NUMERICAL RADIUS AND p-SCHATTEN NORM INEQUALITIES FOR POWER SERIES OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. Assume that the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on the open disk $D(0,R)$, $f_a(z) := \sum_{k=0}^{\infty} |a_k| z^k$ that has the same radius of convergence R and A, B, $C \in B(H)$ with $\|A\| < R$, then we have the following Schwarz type inequality

$$|\langle C^* Af(A) Bx, y\rangle| \leq f_a(\|A\|) \left\langle \left| |A|^{\alpha} B \right|^2 x, x \right\rangle^{1/2} \left\langle \left| |A^*|^{1-\alpha} C \right|^2 y, y \right\rangle^{1/2}$$

for $\alpha \in [0,1]$ and x, $y \in H$. Some natural applications for *numerical radius* and p-*Schatten norm* are also provided.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leq w(T) \|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T + V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{1.3}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [12], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.4}$$

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Utilizing the Cartesian decomposition for operators, F. Kittaneh in [13] improved the inequality (1.3) as follows:

$$\frac{1}{4}\left\|T^{*}T+TT^{*}\right\|\leq\omega^{2}\left(T\right)\leq\frac{1}{2}\left\|T^{*}T+TT^{*}\right\| \tag{1.5}$$

for any operator $T\in B\left(H\right)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [9]:

If for an operator $T\in B\left(H\right)$ we denote $\left|T\right|:=\left(T^{*}T\right)^{1/2}$, then

$$\omega^{r}\left(T\right)\leq\frac{1}{2}\left\|\left|T\right|^{2\alpha r}+\left|T^{*}\right|^{2(1-\alpha)r}\right\| \tag{1.6}$$

and

$$\omega^{2r}\left(T\right)\leq\left\|\alpha\left|T\right|^{2r}+\left(1-\alpha\right)\left|T^{*}\right|^{2r}\right\|, \tag{1.7}$$

where $\alpha\in(0,1)$ and $r\geq1$.

If we take $\alpha=\frac{1}{2}$ and $r=1$ we get from (1.6) that

$$\omega\left(T\right)\leq\frac{1}{2}\left\|\left|T\right|+\left|T^{*}\right|\right\| \tag{1.8}$$

and from (1.7) that

$$\omega^{2}\left(T\right)\leq\frac{1}{2}\left\|\left|T\right|^{2}+\left|T^{*}\right|^{2}\right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [7] and [4].

Let $(H;\langle.,.\rangle)$ be a complex Hilbert space and $\mathcal{B}\left(H\right)$ the Banach algebra of all bounded linear operators on H. If $\left\{e_{i}\right\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}\left(H\right)$ is of *trace class* if

$$\left\|A\right\|_{1}:=\sum_{i\in I}\left\langle\left|A\right|e_{i},e_{i}\right\rangle<\infty. \tag{1.10}$$

The definition of $\left\|A\right\|_{1}$ does not depend on the choice of the orthonormal basis $\left\{e_{i}\right\}_{i\in I}$. We denote by $\mathcal{B}_{1}\left(H\right)$ the set of trace class operators in $\mathcal{B}\left(H\right)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_{1}\left(H\right)$ to be

$$\operatorname{tr}\left(A\right):=\sum_{i\in I}\left\langle Ae_{i},e_{i}\right\rangle, \tag{1.11}$$

where $\left\{e_{i}\right\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.11) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1. *We have:*

(i) If $A\in\mathcal{B}_{1}\left(H\right)$ *then* $A^{*}\in\mathcal{B}_{1}\left(H\right)$ *and*

$$\operatorname{tr}\left(A^{*}\right)=\overline{\operatorname{tr}\left(A\right)}; \tag{1.12}$$

(ii) If $A\in\mathcal{B}_{1}\left(H\right)$ *and* $T\in\mathcal{B}\left(H\right)$, *then* $AT,\ TA\in\mathcal{B}_{1}\left(H\right)$ *and*

$$\operatorname{tr}\left(AT\right)=\operatorname{tr}\left(TA\right)\ \text{ and }\ \left|\operatorname{tr}\left(AT\right)\right|\leq\left\|A\right\|_{1}\left\|T\right\|; \tag{1.13}$$

(iii) $\operatorname{tr}\left(\cdot\right)$ *is a bounded linear functional on* $\mathcal{B}_{1}\left(H\right)$ *with* $\left\|\operatorname{tr}\right\|=1$;

(iv) If $A,\ B\in\mathcal{B}_{2}\left(H\right)$ *then* $AB,\ BA\in\mathcal{B}_{1}\left(H\right)$ *and* $\operatorname{tr}\left(AB\right)=\operatorname{tr}\left(BA\right)$;

(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

For a large number of results concerning trace inequalities, see the recent survey paper [8].

