Updated radial Ostrowski inequalities over a ball

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Abstract

Here we present general multivariate radial mixed Ostrowski type inequalities over balls. The proofs derive by implementation of some essential estimates out of some new trigonometric and hyperbolic Taylor's formulae ([2]) and reducing the multivariate problem to a univariate one via general polar coordinates.

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1 Introduction

We are motivated by the following:

In 1938, A. Ostrwoski [5] proved the following famous inequality.

Theorem 1 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty , \tag{1}$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best.

Ostrowski type inequalities have great applications to numerical analysis and probability and their literature is enormous.

Here $K = \mathbb{R}$ or $\mathbb{C}$.

Recently the author proved:

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Theorem 2 *([2]) Let $f \in C_K^3([c,d])$, $a \in [c,d]$, such that $f'(a) = f''(a) = 0$. Then*

i)

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f(a) \right| \leq \|f''' + f'\|_\infty \frac{\left[(d-a)^3 + (a-c)^3 \right]}{6(d-c)}, \tag{2}$$

ii) when $f'\left(\frac{c+d}{2}\right) = f''\left(\frac{c+d}{2}\right) = 0$, and $a = \frac{c+d}{2}$, we get

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f\left(\frac{c+d}{2}\right) \right| \leq \|f' + f'''\|_\infty \frac{(d-c)^2}{24}. \tag{3}$$

We are also motivated by author's monograph, see chapters 5,6.

This work is based on author's recent article [2], where we developed some new trigonometric and hyperbolic type Taylor's formulae.

We prove here a collection of multivariate Ostrowski type inequalities related to radial functions over a ball in $\mathbb{R}^N$, with respect to all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$, and we give also their generalizations.

2 Main Results

We make

Remark 3 *We define the ball*

$$B(0,R) := \left\{ x \in \mathbb{R}^N : |x| < R \right\} \subseteq \mathbb{R}^N, \quad N \geq 2, \ R > 0,$$

and the sphere

$$S^{N-1} := \left\{ x \in \mathbb{R}^N : |x| = 1 \right\},$$

where $|\cdot|$ is the Euclidean norm.

Let $d\omega$ be the element of surface measure on $\mathbb{S}^{N-1}$ and let

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{4}$$

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$.

Note that

$$\int_{B(0,R)} dy = \frac{\omega_N R^N}{N} \tag{5}$$

is the Lebesgue measure of the ball.

Following [3, pp. 149-150, exercise 6] and [4, pp. 87-88, Theorem 5.2.2] we can write for $F : \overline{B(0,R)} \to \mathbb{R}$ a Lebesgue integrable function that

$$\int_{B(0,R)} F(x)\,dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)\, r^{N-1} dr \right) d\omega; \tag{6}$$

a formula to be used next.

We present the following multivariate radial Ostrowski type inequality.

Theorem 4 *Let the function $f : \overline{B(0,R)} \to \mathbb{R}$ be radial, that is, there exists a function g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0,R]$, $\forall\, x \in \overline{B(0,R)}$. We further assume that $g \in C^3([0,R])$, and $g^{(k)}(r_0) = 0$, $k = 1,2$, where $r_0 \in [0,R]$ is fixed. Then ($\forall\, \omega \in S^{N-1}$)*

$$\left| f(r_0\omega) - \frac{\int_{B(0,R)} f(y)\,dy}{Vol(B(0,R))} \right| = \left| g(r_0) - \frac{N}{R^N} \int_0^R g(s)\, s^{N-1} ds \right| \le \tag{7}$$

$$\frac{N!}{R^N} \left[\|g''' + g'\|_{\infty,[0,r_0]} \frac{r_0^{3+N}}{(3+N)!} + \right.$$

$$\left. \|g''' + g'\|_{\infty,[r_0,R]} \sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}}{k!\,(N-k+3)!} R^k (R-r_0)^{N-k+3} \right].$$

Proof. As in [2], we get that

$$|g(r) - g(r_0)| \le \|g''' + g'\|_{\infty,[r_0,R]} \frac{(r-r_0)^3}{3!}, \tag{8}$$

$\forall\, r \in [r_0, R]$,

and

$$|g(r) - g(r_0)| \le \|g''' + g'\|_{\infty,[0,r_0]} \frac{(r_0-r)^3}{3!}, \tag{9}$$

$\forall\, r \in [0, r_0]$.

