General multiple sigmoid functions relied complex valued multivariate trigonometric and hyperbolic neural network approximations

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Abstract

Here we research the multivariate quantitative approximation of complex valued continuous functions on a box of $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized type neural network operators. We investigate also the case of approximation by iterated multilayer neural network operators. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate moduli of continuity of the engaged function and its partial derivatives. Our multivariate operators are defined by using a multidimensional density function induced by general multiple sigmoid functions. The approximations are pointwise and uniform. The related feed-forward neural network are with one or multi hidden layers. The basis of our theory are the introduced multivariate Taylor formulae of trigonometric and hyperbolic type.

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1 Introduction

The author in [1] and [2], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats

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there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support.

Motivations for this work are the article [14] of Z. Chen and F. Cao, also by [3]-[12], [15], [16].

Here we perform general multiple sigmoid functions based trigonometric and hyperbolic neural network approximations to complex valued continuous functions over boxes in $\mathbb{R}^N$, $N \in \mathbb{N}$ and also iterated, multi layer approximations. All convergences here are with rates expressed via the multivariate moduli of continuity of the involved function and its partial derivatives and given by very tight multidimensional Jackson type inequalities.

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators based on boxes of $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we mention important properties of the basic multivariate density function induced by a set of general multiple sigmoid functions.

Feed-forward neural networks (FNNs) with one hidden layer here are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{C}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is a kind of general sigmoid function. About neural networks read [17] - [19].

2 Basics

2.1 General neural network background

The following come from [12], Ch. 27.

Let $i = 1, ..., N \in \mathbb{N}$ and $h_i : \mathbb{R} \to [-1, 1]$ be a general sigmoid activation function, such that it is strictly increasing, $h_i(0) = 0$, $h_i(-x) = -h_i(x)$, $h_i(+\infty) = 1$, $h_i(-\infty) = -1$. Also h_i is strictly convex over $(-\infty, 0]$ and striclty concave over $[0, +\infty)$, with $h_i^{(2)} \in C(\mathbb{R})$.

We consider the scaled function

$$\psi_i(x) := \frac{1}{4}\left(h_i(x+1) - h_i(x-1)\right), \; x \in \mathbb{R}, \; i = 1, ..., N. \tag{1}$$

As in [10], p. 285, we get that $\psi_i(-x) = \psi_i(x)$, thus ψ_i is an even function. Since $x + 1 > x - 1$, then $h_i(x+1) > h_i(x-1)$, and $\psi_i(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi_i(0) = \frac{h_i(1)}{2}, \quad i = 1, ..., N. \tag{2}$$

Let $x > 1$, we have that

$$\psi_i'(x) = \frac{1}{4}\left(h_i'(x+1) - h_i'(x-1)\right) < 0,$$

by h_i' being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1 - x > 0$ and $0 < 1 - x < 1 + x$. It holds $h_i'(x-1) = h_i'(1-x) > h_i'(x+1)$, so that again $\psi_i'(x) < 0$. Consequently ψ_i is stritly decreasing on $(0, +\infty)$.

Clearly, ψ_i is strictly increasing on $(-\infty, 0)$, and $\psi_i'(0) = 0$.

See that

$$\lim_{x \to +\infty} \psi_i(x) = \frac{1}{4}\left(h_i(+\infty) - h_i(+\infty)\right) = 0, \tag{3}$$

and

$$\lim_{x \to -\infty} \psi_i(x) = \frac{1}{4}\left(h_i(-\infty) - h_i(-\infty)\right) = 0. \tag{4}$$

That is the x-axis is the horizontal asymptote on ψ_i.

Conclusion, ψ is a bell symmetric function with maximum

$$\psi_i(0) = \frac{h_i(1)}{2}.$$

We need

Theorem 1 *([12], Ch. 27) We have that*

$$\sum_{i=-\infty}^{\infty} \psi_i(x - i) = 1, \;\; \forall\, x \in \mathbb{R}, \;\; i = 1, ..., N. \tag{5}$$

Theorem 2 *([12], Ch. 27) It holds*

$$\int_{-\infty}^{\infty} \psi_i(x)\, dx = 1, \quad i = 1, ..., N. \tag{6}$$

Thus $\psi_i(x)$ is a density function on $\mathbb{R}$, $i = 1, ..., N$.

We give

Theorem 3 *([12], Ch. 27) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_i(nx - k) < \frac{\left(1 - h_i\left(n^{1-\alpha} - 2\right)\right)}{2}, \; i = 1, ..., N. \tag{7}$$

Notice that

$$\lim_{n \to +\infty} \frac{\left(1 - h_i\left(n^{1-\alpha} - 2\right)\right)}{2} = 0, \;\; i = 1, ..., N.$$

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number.

We further give

Theorem 4 *([12], Ch. 27) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na\rceil \leq \lfloor nb\rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_i(nx-k)} < \frac{1}{\psi_i(1)}, \quad \forall\ x \in [a,b], \quad i = 1, ..., N. \tag{8}$$

Remark 5 *([12], Ch. 27) We have that*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_i(nx-k) \neq 1, \quad i = 1, ..., N, \tag{9}$$

for at least some $x \in [a,b]$.

