SOME NEW p-SCHATTEN NORM INEQUALITIES FOR OPERATORS IN HILBERT SPACES VIA A KITTANEH RESULT

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Abstract. Let H be a complex Hilbert space and $\alpha \in [0,1]$, $r \geq 1$. In this paper we show, among others that, if $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis of H and $|S^*D|^2, |VE|^2 \in \mathcal{B}_r(H)$, then $D^*SVE \in \mathcal{B}_{2r}(H)$ and

$$\|D^*SVE\|_{\mathcal{E},2r}^{2r} \leq \left\|(1-\alpha)|S^*D|^{2r} + \alpha|VE|^{2r}\right\| \|S^*D\|_{2r}^{2\alpha r} \|VE\|_{2r}^{2(1-\alpha)r},$$

where we define for $A \in \mathcal{B}_p(H)$, $p \geq 1$

$$\|A\|_{\mathcal{E},p} := \left(\sum_{i\in I} |\langle Ae_i, e_i\rangle|^p\right)^{1/p}$$

and $\|\cdot\|_p$ is the usual p-Schatten norm. Some examples concerning the generalized *Aluthge transform* are also provided.

1. Introduction

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.2}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.3}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT, $TA \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.4}$$

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(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;
(iv) If $A,\ B \in \mathcal{B}_2(H)$ *then* $AB,\ BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite [16, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} = \left(\sum_{i\in I} \langle |A|^p e_i, e_i\rangle\right)^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.5}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{1.6}$$

For $p \ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [16, p. 60-64],

$$\|A\|_p = \|A^*\|_p,\ A \in \mathcal{B}_p(H) \tag{1.7}$$

$$\|AB\|_p \le \|A\|_p \|B\|_p,\ A, B \in \mathcal{B}_p(H) \tag{1.8}$$

and

$$\|AB\|_p \le \|A\|_p \|B\|,\ \|BA\|_p \le \|B\| \|A\|_p,\ A \in \mathcal{B}_p(H),\ B \in \mathcal{B}(H). \tag{1.9}$$

This implies that

$$\|CAB\|_p \le \|C\| \|A\|_p \|B\|,\ A \in \mathcal{B}_p(H),\ B,\ C \in \mathcal{B}(H). \tag{1.10}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \le) \|AB\|_1 \le \|A\|_p \|B\|_q,\ A \in \mathcal{B}_p(H),\ B \in \mathcal{B}_q(H). \tag{1.11}$$

For the theory of trace functionals and their applications the reader is referred to [14] and [16].

For $\mathcal{E} := \{e_i\}_{i\in I}$ an orthonormal basis of H we define for $A \in \mathcal{B}_p(H)$, $p \ge 1$

$$\|A\|_{\mathcal{E},p} := \left(\sum_{i\in I} |\langle Ae_i, e_i\rangle|^p\right)^{1/p}.$$

We observe that $\|\cdot\|_{\mathcal{E},p}$ is a norm on $\mathcal{B}_p(H)$ and

$$\|A\|_{\mathcal{E},p} \le \|A\|_p \text{ for } A \in \mathcal{B}_p(H).$$

It is known that, if $\mathcal{E} = \{e_i\}_{i\in I}$ and $\mathcal{F} = \{f_i\}_{i\in I}$ are orthonormal basis, then [15]

$$\sup_{\mathcal{E},\mathcal{F}} \sum_{i\in I} |\langle Te_i, f_i\rangle|^s = \|T\|_s^s \text{ for } s \ge 1. \tag{1.12}$$

The following result for operator matrices was obtained by F. Kittaneh in [10]:

Lemma 1. *Let A, B, $C \in \mathcal{B}(H)$ with A, $B \geq 0$. Then the operator matrix*

$$\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \in \mathcal{B}(H \oplus H)$$

is positive, if and only if

$$|\langle Cx, y\rangle|^2 \leq \langle Ax, x\rangle \langle By, y\rangle \tag{1.13}$$

for all x, $y \in H$.

In [10] the author obtained among others that, if A, B, C satisfy the assumptions in Lemma 1 and $A \in \mathcal{B}_p(H)$, $B \in \mathcal{B}_q(H)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ with p, $q \geq 1$, then $C \in \mathcal{B}_{2r}(H)$ and

$$\|C\|_{2r}^2 \leq \|A\|_p \|B\|_q .$$

In particular, if A, $B \in \mathcal{B}_p(H)$, then $C \in \mathcal{B}_p(H)$ and

$$\|C\|_p^2 \leq \|A\|_p \|B\|_p , \ p \geq 1.$$

In this paper we show, among others that, if $\mathcal{E} := \{e_i\}_{i \in I}$ is an orthonormal basis of H and $|S^*D|^2$, $|VE|^2 \in \mathcal{B}_r(H)$, then $D^*SVE \in \mathcal{B}_{2r}(H)$ and

$$\|D^*SVE\|_{\mathcal{E},2r}^{2r} \leq \left\| (1-\alpha)|S^*D|^{2r} + \alpha |VE|^{2r} \right\| \|S^*D\|_{2r}^{2\alpha r} \|VE\|_{2r}^{2(1-\alpha)r} .$$

Some examples concerning the generalized *Aluthge transform* are also provided.