Next we observe

$$\left| f(r_0\omega) - \frac{\int_{B(0,R)} f(y)\,dy}{Vol(B(0,R))} \right| =$$

$$\left| g(r_0) - \frac{\int_{S^{N-1}} \left(\int_0^R g(s)\, s^{N-1} ds \right) d\omega}{\int_{S^{N-1}} \left(\int_0^R s^{N-1} ds \right) d\omega} \right| =$$

$$\left| g(r_0) - \frac{N}{R^N} \int_0^R g(s)\, s^{N-1} ds \right| = \frac{N}{R^N} \left| \int_0^R s^{N-1} (g(r_0) - g(s))\, ds \right| \le$$

$$
\frac{N}{R^N}\int_0^R s^{N-1}\left|g(r_0)-g(s)\right|ds= \tag{10}
$$

$$
\frac{N}{R^N}\left[\int_0^{r_0} s^{N-1}\left|g(r_0)-g(s)\right|ds+\int_{r_0}^R s^{N-1}\left|g(r_0)-g(s)\right|ds\right]\leq
$$

$$
\frac{N}{6R^N}\left[\left\|g'''+g'\right\|_{\infty,[0,r_0]}\int_0^{r_0} s^{N-1}(r_0-s)^3\,ds+\right.
$$

$$
\left.\left\|g'''+g'\right\|_{\infty,[r_0,R]}\int_{r_0}^R s^{N-1}(s-r_0)^3\,ds\right]=
$$

$$
\frac{N}{6R^N}\left[\left\|g'''+g'\right\|_{\infty,[0,r_0]}\int_0^{r_0}(r_0-s)^{4-1}(s-0)^{N-1}ds+\right. \tag{11}
$$

$$
\left.\left\|g'''+g'\right\|_{\infty,[r_0,R]}(-1)^{N-1}\int_{r_0}^R((R-s)-R)^{N-1}(s-r_0)^3\,ds\right]=
$$

$$
\frac{N}{6R^N}\left[\left\|g'''+g'\right\|_{\infty,[0,r_0]}\frac{\Gamma(4)\Gamma(N)}{\Gamma(4+N)}r_0^{3+N}+\right.
$$

$$
\left.\left\|g'''+g'\right\|_{\infty,[r_0,R]}(-1)^{N-1}\sum_{k=0}^{N-1}(-1)^k R^k\binom{N-1}{k}\int_{r_0}^R(R-s)^{N-k-1}(s-r_0)^{4-1}ds\right]=
$$

$$
\frac{N}{6R^N}\left[\left\|g'''+g'\right\|_{\infty,[0,r_0]}\frac{3!\,(N-1)!}{(3+N)!}r_0^{3+N}+\right. \tag{12}
$$

$$
\left.\left\|g'''+g'\right\|_{\infty,[r_0,R]}(-1)^{N-1}\sum_{k=0}^{N-1}(-1)^k R^k\binom{N-1}{k}\frac{(N-k-1)!3!}{(N-k+3)!}(R-r_0)^{N-k+3}\right]=
$$

$$
\frac{N}{R^N}\left[\left\|g'''+g'\right\|_{\infty,[0,r_0]}\frac{(N-1)!}{(3+N)!}r_0^{3+N}+\right.
$$

$$
\left.\left\|g'''+g'\right\|_{\infty,[r_0,R]}\sum_{k=0}^{N-1}(-1)^{k+N-1}R^k\binom{N-1}{k}\frac{(N-k-1)!}{(N-k+3)!}(R-r_0)^{N-k+3}\right]=
$$

$$
\frac{N!}{R^N}\left[\left\|g'''+g'\right\|_{\infty,[0,r_0]}\frac{r_0^{3+N}}{(3+N)!}+\right.
$$

$$
\left.\left\|g'''+g'\right\|_{\infty,[r_0,R]}\sum_{k=0}^{N-1}(-1)^{k+N-1}\frac{R^k}{k!\,(N-k-1)!}\frac{(N-k-1)!}{(N-k+3)!}(R-r_0)^{N-k+3}\right]=
$$

$$
\frac{N!}{R^N}\left[\left\|g'''+g'\right\|_{\infty,[0,r_0]}\frac{r_0^{3+N}}{(3+N)!}+\right. \tag{13}
$$

$$
\left.\left\|g'''+g'\right\|_{\infty,[r_0,R]}\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}}{k!\,(N-k+3)!}R^k(R-r_0)^{N-k+3}\right].
$$

■

We continue with a more general multivariate radial Ostrowski inequality.