Note 6 *([12], Ch. 27) For large enough n we always obtain $\lceil na\rceil \leq \lfloor nb\rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na\rceil \leq k \leq \lfloor nb\rfloor$. In general it holds (by (5))*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_i(nx-k) \leq 1, \quad i = 1, ..., N. \tag{10}$$

We make

Remark 7 *([12], Ch. 27) We define*

$$Z(x_1, ..., x_N) := Z(x) := \prod_{i=1}^{N}\psi_i(x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}.$$

It has the properties:

(i)

$$Z(x) > 0, \ \forall x \in \mathbb{R}^N, \tag{11}$$

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty} Z(x_1-k_1, ..., x_N-k_N) =$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\prod_{i=1}^{N}\psi_i(x_i-k_i) = \prod_{i=1}^{N}\left(\sum_{k_i=-\infty}^{\infty}\psi_i(x_i-k_i)\right) \overset{(5)}{=} 1.$$

Hence

$$\sum_{k=-\infty}^{\infty} Z(x-k) = 1. \tag{12}$$

That is

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1, \quad \forall x \in \mathbb{R}^N; \; n \in \mathbb{N}. \tag{13}$$

And

(iv)

$$\int_{\mathbb{R}^N} Z(x)\,dx = \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} \psi_i(x_i)\right) dx_1 ... dx_N = \prod_{i=1}^{N} \left(\int_{-\infty}^{\infty} \psi_i(x_i)\,dx_i\right) \overset{(6)}{=} 1, \tag{14}$$

thus

$$\int_{\mathbb{R}^N} Z(x)\,dx = 1, \tag{15}$$

that is Z *is a multivariate density function.*

Here denote $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, *also set* $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ *upon the multivariate context, and*

$$\lceil na \rceil := (\lceil na_1 \rceil, ..., \lceil na_N \rceil),$$

$$\lfloor nb \rfloor := (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor),$$

where $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \psi_i(nx_i - k_i)\right) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^{N} \psi_i(nx_i - k_i)\right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi_i(nx_i - k_i)\right). \tag{16}$$

For $0 < \beta < 1$ *and* $n \in \mathbb{N}$, *a fixed* $x \in \mathbb{R}^N$, *we have that*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) =$$

$$\sum_{\substack{k=\lceil na \rceil \\ \left\|\frac{k}{n}-x\right\|_\infty \le \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k) + \sum_{\substack{k=\lceil na \rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k). \tag{17}$$

In the last two sums the counting is over disjoint vector sets of k*'s, because the condition* $\left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}$ *implies that there exists at least one* $\left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^\beta}$, *where* $r \in \{1, ..., N\}$.

(v) We notice that

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor} Z(nx-k) = \sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \ldots \sum_{\substack{k_N=\lceil na_N\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb_N\rfloor} \left(\prod_{i=1}^{N}\psi_i(nx_i-k_i)\right) =$$

$$\prod_{i=1}^{N}\left(\sum_{\begin{cases} k_i=\lceil na_i\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb_i\rfloor} \psi_i(nx_i-k_i)\right) \leq$$

$$\left(\prod_{\substack{i=1\\ i\neq r}}^{N}\left(\sum_{k_i=-\infty}^{\infty}\psi_i(nx_i-k_i)\right)\right)\left(\sum_{\begin{cases} k_r=\lceil na_r\rceil \\ \left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb_r\rfloor} \psi_r(nx_r-k_r)\right) =$$

$$\left(\sum_{\begin{cases} k_r=\lceil na_r\rceil \\ \left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb_r\rfloor} \psi_r(nx_r-k_r)\right) \leq \tag{18}$$

$$\sum_{\begin{cases} k_r=-\infty \\ \left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^{\beta}}\end{cases}}^{\infty} \psi_r(nx_r-k_r) = \sum_{\begin{cases} k_r=-\infty \\ |nx_r-k_r|>n^{1-\beta}\end{cases}}^{\infty} \psi_r(nx_r-k_r) \overset{(7)}{<}$$

$$\frac{1-h_r\left(n^{1-\beta}-2\right)}{2} \leq \max_{i\in\{1,\ldots,N\}}\left(\frac{1-h_i\left(n^{1-\beta}-2\right)}{2}\right),$$

where $0<\beta<1$.

That is we get:

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor} Z(nx-k) < \max_{i\in\{1,\ldots,N\}}\left(\frac{1-h_i\left(n^{1-\beta}-2\right)}{2}\right), \tag{19}$$

$0<\beta<1$, *with* $n\in\mathbb{N}: n^{1-\beta}>2$, $\forall\ x\in\prod_{i=1}^{N}[a_i,b_i]$.

(vi) It is clear that

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}} \end{array}\right.}^{\infty} Z(nx-k) < \max_{i\in\{1,...,N\}} \left(\frac{1-h_i\left(n^{1-\beta}-2\right)}{2}\right), \tag{20}$$

$0<\beta<1$, $n\in\mathbb{N}: n^{1-\beta}>2$, $\forall\ x\in\prod_{i=1}^{N}[a_i,b_i]$.

(viii) By Theorem 4 we get that

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)} = \frac{1}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\psi_i(nx_i-k_i)\right)} < \frac{1}{\prod_{i=1}^{N}\psi_i(1)},$$

thus

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)} < \frac{1}{\prod_{i=1}^{N}\psi_i(1)}, \tag{21}$$

$\forall\ x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n\in\mathbb{N}$.

Furthermore it holds

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) = \lim_{n\to\infty}\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\psi_i(nx_i-k_i)\right) = \tag{22}$$

$$\prod_{i=1}^{N}\left(\lim_{n\to\infty}\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\psi_i(nx_i-k_i)\right)\neq 1,$$

for at least some $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

We state

Definition 8 *([12], Ch. 27) We denote by*

$$\delta_N(\beta,n) := \max_{i\in\{1,...,N\}}\left(\frac{1-h_i\left(n^{1-\beta}-2\right)}{2}\right), \tag{23}$$

where $0<\beta<1$.

We make

Remark 9 *Let* $f\in C\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$, $x=(x_1,...,x_N)\in\prod_{i=1}^{N}[a_i,b_i]$, $n\in\mathbb{N}$ *such that* $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

We introduce and define the following multivariate linear normalized neural network operator ($x:=(x_1,...,x_N)\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$):

$$A_n(f,x_1,...,x_N) := A_n(f,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)} =$$

$$\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}...\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\psi_i\left(nx_i-k_i\right)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\psi_i\left(nx_i-k_i\right)\right)}. \quad (24)$$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i\rceil \leq \lfloor nb_i\rfloor$, $i = 1,...,N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i\rceil \leq k_i \leq \lfloor nb_i\rfloor$, $i = 1,...,N$.