2. Main Results

We recall the following vector inequality for positive operators $A \geq 0$, obtained by C. A. McCarthy in [13]

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle , \ p \geq 1$$

for $x \in H$, $\|x\| = 1$ and Buzano's inequality [5],

$$\frac{1}{2}\left[\|x\| \|y\| + |\langle x, y\rangle|\right] \geq |\langle x, e\rangle \langle e, y\rangle| \tag{2.1}$$

that holds for any x, y, $e \in H$ with $\|e\| = 1$.

Our first main result is as follows:

Theorem 2. *Let A, B, C, D, $E \in \mathcal{B}(H)$ with A, $B \geq 0$ and such that the operator matrix*

$$\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \in \mathcal{B}(H \oplus H)$$

is positive. Let $\alpha \in [0,1]$ and $r \geq 1$. If $\mathcal{E} := \{e_i\}_{i \in I}$ is an orthonormal basis of H and D^*AD, $E^*BE \in \mathcal{B}_r(H)$, then $E^*CD \in \mathcal{B}_{2r}(H)$ and

$$\begin{aligned} &\|E^*CD\|_{\mathcal{E},2r}^{2r} \\ &\leq \|(1-\alpha)(D^*AD)^r + \alpha (E^*BE)^r\| \|D^*AD\|_{\mathcal{E},r}^{\alpha r} \|E^*BE\|_{\mathcal{E},r}^{(1-\alpha)r} \\ &\leq \|(1-\alpha)(D^*AD)^r + \alpha (E^*BE)^r\| \|D^*AD\|_r^{\alpha r} \|E^*BE\|_r^{(1-\alpha)r} \end{aligned} \tag{2.2}$$

and, for $p,\, q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *we also have for* $(D^{*}AD)^{r},\, (E^{*}BE)^{r}\in\mathcal{B}_{p}(H)\cap\mathcal{B}_{q}(H)$ *that*

$$\begin{aligned}\|E^{*}CD\|_{\mathcal{E},2r}^{2r} &\leq \|(1-\alpha)(D^{*}AD)^{r}+\alpha(E^{*}BE)^{r}\|_{\mathcal{E},p}^{p} \\ &\quad\times\|\alpha(D^{*}AD)^{r}+(1-\alpha)(E^{*}BE)^{r}\|_{\mathcal{E},q}^{q} \\ &\leq \|(1-\alpha)(D^{*}AD)^{r}+\alpha(E^{*}BE)^{r}\|_{p}^{p} \\ &\quad\times\|\alpha(D^{*}AD)^{r}+(1-\alpha)(E^{*}BE)^{r}\|_{q}^{q}.\end{aligned}\tag{2.3}$$

Moreover, we have

$$\begin{aligned}\|E^{*}CD\|_{\mathcal{E},2r}^{2r} &\leq \|D^{*}AD\|^{r\alpha}\|E^{*}BE\|^{r(1-\alpha)} \\ &\quad\times\operatorname{tr}\left[(1-\alpha)(D^{*}AD)^{r}+\alpha(E^{*}BE)^{r}\right]\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}\|E^{*}CD\|_{\mathcal{E},2r}^{2r} &\leq \|(1-\alpha)(D^{*}AD)^{r}+\alpha(E^{*}BE)^{r}\| \\ &\quad\times\operatorname{tr}\left[\alpha(D^{*}AD)^{r}+(1-\alpha)(E^{*}BE)^{r}\right]\end{aligned}\tag{2.5}$$

provided that $D^{*}AD,\, E^{*}BE\in\mathcal{B}_{r}(H)$.

Proof. From (1.13) we get

$$|\langle E^{*}CDx,y\rangle|^{2}\leq\langle D^{*}ADx,x\rangle\langle E^{*}BEy,y\rangle$$

for all $x,\, y\in H$.

If we take $y=x$, then we get by the *A-G-inequality* that

$$\begin{aligned}&|\langle E^{*}CDx,x\rangle|^{2} \\ &\leq\langle D^{*}ADx,x\rangle\langle E^{*}BEx,x\rangle \\ &=\langle D^{*}ADx,x\rangle^{1-\alpha}\langle E^{*}BEx,x\rangle^{\alpha}\langle D^{*}ADx,x\rangle^{\alpha}\langle E^{*}BEx,x\rangle^{1-\alpha} \\ &\leq\left[(1-\alpha)\langle D^{*}ADx,x\rangle+\alpha\langle E^{*}BEx,x\rangle\right]\langle D^{*}ADx,x\rangle^{\alpha}\langle E^{*}BEx,x\rangle^{1-\alpha}\end{aligned}$$

for all $x\in H$.