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite [17, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} = \left(\sum_{i \in I} \langle |A|^p e_i, e_i \rangle \right)^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.14}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{1.15}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [17, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{1.16}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{1.17}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \ \|BA\|_p \leq \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{1.18}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{1.19}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{1.20}$$

For the theory of trace functionals and their applications the reader is referred to [15] and [17].

For $\mathcal{E} := \{e_i\}_{i \in I}$ an orthonormal basis of H we define for $A \in \mathcal{B}_p(H)$, $p \geq 1$

$$\|A\|_{\mathcal{E},p} := \left(\sum_{i \in I} |\langle Ae_i, e_i \rangle|^p \right)^{1/p}.$$

We observe that $\|\cdot\|_{\mathcal{E},p}$ is a norm on $\mathcal{B}_p(H)$ and

$$\|A\|_{\mathcal{E},p} \leq \|A\|_p \text{ for } A \in \mathcal{B}_p(H).$$

Further, we can take the supremum over all orthonormal basis in H we can also define, for $A \in \mathcal{B}_p(H)$, that

$$\omega_p(A) := \sup_{\mathcal{E}} \|A\|_{\mathcal{E},p} \leq \|A\|_p,$$

which is a *norm* on $\mathcal{B}_p(H)$.

It is also known that, if $\mathcal{E}=\{e_i\}_{i\in I}$ and $\mathcal{F}=\{f_i\}_{i\in I}$ are orthonormal basis, then [16]

$$\sup_{\mathcal{E},\mathcal{F}}\sum_{i\in I}\left|\langle Te_i,f_i\rangle\right|^s=\|T\|_s^s \text{ for } s\geq 1. \tag{1.21}$$

2. Vector Inequalities

In 1988 F. Kittaneh [11, Corollary 7] obtained the following Schwarz type inequality for powers of operators:

Lemma 1. *Let $A\in B(H)$ and $\alpha\in[0,1]$. Then for $n\geq 1$ we have*

$$\left|\langle A^n x,y\rangle\right|^2\leq\|A\|^{2n-2}\left\langle |A|^{2\alpha}x,x\right\rangle\left\langle |A^*|^{2(1-\alpha)}y,y\right\rangle \tag{2.1}$$

for all x, $y\in H$.

We can state the following result as well:

Corollary 1. *Let A, B, $C\in B(H)$ and $\alpha\in[0,1]$. Then for $n\geq 1$ we have*

$$\left|\langle C^*A^nBx,y\rangle\right|^2\leq\|A\|^{2n-2}\left\langle \left||A|^{\alpha}B\right|^2x,x\right\rangle\left\langle \left||A^*|^{1-\alpha}C\right|^2y,y\right\rangle \tag{2.2}$$

for all x, $y\in H$.

Proof. If we replace x by Bx and y by Cy in (2.1), then we get

$$\left|\langle C^*A^nBx,y\rangle\right|^2\leq\|A\|^{2n-2}\left\langle B^*|A|^{2\alpha}Bx,x\right\rangle\left\langle C^*|A^*|^{2(1-\alpha)}Cy,y\right\rangle. \tag{2.3}$$

Observe that $B^*|A|^{2\alpha}B=\left||A|^{\alpha}B\right|^2$ and $C^*|A^*|^{2(1-\alpha)}C=\left||A^*|^{1-\alpha}C\right|^2$, then by (2.3) we get (2.2). □

We consider the power series with complex coefficients $f(z):=\sum_{k=0}^{\infty}a_kz^k$ with $a_k\in\mathbb{C}$ for $k\in\mathbb{N}:=\{0,1,...\}$. We assume that this power series in convergent on the open disk $D(0,R):=\{z\in\mathbb{C}|\ z<R\}$. If $R=\infty$ then $D(0,R)=\mathbb{C}$. We define $f_a(z):=\sum_{k=0}^{\infty}|a_k|z^k$ which has the same radius of convergence R.