Theorem 5 *Let the function* $f : \overline{B(0,R)} \to \mathbb{R}$ *be radial, that is there exists a function* g *such that* $f(x) = g(r)$*, where* $r = |x|$*,* $r \in [0,R]$*,* $\forall\ x \in \overline{B(0,R)}$*. We further assume that* $g \in C^5([0,R])$*, and* $g^{(k)}(r_0) = 0$*,* $k = 1,2,3,4$*, where* $r_0 \in [0,R]$ *is fixed. Here* $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$*. Then* ($\forall\ \omega \in S^{N-1}$)

$$\left| f(r_0\omega) - \frac{\int_{B(0,R)} f(y)\,dy}{Vol(B(0,R))} \right| = \left| g(r_0) - \frac{N}{R^N}\int_0^R g(s)\, s^{N-1} ds \right| \leq \tag{14}$$

$$\frac{2N!}{R^N\left|\beta^2 - \alpha^2\right|}\left[\left\| g^{(5)} + \left(\alpha^2+\beta^2\right) g^{(3)} + \alpha^2\beta^2 g' \right\|_{\infty,[0,r_0]} \frac{r_0^{3+N}}{(3+N)!} \right.$$

$$\left. \left\| g^{(5)} + \left(\alpha^2+\beta^2\right) g^{(3)} + \alpha^2\beta^2 g' \right\|_{\infty,[r_0,R]} \sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}}{k!\,(N-k+3)!} R^k (R-r_0)^{N-k+3} \right]. \tag{15}$$

Proof. As in [2], we get that

$$|g(r) - g(r_0)| \leq \frac{1}{3\left|\beta^2-\alpha^2\right|}\left\| g^{(5)} + \left(\alpha^2+\beta^2\right) g^{(3)} + \alpha^2\beta^2 g' \right\|_{\infty,[r_0,R]} (r-r_0)^3$$

$$=: A(r-r_0)^3, \quad \forall\ r \in [r_0, R], \tag{16}$$

and

$$|g(r) - g(r_0)| \leq \frac{1}{3\left|\beta^2-\alpha^2\right|}\left\| g^{(5)} + \left(\alpha^2+\beta^2\right) g^{(3)} + \alpha^2\beta^2 g' \right\|_{\infty,[0,r_0]} (r_0-r)^3$$

$$=: B(r_0-r)^3, \quad \forall\ r \in [0, r_0]. \tag{17}$$

Next we observe

$$\left| f(r_0\omega) - \frac{\int_{B(0,R)} f(y)\,dy}{Vol(B(0,R))} \right| \leq$$

(as in the proof of Theorem 1)

$$\frac{N}{R^N}\left[\int_0^{r_0} s^{N-1}\left|g(r_0)-g(s)\right| ds + \int_{r_0}^R s^{N-1}\left|g(r_0)-g(s)\right| ds \right] \leq$$

$$\frac{N}{R^N}\left[B\int_0^{r_0} s^{N-1}(r_0-s)^3 ds + A\int_{r_0}^R s^{N-1}(s-r_0)^3 ds \right] = \tag{18}$$

(as earlier)

$$\frac{N}{R^N}\left[B\frac{3!\,(N-1)!}{(3+N)!} r_0^{3+N} + \right.$$

$$\left. A(-1)^{N-1}\sum_{k=0}^{N-1}(-1)^k R^k \binom{N-1}{k}\frac{(N-k-1)!3!}{(N-k+3)!}(R-r_0)^{N-k+3} \right] =$$

$$\frac{6N!}{R^N}\left[B\frac{r_0^{3+N}}{(3+N)!}+A\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}}{k!\,(N-k+3)!}R^k\,(R-r_0)^{N-k+3}\right].$$

■

It follows an L_1 Ostrowski inequality.