If we take the power $r\geq 1$, then we get by the convexity of power function and McCarthy inequality that

$$\begin{aligned}|\langle E^{*}CDx,x\rangle|^{2r} &\leq\left[(1-\alpha)\langle D^{*}ADx,x\rangle+\alpha\langle E^{*}BEx,x\rangle\right]^{r} \\ &\quad\times\langle D^{*}ADx,x\rangle^{r\alpha}\langle E^{*}BEx,x\rangle^{r(1-\alpha)} \\ &\leq\left[(1-\alpha)\langle D^{*}ADx,x\rangle^{r}+\alpha\langle E^{*}BEx,x\rangle^{r}\right] \\ &\quad\times\langle D^{*}ADx,x\rangle^{r\alpha}\langle E^{*}BEx,x\rangle^{r(1-\alpha)} \\ &\leq\left[(1-\alpha)\langle(D^{*}AD)^{r}x,x\rangle+\alpha\langle(E^{*}BE)^{r}x,x\rangle\right] \\ &\quad\times\langle D^{*}ADx,x\rangle^{r\alpha}\langle E^{*}BEx,x\rangle^{r(1-\alpha)} \\ &=\langle\left[(1-\alpha)(D^{*}AD)^{r}+\alpha(E^{*}BE)^{r}\right]x,x\rangle \\ &\quad\times\langle D^{*}ADx,x\rangle^{r\alpha}\langle E^{*}BEx,x\rangle^{r(1-\alpha)}\end{aligned}\tag{2.6}$$

for all $x\in H$.

Let $\mathcal{E} := \{e_i\}_{i\in I}$ be an orthonormal basis of H. If we take $x = e_i$ and sum over $i \in I$, then we get

$$\begin{aligned}\|E^*CD\|_{\mathcal{E},2r}^{2r} &= \sum_{i\in I} |\langle E^*CDe_i, e_i\rangle|^{2r} \\ &\le \sum_{i\in I} \langle [(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r] e_i, e_i\rangle \\ &\times \langle D^*ADe_i, e_i\rangle^{r\alpha} \langle E^*BEe_i, e_i\rangle^{r(1-\alpha)} \\ &\le \|(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r\| \\ &\times \sum_{i\in I} \langle D^*ADe_i, e_i\rangle^{r\alpha} \langle E^*BEe_i, e_i\rangle^{r(1-\alpha)}.\end{aligned} \tag{2.7}$$

If we use Hölder's inequality for $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, then we get

$$\begin{aligned}&\sum_{i\in I} \langle D^*ADe_i, e_i\rangle^{r\alpha} \langle E^*BEe_i, e_i\rangle^{r(1-\alpha)} \\ &\le \left(\sum_{i\in I} \langle D^*ADe_i, e_i\rangle^r\right)^{\alpha} \left(\sum_{i\in I} \langle E^*BEe_i, e_i\rangle^r\right)^{1-\alpha} \\ &= \|D^*AD\|_{\mathcal{E},r}^{\alpha r} \|E^*BE\|_{\mathcal{E},r}^{(1-\alpha)r} \le \|D^*AD\|_r^{\alpha r} \|E^*BE\|_r^{(1-\alpha)r}\end{aligned}$$

and by (2.7) we get (2.2).

From (2.6) we also have that

$$\begin{aligned}|\langle E^*CDx, x\rangle|^{2r} &\le [(1-\alpha)\langle D^*ADx, x\rangle + \alpha\langle E^*BEx, x\rangle]^r \\ &\times [\alpha\langle D^*ADx, x\rangle + (1-\alpha)\langle E^*BEx, x\rangle]^r \\ &\le [(1-\alpha)\langle D^*ADx, x\rangle^r + \alpha\langle E^*BEx, x\rangle^r] \\ &\times [\alpha\langle D^*ADx, x\rangle^r + (1-\alpha)\langle E^*BEx, x\rangle^r] \\ &\le [(1-\alpha)\langle (D^*AD)^r x, x\rangle + \alpha\langle (E^*BE)^r x, x\rangle] \\ &\times [\alpha\langle (D^*AD)^r x, x\rangle + (1-\alpha)\langle (E^*BE)^r x, x\rangle] \\ &= \langle [(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r] x, x\rangle \\ &\times \langle [\alpha(D^*AD)^r + (1-\alpha)(E^*BE)^r] x, x\rangle\end{aligned} \tag{2.8}$$

for all $x \in H$.

If we take $x = e_i$ and sum, then we get by using Hölder's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned}
\|E^*CD\|_{\mathcal{E},2r}^{2r} &= \sum_{i\in I} |\langle E^*CDe_i, e_i\rangle|^{2r} \\
&\leq \sum_{i\in I} \langle[(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r] e_i, e_i\rangle \\
&\times \langle[\alpha(D^*AD)^r + (1-\alpha)(E^*BE)^r] e_i, e_i\rangle \\
&\leq \left(\sum_{i\in I} \langle[(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r] e_i, e_i\rangle^p\right)^{1/p} \\
&\times \left(\sum_{i\in I} \langle[\alpha(D^*AD)^r + (1-\alpha)(E^*BE)^r] e_i, e_i\rangle^q\right)^{1/q} \\
&= \|(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r\|_{\mathcal{E},p}^p\,, \\
&\times \|\alpha(D^*AD)^r + (1-\alpha)(E^*BE)^r\|_{\mathcal{E},q}^q\,,
\end{aligned}$$

which proves (2.3).