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1); \tag{2.4}$$
$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C};$$
$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C};$$
$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n}\lambda^n = \ln \frac{1}{1-\lambda},\ \lambda \in D(0,1); \tag{2.5}$$

$$g_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\lambda^{2n} = \cosh \lambda,\ \lambda \in \mathbb{C};$$

$$h_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\lambda^{2n+1} = \sinh \lambda,\ \lambda \in \mathbb{C};$$

$$l_a(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda},\ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \qquad \lambda \in \mathbb{C}, \tag{2.6}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

The following result is of interest:

Theorem 2. *Assume that the power series with complex coefficients* $f(z) := \sum_{k=0}^{\infty} a_k z^k$ *is convergent on* $D(0,R)$ *and* A, B, $C \in B(H)$ *with* $\|A\| < R$, *then*

$$\left|\langle C^* A f(A) Bx, y\rangle\right|^2 \le f_a^2(\|A\|) \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{2.7}$$

for $\alpha \in [0,1]$ *and* x, $y \in H$.

In particular,

$$\left|\langle C^* A f(A) Bx, y\rangle\right|^2 \le f_a^2(\|A\|) \left\langle \left||A|^{1/2} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1/2} C\right|^2 y, y\right\rangle \tag{2.8}$$

for x, $y \in H$.

Proof. If we take $n = k+1$, $k \in \mathbb{N}$ in (2.2) and take the square root, then we get

$$\left|\left\langle C^* A A^k Bx, y\right\rangle\right| \le \|A\|^k \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle^{1/2}$$

for all x, $y \in H$.

Further, if we multiply by $|a_k| \geq 0$, $k \in \{0, 1, ...\}$ and sum over k from 0 to m, then we get

$$\left|\left\langle C^* A \sum_{k=0}^{m} a_k A^k Bx, y\right\rangle\right| \quad (2.9)$$
$$= \left|\sum_{k=0}^{m} a_k \left\langle C^* A A^k Bx, y\right\rangle\right| \leq \sum_{k=0}^{m} |a_k| \left|\left\langle C^* A A^k Bx, y\right\rangle\right|$$
$$\leq \sum_{k=0}^{m} |a_k| \|A\|^k \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle^{1/2}$$

for all x, $y \in H$.

Since $\|A\| < R$ then series $\sum_{k=0}^{\infty} a_k A^k$ and $\sum_{k=0}^{\infty} |a_k| \|A\|^k$ are convergent and

$$\sum_{k=0}^{\infty} a_k A^k = f(A) \text{ and } \sum_{k=0}^{\infty} |a_k| \|A\|^k = f_a(\|A\|).$$

By taking now the limit over $m \to \infty$ in (2.9) we deduce the desired result (2.7). □

Remark 1. *If A, B, $C \in B(H)$ with $\|A\| < 1$, then for $\alpha \in [0, 1]$*

$$\left|\left\langle C^* A (I \pm A)^{-1} Bx, y\right\rangle\right|^2 \quad (2.10)$$
$$\leq (1 - \|A\|)^{-2} \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle$$

and

$$|\langle C^* A \ln (I \pm A) Bx, y\rangle|^2 \quad (2.11)$$
$$\leq [\ln (1 - \|A\|)]^2 \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle$$

for all x, $y \in H$.

For $\alpha = 1/2$ in (2.10) and (2.11) we obtain

$$\left|\left\langle C^* A (I \pm A)^{-1} Bx, y\right\rangle\right|^2 \leq (1 - \|A\|)^{-2} \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \quad (2.12)$$

and

$$|\langle C^* A \ln (I \pm A) Bx, y\rangle|^2 \leq [\ln (1 - \|A\|)]^2 \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \quad (2.13)$$

for all x, $y \in H$.

If A, B, $C \in B(H)$ and $\alpha \in [0, 1]$, then

$$|\langle C^* A \sin (A) Bx, y\rangle|^2 \leq [\sinh (\|A\|)]^2 \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \quad (2.14)$$

and

$$|\langle C^* A \cos (A) Bx, y\rangle|^2 \leq [\cosh (\|A\|)]^2 \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \quad (2.15)$$

for all x, $y \in H$.

For $\alpha = 1/2$ in (2.14) and (2.15) we obtain

$$|\langle C^* A \sin (A) Bx, y\rangle|^2 \leq [\sinh (\|A\|)]^2 \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \quad (2.16)$$

and

$$|\langle C^* A\cos(A) Bx, y\rangle|^2 \le [\cosh(\|A\|)]^2 \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \tag{2.17}$$

for all x, $y \in H$.