When $g \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ we define the companion operator

$$\widetilde{A}_n(g,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} g\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}. \quad (25)$$

Clearly $\widetilde{A}_n$ is a positive linear operator. We have that

$$\widetilde{A}_n(1,x) = 1, \ \forall\, x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right).$$

Notice that $A_n(f) \in C\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$ and $\widetilde{A}_n(g) \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Furthermore it holds

$$|A_n(f,x)| \leq \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left|f\left(\frac{k}{n}\right)\right| Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)} = \widetilde{A}_n(|f|,x), \quad (26)$$

$\forall\, x \in \prod_{i=1}^{N}[a_i,b_i]$.

Clearly $|f| \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

So, we have that

$$|A_n(f,x)| \leq \widetilde{A}_n(|f|,x), \quad (27)$$

$\forall\, x \in \prod_{i=1}^{N}[a_i,b_i]$, $\forall\, n \in \mathbb{N}$, $\forall\, f \in C\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$.

Let $c \in \mathbb{C}$ and $g \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, then $cg \in C\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$.

Furthermore it holds

$$A_n(cg,x) = c\widetilde{A}_n(g,x), \ \forall\, x \in \prod_{i=1}^{N}[a_i,b_i]. \quad (28)$$

Since $\widetilde{A}_n(1) = 1$, we get that

$$A_n(c) = c, \ \forall\, c \in \mathbb{C}. \quad (29)$$

We call $\widetilde{A}_n$ the companion operator of A_n.

For convinience we call

$$A_n^*(f,x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k) =$$

$$\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}\cdots\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\psi_i\left(nx_i-k_i\right)\right), \tag{30}$$

$\forall\ x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

That is

$$A_n(f,x) := \frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}, \tag{31}$$

$\forall\ x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n\in\mathbb{N}$.

Hence

$$A_n(f,x)-f(x) = \frac{A_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}. \tag{32}$$

Consequently we derive

$$|A_n(f,x)-f(x)| \overset{(21)}{\leq} \left(\prod_{i=1}^{N}\psi_i(1)\right)^{-1}\left|A_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right|, \tag{33}$$

$\forall\ x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

We will estimate the right hand side of (33).

For the last we need

Definition 10 *([10], p. 274) Let M be a convex and compact subset of $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $p\in[1,\infty]$. Let $f\in C(M,\mathbb{C})$, we define the first modulus of continuity of f as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in M:\\ \|x-y\|_p\leq\delta}} |f(x)-f(y)|,\quad 0<\delta\leq diam(M). \tag{34}$$

If $\delta > diam(M)$, then

$$\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{35}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta>0$.

Lemma 11 *([10], p. 274) We have $\omega_1(f,\delta)\to 0$ as $\delta\downarrow 0$, iff $f\in C(M,\mathbb{C})$, where M is a convex compact subset of $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $p\in[1,\infty]$.*

In our results we use $p=\infty$.

Let now $f\in C^2\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$, $N\in\mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}_+$, $i=1,...,N$, and $|\alpha|:=\sum_{i=1}^{N}\alpha_i=l$, where $l=0,1,2$. We write also $f_\alpha:=\frac{\partial^n f}{\partial x^n}$ and we say it is of order l.

We denote

$$\omega_1^{\max}(f_\alpha, h) := \max_{\alpha:|\alpha|=2} \omega_1(f_\alpha, h). \tag{36}$$

Call also

$$\|f_\alpha\|_\infty^{\max} := \max_{|\alpha|=2} \{\|f_\alpha\|_\infty\}, \tag{37}$$

where $\|\cdot\|_\infty$ is the supremum norm.

2.2 Multivariate New Taylor formulae

We will use

Theorem 12 *([13]) Let $f \in C^2([c,d], \mathbb{C})$, where $a, x \in [c,d]$. Then*

$$f(x) - f(a) = f'(a)\sin(x-a) + 2f''(a)\sin^2\left(\frac{x-a}{2}\right) + \tag{38}$$

$$\int_a^x [(f''(t) + f(t)) - (f''(a) + f(a))] \sin(x-t)\, dt.$$

We make

Remark 13 *Let now Q be an open convex subset of $\mathbb{R}^k$, $k \geq 2$; $z = (z_1, ..., z_k)$, $x_0 := (x_{01}, ..., x_{0k}) \in Q$. We consider $f \in C^2(Q, \mathbb{C})$ each second order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, ..., \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., k$ and $|\alpha| := \sum_{i=1}^k \alpha_i = 2$. We consider $g_z(t) := f(x_0 + t(z - x_0))$, $0 \leq t \leq 1$. Clearly $x_0 + t(z - x_0) \in Q$. Then*

$$g_z(0) = f(x_0), \quad g_z(1) = f(z),$$

$$g_z'(t) = \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial f}{\partial x_i}(x_{01} + t(z_1 - x_{01}), ..., x_{0k} + t(z_k - x_{0k})), \tag{39}$$

$$g_z'(0) = \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial f}{\partial x_i}(x_{01}, ..., x_{0k}),$$

and

$$g_z''(t) = \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^2 f\right](x_{01} + t(z_1 - x_{01}), ..., x_{0k} + t(z_k - x_{0k})),$$

$$g_z''(0) = \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^2 f\right](x_{01}, ..., x_{0k}). \tag{40}$$

Notice above the second order partials commute.