From (2.6) we also have

$$\begin{aligned}
\|E^*CD\|_{\mathcal{E},2r}^{2r} &= \sum_{i\in I} |\langle E^*CDe_i, e_i\rangle|^{2r} \\
&\leq \sum_{i\in I} \langle[(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r] e_i, e_i\rangle \\
&\times \langle D^*ADe_i, e_i\rangle^{r\alpha} \langle E^*BEe_i, e_i\rangle^{r(1-\alpha)} \\
&\leq \|D^*AD\|^{r\alpha} \|E^*BE\|^{r(1-\alpha)} \\
&\times \sum_{i\in I} \langle[(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r] e_i, e_i\rangle \\
&= \|D^*AD\|^{r\alpha} \|E^*BE\|^{r(1-\alpha)} \operatorname{tr}[(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r]\,,
\end{aligned}$$

which proves (2.4).

From (2.8) we also get

$$\begin{aligned}
\|E^*CD\|_{\mathcal{E},2r}^{2r} &= \sum_{i\in I} |\langle E^*CDe_i, e_i\rangle|^{2r} \\
&\leq \sum_{i\in I} \langle[(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r] e_i, e_i\rangle \\
&\times \langle[\alpha(D^*AD)^r + (1-\alpha)(E^*BE)^r] e_i, e_i\rangle \\
&\leq \|(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r\| \\
&\times \sum_{i\in I} \langle[\alpha(D^*AD)^r + (1-\alpha)(E^*BE)^r] e_i, e_i\rangle \\
&= \|(1-\alpha)(D^*AD)^r + \alpha(E^*BE)^r\| \\
&\times \operatorname{tr}[\alpha(D^*AD)^r + (1-\alpha)(E^*BE)^r]\,,
\end{aligned}$$

which proves (2.5). □

Remark 1. *If we take $r=1$ and assume that $D^*AD,\ E^*BE\in\mathcal{B}_1(H)$, then $E^*CD\in\mathcal{B}_2(H)$ and*

$$\begin{aligned}\|E^*CD\|_{\mathcal{E},2}^2&\le\|(1-\alpha)D^*AD+\alpha E^*BE\|\,\|D^*AD\|_{\mathcal{E},1}^{\alpha}\,\|E^*BE\|_{\mathcal{E},1}^{1-\alpha}\\&\le\|(1-\alpha)D^*AD+\alpha E^*BE\|\,\|D^*AD\|_1^{\alpha}\,\|E^*BE\|_1^{1-\alpha}\end{aligned}$$

and, for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we also have for $D^*AD,\ E^*BE\in\mathcal{B}_p(H)\cap\mathcal{B}_q(H)$ that

$$\begin{aligned}\|E^*CD\|_{\mathcal{E},2}^2&\le\|(1-\alpha)D^*AD+\alpha E^*BE\|_{\mathcal{E},p}^p\\&\quad\times\|\alpha D^*AD+(1-\alpha)E^*BE\|_{\mathcal{E},q}^q\\&\le\|(1-\alpha)D^*AD+\alpha E^*BE\|_p^p\\&\quad\times\|\alpha D^*AD+(1-\alpha)E^*BE\|_q^q.\end{aligned}$$

Moreover, we have for $D^*AD,\ E^*BE\in\mathcal{B}_1(H)$ that

$$\|E^*CD\|_{\mathcal{E},2}^2\le\|D^*AD\|^{\alpha}\,\|E^*BE\|^{1-\alpha}\operatorname{tr}\left[(1-\alpha)D^*AD+\alpha E^*BE\right]$$

and

$$\|E^*CD\|_{\mathcal{E},2}^2\le\|(1-\alpha)D^*AD+\alpha E^*BE\|\operatorname{tr}\left[\alpha D^*AD+(1-\alpha)E^*BE\right]$$

for $\alpha\in[0,1]$.