Also, if A, B, $C \in B(H)$ *and* $\alpha \in [0,1]$, *then*

$$|\langle C^* A\exp(A) Bx, y\rangle|^2 \le \exp(2\|A\|) \left\langle |A|^\alpha B|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle, \tag{2.18}$$

$$\begin{aligned} &|\langle C^* A\sinh(A) Bx, y\rangle|^2 \\ &\le [\sinh(\|A\|)]^2 \left\langle ||A|^\alpha B|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &|\langle C^* A\cosh(A) Bx, y\rangle|^2 \\ &\le [\cosh(\|A\|)]^2 \left\langle ||A|^\alpha B|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \end{aligned} \tag{2.20}$$

for all x, $y \in H$.

For $\alpha = 1/2$ *in (2.18)-(2.20) we obtain some simpler inequalities. We omit the details.*

3. Norm and Numerical Radius Inequalities

The following vector inequality for positive operators $A \ge 0$, obtained by C. A. McCarthy in [14] is well known,

$$\langle Ax, x\rangle^p \le \langle A^p x, x\rangle, \; p \ge 1$$

for $x \in H$, $\|x\| = 1$.

Buzano's inequality [5],

$$\frac{1}{2}[\|x\|\|y\| + |\langle x, y\rangle|] \ge |\langle x, e\rangle \langle e, y\rangle| \tag{3.1}$$

that holds for any x, y, $e \in H$ with $\|e\| = 1$ will also be used in the sequel.

Our first main result is as follows:

Theorem 3. *Assume that the power series with complex coefficients* $f(z) := \sum_{k=0}^{\infty} a_k z^k$ *is convergent on* $D(0,R)$, $\alpha \in [0,1]$ *and* A, B, $C \in B(H)$ *with* $\|A\| < R$, *then we have the norm inequality*

$$\|C^* Af(A) B\| \le f_a(\|A\|) \||A|^\alpha B\| \left\||A^*|^{1-\alpha} C\right\|. \tag{3.2}$$

We also have the numerical radius inequalities

$$\omega(C^* Af(A) B) \le \frac{1}{2} f_a(\|A\|) \left\| ||A|^\alpha B|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right\| \tag{3.3}$$

and

$$\begin{aligned} &\omega^2(C^* Af(A) B) \\ &\le \frac{1}{2} f_a^2(\|A\|) \left[\||A|^\alpha B\|^2 \left\||A^*|^{1-\alpha} C\right\|^2 + \omega\left(\left||A^*|^{1-\alpha} C\right|^2 ||A|^\alpha B|^2 \right) \right]. \end{aligned} \tag{3.4}$$

Proof. We have from (2.7), by taking the supremum over $\|x\| = \|y\| = 1$, that

$$\begin{aligned}\|C^* Af(A) B\|^2 &= \sup_{\|x\|=\|y\|=1} |\langle C^* Af(A) Bx, y\rangle|^2 \\ &\le f_a^2(\|A\|) \sup_{\|x\|=1} \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \sup_{\|y\|=1} \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \\ &= f_a^2(\|A\|) \left\| \left||A|^\alpha B\right|^2 \right\| \left\| \left||A^*|^{1-\alpha} C\right|^2 \right\| \\ &= f_a^2(\|A\|) \left\||A|^\alpha B\right\|^2 \left\||A^*|^{1-\alpha} C\right\|^2,\end{aligned}$$

which gives (3.2).

From (2.7) we get, by taking $y = x$, the square root and using the *A-G-mean inequality*, that

$$\begin{aligned}(3.5)\qquad &|\langle C^* Af(A) Bx, x\rangle| \\ &\le f_a(\|A\|) \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} C\right|^2 x, x\right\rangle^{1/2} \\ &\le \frac{1}{2} f_a(\|A\|) \left(\left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle + \left\langle \left||A^*|^{1-\alpha} C\right|^2 x, x\right\rangle \right) \\ &= \frac{1}{2} f_a(\|A\|) \left\langle \left(\left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right) x, x\right\rangle\end{aligned}$$

for all $x \in H$.

By taking the supremum over $\|x\| = 1$ in (3.5) we get that

$$\begin{aligned}&\omega(C^* Af(A) B) \\ &= \sup_{\|x\|=1} |\langle C^* Af(A) Bx, x\rangle| \\ &\le \frac{1}{2} f_a(\|A\|) \sup_{\|x\|=1} \left\langle \left(\left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right) x, x\right\rangle \\ &= \frac{1}{2} f_a(\|A\|) \left\| \left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right\|,\end{aligned}$$

which proves (3.3).