Clearly $g_z \in C^2([0,1], \mathbb{C})$, and by Theorem 12 we obtain

$$f(z_1, ..., z_k) - f(x_{01}, ..., x_{0k}) = g_z(1) - g_z(0) =$$

$$g_z'(0)\sin(1) + 2g_z''(0)\sin^2\left(\frac{1}{2}\right) + \int_0^1 \left[(g_z''(t) + g_z(t)) - (g_z''(0) + g_z(0))\right]\sin(1-t)\,dt. \tag{41}$$

We also mention

Theorem 14 *([13]) Let $f \in C^2([c,d], \mathbb{C})$, where $a, x \in [c,d]$. Then*

$$f(x) - f(a) = f'(a)\sinh(x-a) + 2f''(a)\sinh^2\left(\frac{x-a}{2}\right) + \tag{42}$$

$$\int_a^x \left[(f''(t) - f(t)) - (f''(a) - f(a))\right]\sinh(x-t)\,dt.$$

We make

Remark 15 *Consequently, we get that*

$$f(z_1, ..., z_k) - f(x_{01}, ..., x_{0k}) = g_z(1) - g_z(0) =$$

$$g_z'(0)\sinh(1) + 2g_z''(0)\sinh^2\left(\frac{1}{2}\right) + \int_0^1 \left[(g_z''(t) - g_z(t)) - (g_z''(0) - g_z(0))\right]\sinh(1-t)\,dt. \tag{43}$$

We make

Remark 16 *Let $f \in C^2\left(\prod_{i=1}^N [a_i, b_i], \mathbb{C}\right)$, $N \in \mathbb{N}$.*

Clearly the mixed partials commute.

Here $\frac{k}{n} := \left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right)$, and $x := (x_1, ..., x_N)$, with $\frac{k}{n}, x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, then (by (41), where $g_{\frac{k}{n}}(t) := f\left(x + t\left(\frac{k}{n} - x\right)\right)$, $0 \le t \le 1$) we have

$$f\left(\frac{k}{n}\right) - f(x) = \left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial f}{\partial x_i}(x)\right)\sin(1) +$$

$$2\left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x)\right\}\sin^2\left(\frac{1}{2}\right) +$$

$$\int_0^1 \left\{\left\{\left[\left(\sum_{i=1}^N \left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right]\left(x + t\left(\frac{k}{n} - x\right)\right) + f\left(x + t\left(\frac{k}{n} - x\right)\right)\right\} -\right.$$

$$\left\{\left[\left(\sum_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x)+f(x)\right\}\right\}\sin(1-t)\,dt. \tag{44}$$

Denote the remainder

$$R:=\int_0^1\left\{\left\{\left[\left(\sum_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right]\left(x+t\left(\frac{k}{n}-x\right)\right)+f\left(x+t\left(\frac{k}{n}-x\right)\right)\right\}\right.$$

$$\left.-\left\{\left[\left(\sum_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x)+f(x)\right\}\right\}\sin(1-t)\,dt= \tag{45}$$

$$\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)\left[f_\alpha\left(x+t\left(\frac{k}{n}-x\right)\right)-f_\alpha(x)\right]\right.$$

$$\left.+\left(f\left(x+t\left(\frac{k}{n}-x\right)\right)-f(x)\right)\right\}\sin(1-t)\,dt.$$

Therefore it holds

$$|R|\le\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\right.$$

$$\left(\prod_{i=1}^{N}\left|\frac{k_i}{n}-x_i\right|^{\alpha_i}\right)\left|f_\alpha\left(x+t\left(\frac{k}{n}-x\right)\right)-f_\alpha(x)\right|$$

$$\left.+\left|f\left(x+t\left(\frac{k}{n}-x\right)\right)-f(x)\right|\right\}|\sin(1-t)|\,dt\le \tag{46}$$

$$\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left|\frac{k_i}{n}-x_i\right|^{\alpha_i}\right)\omega_1\left(f_\alpha,t\left\|\frac{k}{n}-x\right\|_\infty\right)\right.$$

$$\left.+\omega_1\left(f,t\left\|\frac{k}{n}-x\right\|_\infty\right)\right\}|\sin(1-t)|\,dt\le(*).$$

Notice here that $(0<\beta<1)$

$$\left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\Leftrightarrow\left|\frac{k_i}{n}-x_i\right|\leq\frac{1}{n^{\beta}},\ \ i=1,...,N. \tag{47}$$

We further see that

$$(*)\leq\left\{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)\left(\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\frac{1}{n^{\beta\alpha_i}}\right)\right)\right.$$

$$\left.+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}\int_0^1|\sin(1-t)|\,dt=$$

$$\left[\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)\left(\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\right)\frac{1}{n^{2\beta}}\right.$$

$$\left.+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right](1-\cos(1))= \tag{48}$$

$$(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}.$$

We have proved that

$$|R|\leq(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}, \tag{49}$$

given that $\left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}$.

We notice also that

$$|R|\leq\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\right.$$

$$\left.\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)2\|f_{\alpha}\|_{\infty}+2\|f\|_{\infty}\right\}|\sin(1-t)|\,dt\leq \tag{50}$$

$$\left\{ \left(\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+ \\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}} \left(\frac{2}{\prod_{i=1}^{N}\alpha_i!} \right) \right) \right.$$

$$\left. 2\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} + 2\|f\|_\infty \right\} \left(\int_0^1 |\sin(1-t)|\, dt \right) =$$

$$\left(2\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + 2\|f\|_\infty \right)(1-\cos(1)),$$

where $a := (a_1, ..., a_N)$, $b = (b_1, ..., b_N)$.

We have proved that

$$|R| \leq \left(2\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + 2\|f\|_\infty \right)(1-\cos(1)) =: \rho. \tag{51}$$

3 Main Results

Here we discuss the trigonometric approximation by using the smoothness of f.