If we take $\alpha=1/2$, then we obtain

$$\begin{aligned}\|E^*CD\|_{\mathcal{E},2}^2&\le\frac{1}{2}\|D^*AD+E^*BE\|\,\|D^*AD\|_{\mathcal{E},1}^{1/2}\,\|E^*BE\|_{\mathcal{E},1}^{1/2}\\&\le\frac{1}{2}\|D^*AD+E^*BE\|\,\|D^*AD\|_1^{1/2}\,\|E^*BE\|_1^{1/2}\end{aligned}$$

and, for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we also have that

$$\begin{aligned}\|E^*CD\|_{\mathcal{E},2}^2&\le\frac{1}{2^{p+q}}\|D^*AD+E^*BE\|_{\mathcal{E},p}^p\,\|D^*AD+E^*BE\|_{\mathcal{E},q}^q\\&\le\frac{1}{2^{p+q}}\|D^*AD+E^*BE\|_p^p\,\|D^*AD+E^*BE\|_q^q.\end{aligned}$$

Moreover, we have

$$\|E^*CD\|_{\mathcal{E},2}^2\le\frac{1}{2}\|D^*AD\|^{1/2}\,\|E^*BE\|^{1/2}\operatorname{tr}\left(D^*AD+E^*BE\right)$$

and

$$\|E^*CD\|_{\mathcal{E},2}^2\le\frac{1}{4}\|D^*AD+E^*BE\|\operatorname{tr}\left(D^*AD+E^*BE\right).$$

Corollary 1. *Let $\alpha\in[0,1]$ and $r\ge1$. If $\mathcal{E}:=\{e_i\}_{i\in I}$ is an orthonormal basis of H and $|S^*D|^2,\ |VE|^2\in\mathcal{B}_r(H)$, then $D^*SVE\in\mathcal{B}_{2r}(H)$ and*

$$\begin{aligned}&\|D^*SVE\|_{\mathcal{E},2r}^{2r}\\&\le\left\|(1-\alpha)|S^*D|^{2r}+\alpha|VE|^{2r}\right\|\left\||S^*D|^2\right\|_{\mathcal{E},r}^{\alpha r}\left\||VE|^2\right\|_{\mathcal{E},r}^{(1-\alpha)r}\\&\le\left\|(1-\alpha)|S^*D|^{2r}+\alpha|VE|^{2r}\right\|\,\|S^*D\|_{2r}^{2\alpha r}\,\|VE\|_{2r}^{2(1-\alpha)r}\end{aligned}\tag{2.9}$$

and, for p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *we also have for* $|S^*D|^{2r}, |VE|^{2r}\in\mathcal{B}_p(H)\cap\mathcal{B}_q(H)$ *that*

$$\begin{aligned}\|D^*SVE\|_{\mathcal{E},2r}^{2r} &\le \left\|(1-\alpha)|S^*D|^{2r}+\alpha|VE|^{2r}\right\|_{\mathcal{E},p}^{p}\\ &\quad\times\left\|\alpha|S^*D|^{2r}+(1-\alpha)|VE|^{2r}\right\|_{\mathcal{E},q}^{q}\\ &\le \left\|(1-\alpha)|S^*D|^{2r}+\alpha|VE|^{2r}\right\|_{p}^{p}\\ &\quad\times\left\|\alpha|S^*D|^{2r}+(1-\alpha)|VE|^{2r}\right\|_{q}^{q}.\end{aligned}\tag{2.10}$$

Moreover, we have

$$\|D^*SVE\|_{\mathcal{E},2r}^{2r}\le\|S^*D\|^{2r\alpha}\|VE\|^{2r(1-\alpha)}\operatorname{tr}\left[(1-\alpha)|S^*D|^{2r}+\alpha|VE|^{2r}\right]\tag{2.11}$$

and

$$\begin{aligned}&\|D^*SVE\|_{\mathcal{E},2r}^{2r}\\ &\le\left\|(1-\alpha)|S^*D|^{2r}+\alpha|VE|^{2r}\right\|\operatorname{tr}\left[\alpha|S^*D|^{2r}+(1-\alpha)|VE|^{2r}\right],\end{aligned}\tag{2.12}$$

provided that S^*D, $VE\in\mathcal{B}_{2r}(H)$.

Proof. Observe that the operator matrix

$$\begin{bmatrix} SS^* & SV\\ V^*S^* & V^*V\end{bmatrix}\in\mathcal{B}(H\oplus H)$$

is positive. Then by (2.2) for $A=|S^*|^2$, $B=|V|^2$ and $C=V^*S^*$ we get

$$\begin{aligned}&\|E^*V^*S^*D\|_{\mathcal{E},2r}^{2r}\\ &\le\left\|(1-\alpha)|S^*D|^{2r}+\alpha|VE|^{2r}\right\|\left\||S^*D|^2\right\|_{\mathcal{E},r}^{\alpha r}\left\||VE|^2\right\|_{\mathcal{E},r}^{(1-\alpha)r}\\ &\le\left\|(1-\alpha)|S^*D|^{2r}+\alpha|VE|^{2r}\right\|\left\||S^*D|^2\right\|_{r}^{\alpha r}\left\||VE|^2\right\|_{r}^{(1-\alpha)r}.\end{aligned}\tag{2.13}$$

Since

$$\|E^*V^*S^*D\|_{\mathcal{E},2r}^{2r}=\|D^*SVE\|_{\mathcal{E},2r}^{2r}$$

and

$$\left\||S^*D|^2\right\|_{r}^{\alpha r}=\|S^*D\|_{2r}^{2\alpha r},\quad\left\||VE|^2\right\|_{r}^{(1-\alpha)r}=\|VE\|_{2r}^{2(1-\alpha)r},$$

then by (2.13) we obtain (2.9).