From (2.7) for $y = x$ and Buzano's inequality we derive that

$$\begin{aligned}(3.6)\qquad &|\langle C^* Af(A) Bx, x\rangle|^2 \\ &\le f_a^2(\|A\|) \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle \\ &\le \frac{1}{2} f_a^2(\|A\|) \\ &\times \left[\left\| \left||A|^\alpha B\right|^2 x \right\| \left\| \left||A^*|^{1-\alpha} C\right|^2 x \right\| + \left| \left\langle \left||A|^\alpha B\right|^2 x, \left||A^*|^{1-\alpha} C\right|^2 x \right\rangle \right| \right] \\ &= \frac{1}{2} f_a^2(\|A\|) \\ &\times \left[\left\| \left||A|^\alpha B\right|^2 x \right\| \left\| \left||A^*|^{1-\alpha} C\right|^2 x \right\| + \left| \left\langle \left||A^*|^{1-\alpha} C\right|^2 \left||A|^\alpha B\right|^2 x, x \right\rangle \right| \right]\end{aligned}$$

for all $x \in H$.

By taking the supremum over $\|x\|=1$ in (3.6) we get that

$$\begin{aligned}
&\omega^2\left(C^*Af\left(A\right)B\right)\\
&=\sup_{\|x\|=1}\left|\left\langle C^*Af\left(A\right)Bx,x\right\rangle\right|^2\\
&\leq\frac{1}{2}f_a^2\left(\|A\|\right)\\
&\times\sup_{\|x\|=1}\left[\left\|\left|\left|A\right|^\alpha B\right|^2 x\right\|\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2 x\right\|+\left|\left\langle\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^\alpha B\right|^2 x,x\right\rangle\right|\right]\\
&\leq\frac{1}{2}f_a^2\left(\|A\|\right)\\
&\times\left[\sup_{\|x\|=1}\left\{\left\|\left|\left|A\right|^\alpha B\right|^2 x\right\|\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2 x\right\|\right\}\right.\\
&\left.+\sup_{\|x\|=1}\left|\left\langle\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^\alpha B\right|^2 x,x\right\rangle\right|\right]\\
&\leq\frac{1}{2}f_a^2\left(\|A\|\right)\\
&\times\left[\sup_{\|x\|=1}\left\|\left|\left|A\right|^\alpha B\right|^2 x\right\|\sup_{\|x\|=1}\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2 x\right\|\right.\\
&\left.+\sup_{\|x\|=1}\left|\left\langle\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^\alpha B\right|^2 x,x\right\rangle\right|\right]\\
&=\frac{1}{2}f_a^2\left(\|A\|\right)\left[\left\|\left|\left|A\right|^\alpha B\right|^2\right\|\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2\right\|+\omega\left(\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^\alpha B\right|^2\right)\right]\\
&=\frac{1}{2}f_a^2\left(\|A\|\right)\left[\left\|\left|A\right|^\alpha B\right\|^2\left\|\left|A^*\right|^{1-\alpha}C\right\|^2+\omega\left(\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^\alpha B\right|^2\right)\right],
\end{aligned}$$

which proves (3.4). □

Remark 2. *If we take $\alpha=1/2$ in Theorem 3, then we get the norm inequality*

$$\left\|C^*Af\left(A\right)B\right\|\leq f_a\left(\|A\|\right)\left\|\left|A\right|^{1/2}B\right\|\left\|\left|A^*\right|^{1/2}C\right\| \tag{3.7}$$

and the numerical radius inequalities

$$\omega\left(C^*Af\left(A\right)B\right)\leq\frac{1}{2}f_a\left(\|A\|\right)\left\|\left|\left|A\right|^{1/2}B\right|^2+\left|\left|A^*\right|^{1/2}C\right|^2\right\| \tag{3.8}$$

and

$$\begin{aligned}
&\omega^2\left(C^*Af\left(A\right)B\right)\\
&\leq\frac{1}{2}f_a^2\left(\|A\|\right)\left[\left\|\left|A\right|^{1/2}B\right\|^2\left\|\left|A^*\right|^{1/2}C\right\|^2+\omega\left(\left|\left|A^*\right|^{1/2}C\right|^2\left|\left|A\right|^{1/2}B\right|^2\right)\right].
\end{aligned} \tag{3.9}$$

The second main result is as follows:

Theorem 4. *Assume that the conditions of Theorem 3 are satisfied. If* $\alpha \in [0,1]$, $r>0$, p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr \geq 1$, *then*

$$\omega^{2r}\left(C^{*}Af\left(A\right)B\right) \leq f_{a}^{2r}\left(\|A\|\right)\left\|\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}\right\|. \tag{3.10}$$