Theorem 17 *Let* $f \in C^2\left(\prod_{i=1}^{N}[a_i,b_i], \mathbb{C}\right)$, $0 < \beta < 1$, $n, N \in \mathbb{N}$, $n^{1-\beta} > 2$, $x, x_0 \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $a := (a_1,...,a_N)$, $b := (b_1,...,b_N)$. *Then*

(i)

$$\left| A_n(f,x) - f(x) - \left(\sum_{=1}^{N} \frac{\partial f(x)}{\partial x_i} A_n((\cdot - x_i), x) \right) \sin(1) - \right.$$

$$\left. 4 \left\{ \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+ \\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}} f_\alpha(x) \left(\frac{1}{\prod_{i=1}^{N}\alpha_i!} \right) A_n\left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i}, x \right) \right\} \sin^2\left(\frac{1}{2}\right) \right| \leq$$

$$\left(\prod_{i=1}^{N} \psi_i(1) \right)^{-1} \left\{ \left[(1-\cos(1)) \left\{ \frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\} \right] + \right. \tag{52}$$

$$\left. \left[2\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + 2\|f\|_\infty \right] (1-\cos(1))\, \delta_N(\beta,n) \right\},$$

(ii) assume that $\frac{\partial f(x_0)}{\partial x_i} = 0$, $i = 1,...,N$, *and* $f_\alpha(x_0) = 0$, $\alpha : |\alpha| = 2$, *we have that*

$$|A_n(f,x) - f(x)| \leq$$

$$\left(\prod_{i=1}^{N}\psi_i(1)\right)^{-1}\left\{\left[(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]+\right. \quad (53)$$

$$\left.\left[2\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+2\|f\|_\infty\right](1-\cos(1))\,\delta_N(\beta,n)\right\},$$

(iii)

$$|A_n(f,x)-f(x)|\leq\left(\prod_{i=1}^{N}\psi_i(1)\right)^{-1}$$

$$\left\{\left\{\left\{\sum_{i=1}^{N}\left|\frac{\partial f(x)}{\partial x_i}\right|\left\{\frac{1}{n^\beta}+(b_i-a_i)\,\delta_N(\beta,n)\right\}\right\}\sin(1)+\right.\right.$$

$$\left.4\left\{\sum_{\alpha:|\alpha|=2}|f_\alpha(x)|\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\delta_N(\beta,n)\right]\right\}\sin^2\left(\frac{1}{2}\right)\right\}+$$

$$\left\{\left[(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]\right.$$

$$\left.\left.+\left[2\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+2\|f\|_\infty\right](1-\cos(1))\,\delta_N(\beta,n)\right\}\right\}, \quad (54)$$

and

(iv)

$$\|A_n(f)-f\|_\infty\leq\left(\prod_{i=1}^{N}\psi_i(1)\right)^{-1}$$

$$\left\{\left\{\left\{\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty\left\{\frac{1}{n^\beta}+(b_i-a_i)\,\delta_N(\beta,n)\right\}\right\}\sin(1)+\right.\right.$$

$$\left.4\left\{\sum_{\alpha:|\alpha|=2}\|f_\alpha\|_\infty\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\delta_N(\beta,n)\right]\right\}\sin^2\left(\frac{1}{2}\right)\right\} \quad (55)$$

$$+\left\{\left[(1-\cos(1))\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^\beta}\right)\right\}\right]\right.$$

$$\left.\left.+\left[2\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+2\|f\|_\infty\right](1-\cos(1))\,\delta_N(\beta,n)\right\}\right\}=:\xi_n(f).$$

We observe that $A_n\to I$ (unit operator), as $n\to\infty$, pointwise and uniformly.

Proof. Here R is as in (45). We see that

$$U_n := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) R = \tag{56}$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_\infty \le \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) R + \sum_{\begin{cases} k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) R.$$

Therefore

$$|U_n| \le \left(\sum_{\begin{cases} k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_\infty \le \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) \right)$$

$$\left[(1-\cos(1)) \left\{ \frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\} \right] + \rho\delta_N(\beta, n) \le \tag{57}$$

$$\left[(1-\cos(1)) \left\{ \frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\} \right] + \rho\delta_N(\beta, n).$$

We have established that

$$|U_n| \le \left[(1-\cos(1)) \left\{ \frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\} \right]$$

$$+ \left[2\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + 2\|f\|_\infty \right] (1-\cos(1)) \delta_N(\beta, n). \tag{58}$$

By (44) we observe that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \right) =$$

$$(\sum_{i=1}^{N} \left(\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \left(\frac{k_i}{n} - x_i \right) \right) \frac{\partial f}{\partial x_i}(x) \right)) \sin(1) +$$

$$2\{ \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i=1,\dots,N, |\alpha| := \sum_{i=1}^{N} \alpha_i = 2}} f_\alpha(x) \left(\frac{2}{\prod_{i=1}^{N} \alpha_i!} \right) (\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)$$

$$\left(\prod_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)\Bigg)\Bigg\}\sin^2\left(\frac{1}{2}\right)+U_n. \tag{59}$$

The last says

$$A_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)\right)-$$

$$\left(\sum_{=1}^{N}\frac{\partial f(x)}{\partial x_i}A_n^*((\cdot-x_i),x)\right)\sin(1)-$$

$$2\left\{\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}f_\alpha(x)\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)A_n^*\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right\}\sin^2\left(\frac{1}{2}\right)=U_n. \tag{60}$$

We notice that

$$|A_n^*((\cdot-x_i),x)|\le A_n^*(|\cdot-x_i|,x)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right|Z(nx-k)=$$

$$\sum_{\begin{cases}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_\infty\le\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right|Z(nx-k)+$$

$$\sum_{\begin{cases}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}\left|\frac{k_i}{n}-x_i\right|Z(nx-k)\le$$

$$\frac{1}{n^\beta}+(b_i-a_i)\sum_{\begin{cases}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}Z(nx-k)\le \tag{61}$$

$$\frac{1}{n^\beta}+(b_i-a_i)\,\delta_N(\beta,n).$$

We have proved that

$$|A_n^*((\cdot-x_i),x)|\le\frac{1}{n^\beta}+(b_i-a_i)\,\delta_N(\beta,n), \tag{62}$$

$i=1,...,N$.