The other inequalities follow from (2.3)-(2.5) in a similar manner. □

Remark 2. *If we take* $r=1$ *in Corollary 1 and assume that* $\alpha\in[0,1]$, $|S^*D|^2$, $|VE|^2\in\mathcal{B}_1(H)$, *then* $D^*SVE\in\mathcal{B}_2(H)$ *and*

$$\|D^*SVE\|_{\mathcal{E},2}^{2}\le\left\|(1-\alpha)|S^*D|^2+\alpha|VE|^2\right\|\|S^*D\|_2^{2\alpha}\|VE\|_2^{2(1-\alpha)}$$

and, for p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *we also have for* S^*D, $VE\in\mathcal{B}_p(H)\cap\mathcal{B}_q(H)$ *that*

$$\|D^*SVE\|_{\mathcal{E},2}^{2}\le\left\|(1-\alpha)|S^*D|^2+\alpha|VE|^2\right\|_p^p\left\|\alpha|S^*D|^2+(1-\alpha)|VE|^2\right\|_q^q.$$

Moreover, we have

$$\|D^*SVE\|_{\mathcal{E},2}^{2}\le\|S^*D\|^{2\alpha}\|VE\|^{2(1-\alpha)}\operatorname{tr}\left[(1-\alpha)|S^*D|^2+\alpha|VE|^2\right]$$

and

$$\|D^*SVE\|_{\mathcal{E},2}^2 \le \left\|(1-\alpha)\left|S^*D\right|^2+\alpha\left|VE\right|^2\right\| \operatorname{tr}\left[\alpha\left|S^*D\right|^2+(1-\alpha)\left|VE\right|^2\right]$$

for S^*D, $VE \in \mathcal{B}_2(H)$.

If we take $\alpha = 1/2$, *then we obtain for* S^*D, $VE \in \mathcal{B}_2(H)$ *that*

$$\|D^*SVE\|_{\mathcal{E},2}^2 \le \frac{1}{2}\left\|\left|S^*D\right|^2+\left|VE\right|^2\right\| \|S^*D\|_2 \|VE\|_2$$

and, for p, $q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *we also have for* S^*D, $VE \in \mathcal{B}_p(H) \cap \mathcal{B}_q(H)$ *that*

$$\|D^*SVE\|_{\mathcal{E},2}^2 \le \frac{1}{2^{p+q}}\left\|\left|S^*D\right|^2+\left|VE\right|^2\right\|_p^p \left\|\left|S^*D\right|^2+\left|VE\right|^2\right\|_q^q.$$

Moreover, we have

$$\|D^*SVE\|_{\mathcal{E},2}^2 \le \frac{1}{2}\|S^*D\|\|VE\| \operatorname{tr}\left(\left|S^*D\right|^2+\left|VE\right|^2\right)$$

and

$$\|D^*SVE\|_{\mathcal{E},2}^2 \le \frac{1}{4}\left\|\left|S^*D\right|^2+\left|VE\right|^2\right\| \operatorname{tr}\left(\left|S^*D\right|^2+\left|VE\right|^2\right)$$

for S^*D, $VE \in \mathcal{B}_2(H)$.

3. Inequalities Via Polar Decomposition

If the operator T has the *polar decomposition* $T = U|T|$ with U *a partial isometry*, we define the transform

$$\Delta_{p,q}(T) := |T|^p U |T|^q,$$

for p, $q \ge 0$. Here we assume that $|T|^0 = I$.

The p-*generalized Dougal transform* is defined by

$$\widehat{T}_p := |T|^p U,$$

the usual *Dougal transform* is then

$$\widehat{T} := |T| U,$$

and the p-*generalized Aluthge transform*

$$\widetilde{T}_p := |T|^p U |T|^p,$$

which for $p = 1/2$ gives the usual Aluthge transform [1]

$$\widetilde{T} := |T|^{1/2} U |T|^{1/2}.$$

Also

$$T_q := U|T|^q,$$

which gives for $q = 1$ the usual polar decomposition $T = U|T|$.

For $p = t$, $q = 1-t$, where $t \in [0,1]$ we have

$$\Delta_t(T) := \Delta_{t,1-t}(T) = |T|^t V |T|^{1-t}.$$

The transform $\Delta_t(T)$ was introduced and studied in [6].

For some recent result concerning these transforms, see [2]-[4], [6]-[8] and [10]-[12].