If $r\geq 1$, *then*

$$\begin{aligned}\omega^{2r}\left(C^{*}Af\left(A\right)B\right) \leq \frac{1}{2}f_{a}^{2r}\left(\|A\|\right)\Big[&\left\|\left|A\right|^{\alpha}B\right\|^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2r}\\ &+\omega^{r}\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\Big].\end{aligned} \tag{3.11}$$

If $r\geq 1$, p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* pr, $qr\geq 2$, *then also*

$$\begin{aligned}\omega^{2r}\left(C^{*}Af\left(A\right)B\right) \leq \frac{1}{2}f_{a}^{2r}\left(\|A\|\right)\Bigg(&\left\|\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right\|\\ &+\omega^{r}\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\Bigg).\end{aligned} \tag{3.12}$$

Proof. If we take the power $r>0$ in (2.7) written for $y=x$ then we get, by Young and McCarthy inequalities that

$$\begin{aligned}&\left|\left\langle C^{*}Af\left(A\right)Bx,x\right\rangle\right|^{2r}\\ &\leq f_{a}^{2r}\left(\|A\|\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{r}\\ &\leq f_{a}^{2r}\left(\|A\|\right)\left[\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{rp}+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{rq}\right]\\ &\leq f_{a}^{2r}\left(\|A\|\right)\left[\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2rp}x,x\right\rangle+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}x,x\right\rangle\right]\\ &= f_{a}^{2r}\left(\|A\|\right)\left[\left\langle\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}x,x\right\rangle\right]\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, then we get that

$$\begin{aligned}&\omega^{2r}\left(C^{*}Af\left(A\right)B\right)\\ &=\sup_{\|x\|=1}\left|\left\langle C^{*}Af\left(A\right)Bx,x\right\rangle\right|^{2r}\\ &\leq f_{a}^{2r}\left(\|A\|\right)\sup_{\|x\|=1}\left[\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}\right)x,x\right\rangle\right]\\ &= f_{a}^{2r}\left(\|A\|\right)\left\|\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}\right\|,\end{aligned}$$

which proves (3.10).

If we take the power $r \geq 1$ in (3.6) and by using the convexity of the power function, we get

$$
\begin{aligned}
(3.13)\quad & |\langle C^* A f(A) Bx, x\rangle|^{2r} \\
&= f_a^{2r}(\|A\|) \\
&\times \left[\frac{\left\| |\,|A|^\alpha B|^2 x\right\| \left\| \left||A^*|^{1-\alpha} C\right|^2 x\right\| + \left|\left\langle \left||A^*|^{1-\alpha} C\right|^2 |\,|A|^\alpha B|^2 x, x\right\rangle\right|}{2}\right]^r \\
&\leq f_a^{2r}(\|A\|) \\
&\times \frac{\left\| |\,|A|^\alpha B|^2 x\right\|^r \left\| \left||A^*|^{1-\alpha} C\right|^2 x\right\|^r + \left|\left\langle \left||A^*|^{1-\alpha} C\right|^2 |\,|A|^\alpha B|^2 x, x\right\rangle\right|^r}{2}
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$
\begin{aligned}
& \omega^{2r}(C^* A f(A) B) \\
&\leq f_a^{2r}(\|A\|) \\
&\times \frac{\left\| |\,|A|^\alpha B|^2\right\|^r \left\| \left||A^*|^{1-\alpha} C\right|^2\right\|^r + \omega^r\left(\left||A^*|^{1-\alpha} C\right|^2 |\,|A|^\alpha B|^2\right)}{2} \\
&= f_a^{2r}(\|A\|) \\
&\times \frac{\|\,|A|^\alpha B\|^{2r} \left\| |A^*|^{1-\alpha} C\right\|^{2r} + \omega^r\left(\left||A^*|^{1-\alpha} C\right|^2 |\,|A|^\alpha B|^2\right)}{2},
\end{aligned}
$$

which proves (3.11).