Theorem 6 *All as in Theorem 4, except now* $g\in C^2([0,R])$. *Then* ($\forall\ \omega\in S^{N-1}$)

$$\left|f(r_0\omega)-\frac{\int_{B(0,R)}f(y)\,dy}{Vol\,(B(0,R))}\right|=\left|g(r_0)-\frac{N}{R^N}\int_0^R g(s)\,s^{N-1}ds\right|\le$$

$$\frac{1}{R^N}\left[\|g''+g-g(r_0)\|_{L_1([0,r_0])}\frac{r_0^{N+1}}{(N+1)}+\right.$$

$$\left.\|g''+g-g(r_0)\|_{L_1([r_0,R])}\left[\left(\frac{N}{N+1}\right)\left(R^{N+1}-r_0^{N+1}\right)-r_0\left(R^N-r_0^N\right)\right]\right].\tag{19}$$

Proof. As in [2], we get that

$$|g(r)-g(r_0)|\le\|g''+g-g(r_0)\|_{L_1([0,r_0])}(r_0-r),\tag{20}$$

$\forall\ r\in[0,r_0]$,

and

$$|g(r)-g(r_0)|\le\|g''+g-g(r_0)\|_{L_1([r_0,R])}(r-r_0),\tag{21}$$

$\forall\ r\in[r_0,R]$.

Next we observe

$$\left|f(r_0\omega)-\frac{\int_{B(0,R)}f(y)\,dy}{Vol\,(B(0,R))}\right|=\left|g(r_0)-\frac{N}{R^N}\int_0^R g(s)\,s^{N-1}ds\right|\le$$

$$\frac{N}{R^N}\left[\int_0^{r_0}s^{N-1}|g(r_0)-g(s)|\,ds+\int_{r_0}^R s^{N-1}|g(r_0)-g(s)|\,ds\right]\le$$

$$\frac{N}{R^N}\left[\|g''+g-g(r_0)\|_{L_1([0,r_0])}\int_0^{r_0}s^{N-1}(r_0-s)^{2-1}\,ds+\right.\tag{22}$$

$$\left.\|g''+g-g(r_0)\|_{L_1([r_0,R])}\int_{r_0}^R s^{N-1}(s-r_0)\,ds\right]=$$

$$\frac{N}{R^N}\left[\|g''+g-g(r_0)\|_{L_1([0,r_0])}\frac{r_0^{N+1}}{N(N+1)}+\right.$$

$$\left.\|g''+g-g(r_0)\|_{L_1([r_0,R])}\left[\left(\frac{R^{N+1}-r_0^{N+1}}{N+1}\right)-r_0\left(\frac{R^N-r_0^N}{N}\right)\right]\right]=$$

$$\frac{1}{R^N}\left[\|g''+g-g(r_0)\|_{L_1([0,r_0])}\frac{r_0^{N+1}}{(N+1)}+\right.$$

$$\left.\|g''+g-g(r_0)\|_{L_1([r_0,R])}\left[\left(\frac{N}{N+1}\right)\left(R^{N+1}-r_0^{N+1}\right)-r_0\left(R^N-r_0^N\right)\right]\right]. \tag{23}$$

■

Next comes an Ostrowski multivariate radial inequality for $\|\cdot\|_p$, $p>1$.

Theorem 7 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Let all as in Theorem 6. Then ($\forall$ $\omega\in S^{N-1}$)*

$$\left|f(r_0\omega)-\frac{\int_{B(0,R)}f(y)\,dy}{Vol(B(0,R))}\right|=\left|g(r_0)-\frac{N}{R^N}\int_0^R g(s)\,s^{N-1}ds\right|\leq \tag{24}$$

$$\frac{N!\Gamma\left(2+\frac{1}{q}\right)}{R^N(q+1)^{\frac{1}{q}}}\left[\|g''+g-g(r_0)\|_{L_p([0,r_0])}\frac{r_0^{N+1+\frac{1}{q}}}{\Gamma\left(N+2+\frac{1}{q}\right)}+\right.$$

$$\left.\|g''+g-g(r_0)\|_{L_p([r_0,R])}\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R^k(R-r_0)^{N-k+1+\frac{1}{q}}}{k!\Gamma\left(N-k+2+\frac{1}{q}\right)}\right].$$

Proof. As in [2], we get that

$$|g(r)-g(r_0)|\leq\|g''+g-g(r_0)\|_{L_p([r_0,R])}\frac{(r-r_0)^{\frac{q+1}{1}}}{(q+1)^{\frac{1}{q}}}, \tag{25}$$

$\forall\ r\in[r_0,R]$,

and

$$|g(r)-g(r_0)|\leq\|g''+g-g(r_0)\|_{L_p([0,r_0])}\frac{(r_0-r)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}, \tag{26}$$

$\forall\ r\in[0,r_0]$.