Next we see that

$$\left|A_n^*\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right|\leq A_n^*\left(\prod_{i=1}^{N}|\cdot-x_i|^{\alpha_i},x\right)=$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N}\left|\frac{k_i}{n}-x_i\right|^{\alpha_i}\right)Z(nx-k)=$$

$$\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^\beta}\end{array}\right.}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N}\left|\frac{k_i}{n}-x_i\right|^{\alpha_i}\right)Z(nx-k)+ \tag{63}$$

$$\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{array}\right.}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N}\left|\frac{k_i}{n}-x_i\right|^{\alpha_i}\right)Z(nx-k)\leq$$

$$\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\delta_N(\beta,n).$$

We have proved that

$$\left|A_n^*\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right|\leq\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\delta_N(\beta,n). \tag{64}$$

At last we observe that

$$\left|A_n(f,x)-f(x)-\left(\sum_{=1}^{N}\frac{\partial f(x)}{\partial x_i}A_n((\cdot-x_i),x)\right)\sin(1)-\right.$$

$$\left.4\left\{\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}f_\alpha(x)\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)A_n\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right\}\sin^2\left(\frac{1}{2}\right)\right|\leq$$

$$\left(\prod_{i=1}^{N}\psi_i(1)\right)^{-1}|U_n|=$$

$$\left(\prod_{i=1}^{N}\psi_i(1)\right)^{-1}\left|A_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)\right)-\right.$$

$$\left(\sum_{=1}^{N}\frac{\partial f(x)}{\partial x_i}A_n^*\left(\left(\cdot-x_i\right),x\right)\right)\sin(1)-$$

$$\left.4\left\{\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}f_\alpha(x)\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)A_n^*\left(\prod_{i=1}^{N}\left(\cdot-x_i\right)^{\alpha_i},x\right)\right\}\sin^2\left(\frac{1}{2}\right)\right|. \tag{65}$$

Putting all of the above together we prove the theorem. ■

We make

Remark 18 *Let* $f\in C^2\left(\prod_{i=1}^{N}[a_i,b_i],\mathbb{C}\right)$, $N\in\mathbb{N}$. *By the mean value theorem we have that* $\sinh x=\sinh x-\sinh 0=(\cosh\xi)(x-0)$, *for some* ξ *between* $\{0,x\}$, *for any* $x\in\mathbb{R}$.

Hence

$$|\sinh x|\leq\|\cosh\|_{\infty,[-1,1]}|x|,\ \ \forall\ x\in[-1,1].$$

But

$$\|\cosh\|_{\infty,[-1,1]}=\cosh(1).$$

Thus, we have

$$|\sinh x|\leq\cosh(1)|x|,\ \ \forall\ x\in[-1,1].$$

Let $\frac{k}{n}:=\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)$, *and* $x:=(x_1,...,x_N)$, *with* $\frac{k}{n},x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, *then (by (43), where* $g_{\frac{k}{n}}(t):=f\left(x+t\left(\frac{k}{n}-x\right)\right)$, $0\leq t\leq 1$*) we have*

$$f\left(\frac{k}{n}\right)-f(x)=\left(\sum_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)\frac{\partial f}{\partial x_i}(x)\right)\sinh(1)+$$

$$2\left\{\left[\left(\sum_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x)\right\}\sinh^2\left(\frac{1}{2}\right)+$$

$$\int_0^1\left\{\left\{\left[\left(\sum_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right]\left(x+t\left(\frac{k}{n}-x\right)\right)-f\left(x+t\left(\frac{k}{n}-x\right)\right)\right\}\right.$$

$$\left.-\left\{\left[\left(\sum_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x)-f(x)\right\}\right\}\sinh(1-t)\,dt. \tag{66}$$

Denote the remainder

$$R:=\int_0^1\left\{\left\{\left[\left(\sum_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right]\left(x+t\left(\frac{k}{n}-x\right)\right)-f\left(x+t\left(\frac{k}{n}-x\right)\right)\right\}\right.$$

$$-\left\{\left[\left(\sum_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^2 f\right](x)-f(x)\right\}\right\}\sinh(1-t)\,dt= \tag{67}$$

$$\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)\left[f_\alpha\left(x+t\left(\frac{k}{n}-x\right)\right)-f_\alpha(x)\right]\right.$$

$$\left.-\left(f\left(x+t\left(\frac{k}{n}-x\right)\right)-f(x)\right)\right\}\sinh(1-t)\,dt.$$

Therefore it holds

$$|R|\leq\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\right.$$

$$\left(\prod_{i=1}^{N}\left|\frac{k_i}{n}-x_i\right|^{\alpha_i}\right)\left|f_\alpha\left(x+t\left(\frac{k}{n}-x\right)\right)-f_\alpha(x)\right|+$$

$$\left.+\left|f\left(x+t\left(\frac{k}{n}-x\right)\right)-f(x)\right|\right\}|\sinh(1-t)|\,dt\leq \tag{68}$$

$$\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left|\frac{k_i}{n}-x_i\right|^{\alpha_i}\right)\omega_1\left(f_\alpha,t\left\|\frac{k}{n}-x\right\|_\infty\right)\right.$$

$$\left.+\omega_1\left(f,t\left\|\frac{k}{n}-x\right\|_\infty\right)\right\}\cosh(1)(1-t)\,dt\leq(*).$$

Notice here that $(0<\beta<1)$

$$\left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^\beta}\Leftrightarrow\left|\frac{k_i}{n}-x_i\right|\leq\frac{1}{n^\beta},\quad i=1,...,N. \tag{69}$$

We further see that

$$(*)\leq\cosh(1)\left\{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)\left(\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\frac{1}{n^{\beta\alpha_i}}\right)\right)\right.$$

$$+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}\int_0^1(1-t)\,dt=$$

$$\cosh(1)\left\{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^{\beta}}\right)\left(\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^N\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^N\alpha_i!}\right)\right)\frac{1}{n^{2\beta}}\right.$$

$$\left.+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}\frac{1}{2}= \tag{70}$$

$$\frac{\cosh(1)}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^{\beta}}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}.$$

We have proved that

$$|R|\leq\frac{\cosh(1)}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_\alpha,\frac{1}{n^{\beta}}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}, \tag{71}$$

given that $\left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^{\beta}}$.