Also, observe that

$$
\begin{aligned}
& \left\| |\,|A|^\alpha B|^2 x\right\|^r \left\| \left||A^*|^{1-\alpha} C\right|^2 x\right\|^r \\
&\leq \frac{1}{p}\left\| |\,|A|^\alpha B|^2 x\right\|^{pr} + \frac{1}{q}\left\| \left||A^*|^{1-\alpha} C\right|^2 x\right\|^{qr} \\
&= \frac{1}{p}\left\| |\,|A|^\alpha B|^2 x\right\|^{2\frac{pr}{2}} + \frac{1}{q}\left\| \left||A^*|^{1-\alpha} C\right|^2 x\right\|^{2\frac{qr}{2}} \\
&= \frac{1}{p}\left\langle |\,|A|^\alpha B|^4 x, x\right\rangle^{\frac{pr}{2}} + \frac{1}{q}\left\langle \left||A^*|^{1-\alpha} C\right|^4 x, x\right\rangle^{\frac{qr}{2}} \\
&\leq \frac{1}{p}\left\langle |\,|A|^\alpha B|^{2pr} x, x\right\rangle + \frac{1}{q}\left\langle \left||A^*|^{1-\alpha} C\right|^{2qr} x, x\right\rangle \\
&= \left\langle \left(\frac{1}{p}|\,|A|^\alpha B|^{2pr} + \frac{1}{q}\left||A^*|^{1-\alpha} C\right|^{2qr}\right) x, x\right\rangle,
\end{aligned}
$$

for $x\in H$ with $\|x\|=1$. Then

$$\frac{\left\| \left| |A|^{\alpha} B \right|^{2} x \right\|^{r} \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\|^{r} + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right|^{r}}{2}$$
$$\leq \frac{1}{2} \left[\left\langle \left(\frac{1}{p} \left| |A|^{\alpha} B \right|^{2pr} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} C \right|^{2qr} \right) x, x \right\rangle + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right|^{r} \right]$$

and by (3.13)

$$\left| \left\langle C^{*} A f(A) Bx, x \right\rangle \right|^{2r} \leq \frac{1}{2} f_{a}^{2r}(\|A\|) \left[\left\langle \left(\frac{1}{p} \left| |A|^{\alpha} B \right|^{2pr} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} C \right|^{2qr} \right) x, x \right\rangle + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right|^{r} \right]$$

for $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, we derive (3.12). □

Remark 3. *If we take $r=1$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ in (3.10), then we obtain*

$$\omega^{2}\left(C^{*} A f(A) B\right) \leq f_{a}^{2}(\|A\|) \left\| \frac{1}{p} \left| |A|^{\alpha} B \right|^{2p} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} C \right|^{2q} \right\|, \tag{3.14}$$

which for $p=q=2$ gives

$$\omega^{2}\left(C^{*} A f(A) B\right) \leq \frac{1}{2} f_{a}^{2}(\|A\|) \left\| \left| |A|^{\alpha} B \right|^{4} + \left| |A^{*}|^{1-\alpha} C \right|^{4} \right\|. \tag{3.15}$$

If we take $r=1$ and $p=q=2$ in (3.12), then we get

$$\omega^{2}\left(C^{*} A f(A) B\right) \leq \frac{1}{2} f_{a}^{2}(\|A\|) \left(\frac{1}{2} \left\| \left| |A|^{\alpha} B \right|^{4} + \left| |A^{*}|^{1-\alpha} C \right|^{4} \right\| + \omega\left(\left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} \right) \right). \tag{3.16}$$

If we take $r=2$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ in (3.12), then we get

$$\omega^{4}\left(C^{*} A f(A) B\right) \leq \frac{1}{2} f_{a}^{4}(\|A\|) \left(\left\| \frac{1}{p} \left| |A|^{\alpha} B \right|^{4p} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} C \right|^{4q} \right\| + \omega^{2}\left(\left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} \right) \right). \tag{3.17}$$

We also have:

Theorem 5. *With the assumptions of Theorem 3, we have for $r\geq 1$, $\lambda\in[0,1]$ that*

$$\omega^{2}\left(C^{*} A f(A) B\right) \leq f_{a}^{2}(\|A\|) \left\| (1-\lambda) \left| |A|^{\alpha} B \right|^{2r} + \lambda \left| |A^{*}|^{1-\alpha} C \right|^{2r} \right\|^{1/r} \times \left\| |A|^{\alpha} B \right\|^{2\lambda} \left\| |A^{*}|^{1-\alpha} C \right\|^{2(1-\lambda)} \tag{3.18}$$

for all $\alpha\in[0,1]$.

Also, we have

$$\omega^2\left(C^*Af\left(A\right)B\right)\leq f_a^2\left(\|A\|\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r} \times\left\|\lambda\left|\left|A\right|^{\alpha}B\right|^{2r}+\left(1-\lambda\right)\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r} \tag{3.19}$$

for all $\alpha\in[0,1]$ *and* $r\geq 1$.