Next we observe

$$\left|f(r_0\omega)-\frac{\int_{B(0,R)}f(y)\,dy}{Vol(B(0,R))}\right|=$$

(as earlier)

$$\left|g(r_0)-\frac{N}{R^N}\int_0^R g(s)\,s^{N-1}ds\right|\leq$$

$$\frac{N}{R^N}\left[\int_0^{r_0}s^{N-1}|g(r_0)-g(s)|\,ds+\int_{r_0}^R s^{N-1}|g(r_0)-g(s)|\,ds\right]\leq$$

$$\frac{N}{R^N(q+1)^{\frac{1}{q}}}\left[\|g''+g-g(r_0)\|_{L_p([0,r_0])}\int_0^{r_0} s^{N-1}(r_0-s)^{\left(2+\frac{1}{q}\right)-1}ds+\right. \tag{27}$$

$$\left.\|g''+g-g(r_0)\|_{L_p([r_0,R])}\int_{r_0}^{R} s^{N-1}(s-r_0)^{1+\frac{1}{q}}ds\right]=$$

$$\frac{N}{R^N(q+1)^{\frac{1}{q}}}\left[\|g''+g-g(r_0)\|_{L_p([0,r_0])}\frac{(N-1)!\Gamma\left(2+\frac{1}{q}\right)}{\Gamma\left(N+2+\frac{1}{q}\right)}r_0^{N+1+\frac{1}{q}}+\right.$$

$$\left.\|g''+g-g(r_0)\|_{L_p([r_0,R])}(-1)^{N-1}\int_{r_0}^{R}((R-s)-R)^{N-1}(s-r_0)^{1+\frac{1}{q}}ds\right]= \tag{28}$$

$$\frac{N}{R^N(q+1)^{\frac{1}{q}}}\left[\|g''+g-g(r_0)\|_{L_p([0,r_0])}\frac{(N-1)!\Gamma\left(2+\frac{1}{q}\right)}{\Gamma\left(N+2+\frac{1}{q}\right)}r_0^{N+1+\frac{1}{q}}+\right.$$

$$\|g''+g-g(r_0)\|_{L_p([r_0,R])}$$

$$\left.\sum_{k=0}^{N-1}\frac{(N-1)!}{k!(N-k-1)!}(-1)^{N+k-1}R^k\frac{(N-k-1)!\Gamma\left(2+\frac{1}{q}\right)}{\Gamma\left(N-k+2+\frac{1}{q}\right)}(R-r_0)^{N-k+1+\frac{1}{q}}\right]= \tag{29}$$

$$\frac{N!\Gamma\left(2+\frac{1}{q}\right)}{R^N(q+1)^{\frac{1}{q}}}\left[\|g''+g-g(r_0)\|_{L_p([0,r_0])}\frac{r_0^{N+1+\frac{1}{q}}}{\Gamma\left(N+2+\frac{1}{q}\right)}+\right.$$

$$\left.\|g''+g-g(r_0)\|_{L_p([r_0,R])}\sum_{k=0}^{N-1}(-1)^{N+k-1}R^k\frac{(R-r_0)^{N-k+1+\frac{1}{q}}}{k!\Gamma\left(N-k+2+\frac{1}{q}\right)}\right].$$

■

It follows a general L_1 estimate.

Theorem 8 *Let the function* $f:\overline{B(0,R)}\to\mathbb{R}$ *be radial, that is, there exists a function* g *such that* $f(x)=g(r)$, *where* $r=|x|$, $r\in[0,R]$, $\forall\ x\in\overline{B(0,R)}$. *We further assume that* $g\in C^4([0,R])$, *and* $g^{(k)}(r_0)=0$, $k=1,2,3,4$, *where* $r_0\in[0,R]$ *is fixed. Let also* $\alpha,\beta\in\mathbb{R}:\alpha\beta\left(\alpha^2-\beta^2\right)\neq0$. *Then* ($\forall\ \omega\in S^{N-1}$)

$$\left|f(r_0\omega)-\frac{\int_{B(0,R)}f(y)\,dy}{Vol(B(0,R))}\right|=\left|g(r_0)-\frac{N}{R^N}\int_0^R g(s)\,s^{N-1}ds\right|\leq$$