We also have:

Proposition 1. *Let* $\alpha,\ t\in[0,1]$ *and* $r,\ v\geq 1$. *If* $\mathcal{E}:=\{e_i\}_{i\in I}$ *is an orthonormal basis of* H *and* $|T|^s|T^*|^{2t}|T|^s,\ |T|^{2(v-t)}\in\mathcal{B}_r(H)$, *then* $\Delta_{s,v}(T)\in\mathcal{B}_{2r}(H)$ *and we have*

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2r}^{2r}\leq\left\|(1-\alpha)\left(|T|^s|T^*|^{2t}|T|^s\right)^r+\alpha|T|^{2r(v-t)}\right\|\times\left\||T|^s|T^*|^{2t}|T|^s\right\|_r^{\alpha r}\|T\|_{2(v-t)r}^{2(v-t)(1-\alpha)r}.$$

For $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *we also have for* $|T|^s|T^*|^{2t}|T|^s,\ |T|^{2(v-t)}\in\mathcal{B}_{rp}(H)\cap\mathcal{B}_{rq}(H)$ *that*

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2r}^{2r}\leq\left\|(1-\alpha)\left(|T|^s|T^*|^{2t}|T|^s\right)^r+\alpha|T|^{2r(v-t)}\right\|_p^p\times\left\|\alpha\left(|T|^s|T^*|^{2t}|T|^s\right)^r+(1-\alpha)|T|^{2r(v-t)}\right\|_q^q.$$

Moreover, we have

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2r}^{2r}\leq\left\||T|^s|T^*|^{2t}|T|^s\right\|_r^{\alpha r}\|T\|_{2(v-t)r}^{2(v-t)(1-\alpha)r}\times\operatorname{tr}\left[(1-\alpha)\left(|T|^s|T^*|^{2t}|T|^s\right)^r+\alpha|T|^{2r(v-t)}\right]$$

and

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2r}^{2r}\leq\left\|(1-\alpha)\left(|T|^s|T^*|^{2t}|T|^s\right)^r+\alpha|T|^{2r(v-t)}\right\|\times\operatorname{tr}\left[\alpha\left(|T|^s|T^*|^{2t}|T|^s\right)^r+(1-\alpha)|T|^{2r(v-t)}\right],$$

provided that $|T|^s|T^*|^{2t}|T|^s,\ |T|^{2(v-t)}\in\mathcal{B}_r(H)$.

Proof. If we take $S=U|T|^t$ and $V=|T|^{1-t}$, $t\in[0,1]$, then observe that $SV=U|T|=T$ and

$$SS^*=U|T|^t|T|^tU^*=U|T|^{2t}U^*=|T^*|^{2t}.$$

If we take $D=|T|^s$ and $E=|T|^{v-1}$ for $s\geq 0$ and $v\geq 1$, then we get

$$|S^*D|^2=|S^*|T|^s|^2=|T|^sSS^*|T|^s=|T|^s|T^*|^{2t}|T|^s,$$

$$|VE|^2=\left||T|^{1-t}|T|^{v-1}\right|^2=|T|^{2(v-t)}$$

Also

$$\|D^*TE\|_{\mathcal{E},2r}^{2r}=\left\||T|^sU|T||T|^{v-1}\right\|_{\mathcal{E},2r}^{2r}=\||T|^sU|T|^v\|_{\mathcal{E},2r}^{2r}=\|\Delta_{s,v}(T)\|_{\mathcal{E},2r}^{2r},$$

$$\left\||S^*D|^2\right\|_r^{\alpha r}=\left\||T|^s|T^*|^{2t}|T|^s\right\|_r^{\alpha r}$$

and

$$\left\||VE|^2\right\|_r^{(1-\alpha)r}=\left\||T|^{2(v-t)}\right\|_r^{(1-\alpha)r}=\|T\|_{2(v-t)r}^{2(v-t)(1-\alpha)r},$$

then by (2.9) we get

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2r}^{2r}\leq\left\|(1-\alpha)\left(|T|^s|T^*|^{2t}|T|^s\right)^r+\alpha|T|^{2r(v-t)}\right\|\times\left\||T|^s|T^*|^{2t}|T|^s\right\|_r^{\alpha r}\|T\|_{2(v-t)r}^{2(v-t)(1-\alpha)r}.$$

The other inequalities follow from (2.10)-(2.12). □

For $r=1$ we get for $|T|^{s}|T^{*}|^{2t}|T|^{s}$, $|T|^{2(v-t)}\in\mathcal{B}_{1}(H)$, that $\Delta_{s,v}(T)\in\mathcal{B}_{2}(H)$ and

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2}^{2}\leq\left\|(1-\alpha)|T|^{s}|T^{*}|^{2t}|T|^{s}+\alpha|T|^{2(v-t)}\right\|\times\left\||T|^{s}|T^{*}|^{2t}|T|^{s}\right\|_{1}^{\alpha}\|T\|_{2(v-t)}^{2(v-t)(1-\alpha)}.$$

For $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we also have for $|T|^{s}|T^{*}|^{2t}|T|^{s}$, $|T|^{2(v-t)}\in\mathcal{B}_{p}(H)\cap\mathcal{B}_{q}(H)$ that

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2}^{2}\leq\left\|(1-\alpha)|T|^{s}|T^{*}|^{2t}|T|^{s}+\alpha|T|^{2(v-t)}\right\|_{p}^{p}\times\left\|\alpha|T|^{s}|T^{*}|^{2t}|T|^{s}+(1-\alpha)|T|^{2(v-t)}\right\|_{q}^{q}.$$

