2.

Proposition 3. *Let $\mathcal{E}=\{e_i\}_{i\in I}$ be an orthonormal basis. If $r \geq 1/2$, p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, pr, $qr \geq 1$, $s \geq 0$, $v \geq 1$ and $t \in [0,1]$, assume that $\left(|T|^s |T^*|^{2t} |T|^s\right)^{pr}$, $|T|^{2(v-t)qr} \in \mathcal{B}_1(H)$, then $\Delta_{s,v}(T) \in \mathcal{B}_{2r}(H)$ and*

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2r}^{2r} \leq \operatorname{tr}\left(\frac{1}{p}\left(|T|^s |T^*|^{2t} |T|^s\right)^{pr} + \frac{1}{q}|T|^{2(v-t)qr}\right). \tag{3.6}$$

If $r \geq 1$ and $\Delta_{s,t}(T) \in \mathcal{B}_{2r}(H)$, $T \in \mathcal{B}_{4(v-t)r}(H)$, $|T|^{2(v-t)+s}|T^*|^{2t}|T|^s \in \mathcal{B}_r(H)$, then $\Delta_{s,v}(T) \in \mathcal{B}_{2r}(H)$ and

$$\begin{aligned} &\|\Delta_{s,v}(T)\|_{\mathcal{E},2r}^{2r} \\ &\leq \frac{1}{2}\left(\|\Delta_{s,t}(T)\|_{2r}^{2r}\|T\|_{4(v-t)r}^{4(v-t)r} + \left\||T|^{2(v-t)+s}|T^*|^{2t}|T|^s\right\|_{\mathcal{E},r}^{r}\right) \\ &\leq \frac{1}{2}\left(\|\Delta_{s,t}(T)\|_{2r}^{2r}\|T\|_{4(v-t)r}^{4(v-t)r} + \left\||T|^{2(v-t)+s}|T^*|^{2t}|T|^s\right\|_{r}^{r}\right). \end{aligned} \tag{3.7}$$

If $r \geq 1$, p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, pr, $qr \geq 2$ and $\left(|T|^s |T^*|^{2t} |T|^s\right)^{p}$, $|T|^{2(v-t)q} \in \mathcal{B}_r(H)$, $|T|^{2(v-t)+s}|T^*|^{2t}|T|^s \in \mathcal{B}_r(H)$ then $\Delta_{s,v}(T) \in \mathcal{B}_{2r}(H)$ and

$$\begin{aligned} \|\Delta_{s,v}(T)\|_{\mathcal{E},2r}^{2r} &\leq \frac{1}{2}\left[\operatorname{tr}\left(\frac{1}{p}\left(|T|^s |T^*|^{2t} |T|^s\right)^{pr} + \frac{1}{q}|T|^{2(v-t)qr}\right)\right. \\ &\quad \left. + \left\||T|^{2(v-t)+s}|T^*|^{2t}|T|^s\right\|_{\mathcal{E},r}^{r}\right] \\ &\leq \frac{1}{2}\left[\operatorname{tr}\left(\frac{1}{p}\left(|T|^s |T^*|^{2t} |T|^s\right)^{pr} + \frac{1}{q}|T|^{2(v-t)qr}\right)\right. \\ &\quad \left. + \left\||T|^{2(v-t)+s}|T^*|^{2t}|T|^s\right\|_{r}^{r}\right]. \end{aligned} \tag{3.8}$$

Proof. If we take $S=U|T|^t$ and $V=|T|^{1-t}$, $t\in[0,1]$ and observe that $SV=U|T|=T$, then

$$SS^* = U|T|^t|T|^tU^* = U|T|^{2t}U^* = |T^*|^{2t}.$$

If we take $D=|T|^s$ and $E=|T|^{v-1}$ for $s\geq 0$ and $v \geq 1$, then we get

$$|S^*D|^2 = |S^*|T|^s|^2 = |T|^s SS^* |T|^s = |T|^s|T^*|^{2t}|T|^s,$$

$$|VE|^2 = \left||T|^{1-t}|T|^{v-1}\right|^2 = |T|^{2(v-t)}$$

and, by (2.24),

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2r}^{2r} \leq \operatorname{tr}\left(\frac{1}{p}\left(|T|^s |T^*|^{2t} |T|^s\right)^{pr} + \frac{1}{q}\left||T|^{v-t}\right|^{2qr}\right)$$
$$= \operatorname{tr}\left(\frac{1}{p}\left(|T|^s |T^*|^{2t} |T|^s\right)^{pr} + \frac{1}{q}|T|^{2(v-t)qr}\right),$$

which proves (3.6).

Now, observe that

$$\|S^*D\|_{2r}^{2r} = \|DS\|_{2r}^{2r} = \left\||T|^s U |T|^t\right\|_{2r}^{2r} = \|\Delta_{s,t}(T)\|_{2r}^{2r},$$
$$|VE|^2 |S^*D|^2 = |T|^{2(v-t)} |T|^s |T^*|^{2t} |T|^s = |T|^{2(v-t)+s} |T^*|^{2t} |T|^s$$

and by (2.25) we get

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2r}^{2r} \leq \frac{1}{2}\left(\|\Delta_{s,t}(T)\|_{2r}^{2r} \|T\|_{4(v-t)r}^{4(v-t)r} + \left\||T|^{2(v-t)+s} |T^*|^{2t} |T|^s\right\|_{\mathcal{E},r}^{r}\right)$$
$$\leq \frac{1}{2}\left(\|\Delta_{s,t}(T)\|_{2r}^{2r} \|T\|_{4(v-t)r}^{4(v-t)r} + \left\||T|^{2(v-t)+s} |T^*|^{2t} |T|^s\right\|_{r}^{r}\right),$$

which proves (3.7).

The last inequality follows by (2.26). □

If in (3.6) we take $r = 1/2$ and $p = q = 2$, then we get for $s \geq 0$, $v \geq 1$ and $t \in [0,1]$,

$$(|\operatorname{tr}[\Delta_{s,v}(T)]| \leq)\, \|\Delta_{s,v}(T)\|_{\mathcal{E},1} \leq \frac{1}{2}\operatorname{tr}\left(|T^*|^{2t} |T|^{2s} + |T|^{2(v-t)}\right) \tag{3.9}$$

since $\operatorname{tr}\left(|T|^s |T^*|^{2t} |T|^s\right) = \operatorname{tr}\left(|T^*|^{2t} |T|^{2s}\right)$.

For $r = 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ in (3.6) we get

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2}^{2} \leq \operatorname{tr}\left(\frac{1}{p}\left(|T|^s |T^*|^{2t} |T|^s\right)^{p} + \frac{1}{q}|T|^{2(v-t)q}\right) \tag{3.10}$$

for $s \geq 0$, $v \geq 1$ and $t \in [0,1]$.

Similar inequalities can be obtained from (3.7) and (3.8). We omit the details.

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