We notice also that

$$|R|\leq\cosh(1)\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^N\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^N\alpha_i!}\right)\right.$$

$$\left.\left(\prod_{i=1}^N(b_i-a_i)^{\alpha_i}\right)2\|f_\alpha\|_\infty+2\|f\|_\infty\right\}(1-t)\,dt\leq \tag{72}$$

$$\cosh(1)\left\{\left(\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^N\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^N\alpha_i!}\right)\right)\right.$$

$$\left.2\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}+2\|f\|_\infty\right\}\left(\int_0^1(1-t)\,dt\right)=$$

$$\cosh(1)\left\{2\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+2\|f\|_\infty\right\}\frac{1}{2}=$$

$$\cosh(1)\left(\|b-a\|_\infty^2\|f_\alpha\|_{\infty,2}^{\max}N^2+\|f\|_\infty\right),$$

where $a := (a_1, ..., a_N)$, $b = (b_1, ..., b_N)$.

We have proved that

$$|R| \leq \cosh(1) \left(\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + \|f\|_\infty \right) =: \rho. \tag{73}$$

We continue with the hyperbolic approximation.

Theorem 19 *Let* $f \in C^2\left(\prod_{i=1}^N [a_i, b_i], \mathbb{C}\right)$, $0 < \beta < 1$, $n, N \in \mathbb{N}$, $n^{1-\beta} > 2$, $x, x_0 \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$. *Then*

(i)

$$\left| A_n(f,x) - f(x) - \left(\sum_{=1}^N \frac{\partial f(x)}{\partial x_i} A_n((\cdot - x_i), x) \right) \sinh(1) - \right.$$

$$\left. 4 \left\{ \sum_{\substack{\alpha := (\alpha_1,...,\alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i=1,...,N, |\alpha| := \sum_{i=1}^N \alpha_i = 2}} f_\alpha(x) \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) A_n\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right\} \sinh^2\left(\frac{1}{2}\right) \right| \leq$$

$$\left(\prod_{i=1}^N \psi_i(1) \right)^{-1} \cosh(1) \left\{ \left[\frac{1}{2} \left\{ \frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\} \right] + \right. \tag{74}$$

$$\left. \left[\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + \|f\|_\infty \right] \delta_N(\beta, n) \right\},$$

(ii) assume that $\frac{\partial f(x_0)}{\partial x_i} = 0$, $i = 1, ..., N$, *and* $f_\alpha(x_0) = 0$, $\alpha : |\alpha| = 2$, *we have that*

$$|A_n(f,x) - f(x)| \leq$$

$$\left(\prod_{i=1}^N \psi_i(1) \right)^{-1} (\cosh(1)) \left\{ \left[\frac{1}{2} \left\{ \frac{\omega_{1,2}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) N^2}{n^{2\beta}} + \omega_1\left(f, \frac{1}{n^\beta}\right) \right\} \right] + \right. \tag{75}$$

$$\left. \left[\|b-a\|_\infty^2 \|f_\alpha\|_{\infty,2}^{\max} N^2 + \|f\|_\infty \right] \delta_N(\beta, n) \right\},$$

(iii)

$$|A_n(f,x) - f(x)| \leq \left(\prod_{i=1}^N \psi_i(1) \right)^{-1}$$

$$\left\{ \left\{ \left\{ \sum_{i=1}^N \left| \frac{\partial f(x)}{\partial x_i} \right| \left\{ \frac{1}{n^\beta} + (b_i - a_i) \delta_N(\beta, n) \right\} \right\} \sinh(1) + \right.\right.$$

$$4\left\{\sum_{\alpha:|\alpha|=2}|f_{\alpha}(x)|\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\delta_N(\beta,n)\right]\right\}\sinh^2\left(\frac{1}{2}\right)\Bigg\}$$

$$+\cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\right.$$

$$\left.\left[\|b-a\|_{\infty}^{2}\|f_{\alpha}\|_{\infty,2}^{\max}N^2+\|f\|_{\infty}\right]\delta_N(\beta,n)\right\}\Bigg\}, \tag{76}$$

and

(iv)

$$\|A_n(f)-f\|_{\infty}\leq\left(\prod_{i=1}^{N}\psi_i(1)\right)^{-1}$$

$$\left\{\left\{\left\{\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}\left\{\frac{1}{n^{\beta}}+(b_i-a_i)\delta_N(\beta,n)\right\}\right\}\sinh(1)+\right.\right.$$

$$4\left\{\sum_{\alpha:|\alpha|=2}\|f_{\alpha}\|_{\infty}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{2\beta}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\delta_N(\beta,n)\right]\right\}\sinh^2\left(\frac{1}{2}\right)\Bigg\}$$

$$+\cosh(1)\left\{\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^2}{n^{2\beta}}+\omega_1\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\right. \tag{77}$$

$$\left.\left[\|b-a\|_{\infty}^{2}\|f_{\alpha}\|_{\infty,2}^{\max}N^2+\|f\|_{\infty}\right]\delta_N(\beta,n)\right\}\Bigg\}=:\psi_n(f).$$

We observe that $A_n\to I$ (unit operator), as $n\to\infty$, pointwise and uniformly.