Moreover, we have

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2}^{2}\leq\left\||T|^{s}|T^{*}|^{2t}|T|^{s}\right\|_{1}^{\alpha}\|T\|_{2(v-t)}^{2(v-t)(1-\alpha)}\times\operatorname{tr}\left[(1-\alpha)|T|^{s}|T^{*}|^{2t}|T|^{s}+\alpha|T|^{2(v-t)}\right]$$

and

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2}^{2}\leq\left\|(1-\alpha)|T|^{s}|T^{*}|^{2t}|T|^{s}+\alpha|T|^{2(v-t)}\right\|\times\operatorname{tr}\left[\alpha|T|^{s}|T^{*}|^{2t}|T|^{s}+(1-\alpha)|T|^{2(v-t)}\right],$$

provided that $|T|^{s}|T^{*}|^{2t}|T|^{s}$, $|T|^{2(v-t)}\in\mathcal{B}_{1}(H)$.

The choice $\alpha=1/2$ gives some simpler inequalities,

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2}^{2}\leq\frac{1}{2}\left\||T|^{s}|T^{*}|^{2t}|T|^{s}+|T|^{2(v-t)}\right\|\left\||T|^{s}|T^{*}|^{2t}|T|^{s}\right\|_{1}^{1/2}\|T\|_{2(v-t)}^{v-t}$$

and

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2}^{2}\leq\frac{1}{2^{p+q}}\left\||T|^{s}|T^{*}|^{2t}|T|^{s}+|T|^{2(v-t)}\right\|_{p}^{p}\left\||T|^{s}|T^{*}|^{2t}|T|^{s}+|T|^{2(v-t)}\right\|_{q}^{q}.$$

for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Moreover, we have

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2}^{2}\leq\frac{1}{2}\left\||T|^{s}|T^{*}|^{2t}|T|^{s}\right\|_{1}^{1/2}\|T\|_{2(v-t)}^{v-t}\operatorname{tr}\left[|T^{*}|^{2t}|T|^{2s}+|T|^{2(v-t)}\right]$$

and

$$\|\Delta_{s,v}(T)\|_{\mathcal{E},2}^{2}\leq\frac{1}{4}\left\||T|^{s}|T^{*}|^{2t}|T|^{s}+|T|^{2(v-t)}\right\|\operatorname{tr}\left[|T^{*}|^{2t}|T|^{2s}+|T|^{2(v-t)}\right],$$

since $\operatorname{tr}\left(|T|^{s}|T^{*}|^{2t}|T|^{s}\right)=\operatorname{tr}\left(|T^{*}|^{2t}|T|^{2s}\right)$.

References

[1] A. Aluthge, Some generalized theorems on p-hyponormal operators, *Integral Equations Operator Theory* **24** (1996), 497-501.

[2] A. Abu-Omar and F. Kittaneh, A numerical radius inequality involving the generalized Aluthge transform, *Studia Math.* **216** (1) (2013) 69-75.

[3] P. Bhunia, S. Bag, and K. Paul, Numerical radius inequalities and its applications in estimation of zeros of polynomials, *Linear Algebra and its Applications*, vol. **573** (2019) pp. 166–177.

[4] P. Bhunia , S. S. Dragomir , M. S. Moslehian , K. Paul, *Lectures on Numerical Radius Inequalities*, Springer Cham, 2022. https://doi.org/10.1007/978-3-031-13670-2.

[5] M. L. Buzano, Generalizzazione della diseguaglianza di Cauchy-Schwarz. (Italian), *Rend. Sem. Mat. Univ. e Politech. Torino*, **31** (1971/73), 405–409 (1974).

[6] M. Cho and K. Tanahashi, Spectral relations for Aluthge transform, *Scientiae Mathematicae Japonicae*, **55** (1) (2002), 77-83.

[7] S. S. Dragomir, *Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces*, SpringerBriefs in Mathematics, 2013. https://doi.org/10.1007/978-3-319-01448-7.

[8] M. El-Haddad and F. Kittaneh, Numerical radius inequalities for Hilbert space operators. II, *Studia Math.* **182** (2007), No. 2, 133-140.

[9] T. Kato, Notes on some inequalities for linear operators, *Math. Ann.*, **125** (1952), 208-212.

[10] F. Kittaneh, Notes on some inequalities for Hilbert space operators, *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 2, 283–293.

[11] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, *Studia Math.* **158** (2003), No. 1, 11-17.

[12] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, *Studia Math.*, **168** (2005), No. 1, 73-80.

[13] C. A. McCarthy, C_p, *Israel J. Math.* **5** (1967), 249–271.

[14] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.

[15] J. R. Ringrose, *Compact Non-self-adjoint Operators,* Van Nostrand Reinhold, New York, 1971.

[16] V. A. Zagrebnov, *Gibbs Semigroups,* Operator Theory: Advances and Applications, Volume 273, Birkhäuser, 2019

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