$$\frac{2}{|\beta^2-\alpha^2|R^N}\left[\left\|g''''+\left(\alpha^2+\beta^2\right)g''+\alpha^2\beta^2g-\alpha^2\beta^2g(r_0)\right\|_{L_1([0,r_0])}\frac{r_0^{N+1}}{(N+1)}+\right.$$

$$\left\|g''''+\left(\alpha^2+\beta^2\right)g''+\alpha^2\beta^2g-\alpha^2\beta^2g(r_0)\right\|_{L_1([r_0,R])}$$

$$\left.\left[\left(\frac{N}{N+1}\right)\left(R^{N+1}-r_0^{N+1}\right)-r_0\left(R^N-r_0^N\right)\right]\right]. \tag{30}$$

Proof. As in [2], we get that

$$|g(r)-g(r_0)| \leq$$

$$\frac{2}{|\beta^2-\alpha^2|}\left\|g^{''''}+\left(\alpha^2+\beta^2\right)g^{''}+\alpha^2\beta^2 g-\alpha^2\beta^2 g(r_0)\right\|_{L_1([0,r_0])}(r_0-r), \quad (31)$$

$\forall\ r\in[0,r_0]$,

and

$$|g(r)-g(r_0)| \leq$$

$$\frac{2}{|\beta^2-\alpha^2|}\left\|g^{''''}+\left(\alpha^2+\beta^2\right)g^{''}+\alpha^2\beta^2 g-\alpha^2\beta^2 g(r_0)\right\|_{L_1([r_0,R])}(r-r_0), \quad (32)$$

$\forall\ r\in[r_0,R]$.

The rest of the proof goes as in Theorem 6. ■

Next comes a general L_p, $p>1$, estimate.

Theorem 9 *All as in Theorem 8 and let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Then* ($\forall\ \omega\in S^{N-1}$)

$$\left|f(r_0\omega)-\frac{\int_{B(0,R)}f(y)\,dy}{Vol(B(0,R))}\right|=\left|g(r_0)-\frac{N}{R^N}\int_0^R g(s)\,s^{N-1}ds\right|\leq$$

$$\frac{2N!\Gamma\left(2+\frac{1}{q}\right)}{|\beta^2-\alpha^2|\,R^N(q+1)^{\frac{1}{q}}}$$

$$\left[\left\|g^{''''}+\left(\alpha^2+\beta^2\right)g^{''}+\alpha^2\beta^2 g-\alpha^2\beta^2 g(r_0)\right\|_{L_p([0,r_0])}\frac{r_0^{N+1+\frac{1}{q}}}{\Gamma\left(N+2+\frac{1}{q}\right)}+\right. \quad (33)$$

$$\left\|g^{''''}+\left(\alpha^2+\beta^2\right)g^{''}+\alpha^2\beta^2 g-\alpha^2\beta^2 g(r_0)\right\|_{L_p([r_0,R])}$$

$$\left.\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R^k(R-r_0)^{N-k+1+\frac{1}{q}}}{k!\Gamma\left(N-k+2+\frac{1}{q}\right)}\right].$$

Proof. As in [2], we have that

$$|g(r)-g(r_0)|\leq\frac{2}{|\beta^2-\alpha^2|}$$

$$\left\|g^{''''}+\left(\alpha^2+\beta^2\right)g^{''}+\alpha^2\beta^2 g-\alpha^2\beta^2 g(r_0)\right\|_{L_p([0,r_0])}\frac{(r_0-r)^{\frac{q+1}{1}}}{(q+1)^{\frac{1}{q}}}, \quad (34)$$

$\forall\ r\in[0,r_0]$,

and

$$|g(r) - g(r_0)| \leq \frac{2}{|\beta^2 - \alpha^2|}$$

$$\left\| g^{''''} + \left(\alpha^2 + \beta^2\right) g^{''} + \alpha^2 \beta^2 g - \alpha^2 \beta^2 g(r_0) \right\|_{L_p([r_0, R])} \frac{(r - r_0)^{\frac{q+1}{1}}}{(q+1)^{\frac{1}{q}}}, \qquad (35)$$

$\forall\ r \in [r_0, R]$.

The rest of the proof goes as in Theorem 7. ■

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