Proof. Here R is as in (67). We see that

$$U_n:=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)R= \tag{78}$$

$$\sum_{\begin{cases}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}Z(nx-k)R+\sum_{\begin{cases}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}Z(nx-k)R.$$

Therefore

$$|U_n|\leq\left(\sum_{\begin{cases}k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}Z(nx-k)\right)$$

$$\cosh(1)\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\rho\delta_{N}(\beta,n)\leq \tag{79}$$

$$\cosh(1)\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]+\rho\delta_{N}(\beta,n).$$

We have established that

$$|U_n|\leq\cosh(1)\left[\frac{1}{2}\left\{\frac{\omega_{1,2}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)N^{2}}{n^{2\beta}}+\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)\right\}\right]$$

$$+\cosh(1)\left[\|b-a\|_{\infty}^{2}\|f_{\alpha}\|_{\infty,2}^{\max}N^{2}+\|f\|_{\infty}\right]\delta_{N}(\beta,n). \tag{80}$$

By (66) we observe that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)Z(nx-k)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)\right)=$$

$$(\sum_{i=1}^{N}\left(\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)\left(\frac{k_i}{n}-x_i\right)\right)\frac{\partial f}{\partial x_i}(x)\right))\sinh(1)+$$

$$2\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}f_{\alpha}(x)\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)\right.$$

$$\left.\left(\prod_{i=1}^{N}\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right))\right\}\sinh^{2}\left(\frac{1}{2}\right)+U_n.$$

The last says

$$A_n^{*}(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)\right)-$$

$$\left(\sum_{=1}^{N}\frac{\partial f(x)}{\partial x_i}A_n^{*}\left((\cdot-x_i),x\right)\right)\sinh(1)-$$

$$2\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}f_{\alpha}(x)\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)A_n^{*}\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right\}\sinh^{2}\left(\frac{1}{2}\right)=U_n. \tag{81}$$

As earlier it holds

$$\left|A_n^*\left(\left(\cdot - x_i\right), x\right)\right| \leq \frac{1}{n^\beta} + \left(b_i - a_i\right)\delta_N\left(\beta, n\right), \tag{82}$$

$i = 1, ..., N$.

Also, as earlier we have

$$\left|A_n^*\left(\prod_{i=1}^N \left(\cdot - x_i\right)^{\alpha_i}, x\right)\right| \leq \frac{1}{n^{2\beta}} + \left(\prod_{i=1}^N \left(b_i - a_i\right)^{\alpha_i}\right)\delta_n\left(\beta, n\right). \tag{83}$$

At last we observe that

$$\left|A_n\left(f, x\right) - f\left(x\right) - \left(\sum_{=1}^N \frac{\partial f\left(x\right)}{\partial x_i} A_n\left(\left(\cdot - x_i\right), x\right)\right)\sinh\left(1\right) - \right.$$

$$\left. 4\left\{\sum_{\substack{\alpha:=\left(\alpha_1,...,\alpha_N\right), \alpha_i \in \mathbb{Z}_+ \\ i=1,...,N, \left|\alpha\right| := \sum_{i=1}^N \alpha_i = 2}} f_\alpha\left(x\right)\left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right) A_n\left(\prod_{i=1}^N \left(\cdot - x_i\right)^{\alpha_i}, x\right)\right\}\sinh^2\left(\frac{1}{2}\right)\right| \leq$$

$$\left(\prod_{i=1}^N \psi_i\left(1\right)\right)^{-1} \left|U_n\right| =$$

$$\left(\prod_{i=1}^N \psi_i\left(1\right)\right)^{-1} \left|A_n^*\left(f, x\right) - f\left(x\right)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx - k\right)\right) - \right.$$

$$\left(\sum_{=1}^N \frac{\partial f\left(x\right)}{\partial x_i} A_n^*\left(\left(\cdot - x_i\right), x\right)\right)\sinh\left(1\right) -$$

$$\left. 4\left\{\sum_{\substack{\alpha:=\left(\alpha_1,...,\alpha_N\right), \alpha_i \in \mathbb{Z}_+ \\ i=1,...,N, \left|\alpha\right| := \sum_{i=1}^N \alpha_i = 2}} f_\alpha\left(x\right)\left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right) A_n^*\left(\prod_{i=1}^N \left(\cdot - x_i\right)^{\alpha_i}, x\right)\right\}\sinh^2\left(\frac{1}{2}\right)\right|. \tag{84}$$

Putting all of the above together we prove theorem. ■

We make

Remark 20 *By (24) we get that* $\left\|A_n\left(f\right)\right\|_\infty \leq \left\|f\right\|_\infty < \infty$, *and* $A_n\left(f\right) \in C\left(\prod_{i=1}^N \left[a_i, b_i\right], \mathbb{C}\right)$, *given that* $f \in C\left(\prod_{i=1}^N \left[a_i, b_i\right], \mathbb{C}\right)$.

Clearly then

$$\left\| A_n^2 (f) \right\|_\infty = \| A_n (A_n (f)) \|_\infty \leq \| A_n (f) \|_\infty \leq \| f \|_\infty , \tag{85}$$

etc.

Therefore we get

$$\left\| A_n^k (f) \right\|_\infty \leq \| f \|_\infty , \quad \forall \ k \in \mathbb{N}, \tag{86}$$

the contraction property.

Also we see that

$$\left\| A_n^k (f) \right\|_\infty \leq \left\| A_n^{k-1} (f) \right\|_\infty \leq ... \leq \| A_n (f) \|_\infty \leq \| f \|_\infty . \tag{87}$$

Also $A_n (1) = 1$, $A_n^k (1) = 1$, $\forall \ k \in \mathbb{N}$.

Following 18.14, pp. 401-402, of [9], similarly we obtain that

$$\| A_n^r f - f \|_\infty \leq r \| A_n (f) - f \|_\infty , \quad r \in \mathbb{N}. \tag{88}$$

We give

Theorem 21 *All as in Theorems 17, 19. Then*

(i)

$$\| A_n^r f - f \|_\infty \leq r \xi_n (f) , \tag{89}$$

where $\xi_n (f)$ *as in (55).*

(ii)

$$\| A_n^r f - f \|_\infty \leq r \psi_n (f) , \tag{90}$$

where $\psi_n (f)$ *as in (77).*

So that the speed of convergence to the unit operator of A_n^r *is not worse than of* A_n*, see also [8].*

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