Proof. From the first part of (3.6) we have

$$\begin{aligned}
&\left|\left\langle C^*Af\left(A\right)Bx,x\right\rangle\right|^2\\
&\leq f_a^2\left(\|A\|\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle\\
&=f_a^2\left(\|A\|\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle^{1-\lambda}\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^{\lambda}\\
&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle^{\alpha}\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^{1-\lambda}\\
&\leq f_a^2\left(\|A\|\right)\left[\left(1-\lambda\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle+\lambda\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle\right]\\
&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle^{\lambda}\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^{1-\lambda}
\end{aligned}$$

for all $x\in H$, $\|x\|=1$.

If we take the power $r\geq 1$, then we get by the convexity of power r that

$$\begin{aligned}
&\left|\left\langle C^*Af\left(A\right)Bx,x\right\rangle\right|^{2r}\\
&\leq f_a^{2r}\left(\|A\|\right)\left[\left(1-\lambda\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle+\lambda\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle\right]^r\\
&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle^{r\lambda}\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^{r(1-\lambda)}\\
&\leq f_a^{2r}\left(\|A\|\right)\left[\left(1-\lambda\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle^r+\lambda\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^r\right]\\
&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle^{r\lambda}\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^{r(1-\lambda)}
\end{aligned} \tag{3.20}$$

for all $x\in H$, $\|x\|=1$.

If we use McCarthy inequality for power $r\geq 1$, then we get

$$\begin{aligned}
&\left(1-\lambda\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle^r+\lambda\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^r\\
&\leq\left(1-\lambda\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2r}x,x\right\rangle+\lambda\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}x\right\rangle\\
&=\left\langle\left[\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle
\end{aligned}$$

and by (3.20)

$$|\langle C^{*}Af(A)Bx,x\rangle|^{2r} \le f_a^{2r}(\|A\|)\left[\left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right] \times\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{r\lambda}\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle^{r(1-\lambda)} \tag{3.21}$$

for all $x\in H$, $\|x\|=1$.

If we take the supremum over $\|x\|=1$, then we get

$$\begin{aligned}&\omega^{2r}(C^{*}Af(A)B)\\&=\sup_{\|x\|=1}|\langle C^{*}Af(A)Bx,x\rangle|^{2r}\\&\le f_a^{2r}(\|A\|)\sup_{\|x\|=1}\left[\left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\\&\times\sup_{\|x\|=1}\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{r\lambda}\sup_{\|x\|=1}\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle^{r(1-\lambda)}\\&=f_a^{2r}(\|A\|)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right\|\\&\times\left\||A|^{\alpha}B\right\|^{2r\lambda}\left\||A^{*}|^{1-\alpha}C\right\|^{2r(1-\lambda)},\end{aligned}$$

which gives (3.18).

We also have

$$\begin{aligned}&|\langle C^{*}Af(A)Bx,x\rangle|^{2r}\\&\le f_a^{2r}(\|A\|)\left[\left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\\&\times\left[\left\langle\left[\lambda\left||A|^{\alpha}B\right|^{2r}+(1-\lambda)\left||A^{*}|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\end{aligned}$$

for all $x\in H$, $\|x\|=1$, which proves (3.19). □

Remark 4. *If we take $r=1$ in Theorem 5, then we get*

$$\omega^{2}(C^{*}Af(A)B)\le f_a^{2}(\|A\|)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right\| \times\left\||A|^{\alpha}B\right\|^{2\lambda}\left\||A^{*}|^{1-\alpha}C\right\|^{2(1-\lambda)} \tag{3.22}$$

and

$$\omega^{2}(C^{*}Af(A)B)\le f_a^{2}(\|A\|)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2}\right\| \times\left\|\lambda\left||A|^{\alpha}B\right|^{2}+(1-\lambda)\left||A^{*}|^{1-\alpha}C\right|^{2}\right\| \tag{3.23}$$

for all α, $\lambda\in[0,1]$.

If we take $\lambda = 1/2$ in (3.22), then we obtain

$$\omega^2\left(C^*Af\left(A\right)B\right) \leq \frac{1}{2}f_a^2\left(\|A\|\right)\left\|\left|\left|A\right|^\alpha B\right|^2 + \left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\| \left\|\left|A\right|^\alpha B\right\| \left\|\left|A^*\right|^{1-\alpha}C\right\| \tag{3.24}$$

If we take $r = 2$ in Theorem 5, then we get

$$\omega^2\left(C^*Af\left(A\right)B\right) \leq f_a^2\left(\|A\|\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^\alpha B\right|^4 + \lambda\left|\left|A^*\right|^{1-\alpha}C\right|^4\right\|^{1/2} \times \left\|\left|A\right|^\alpha B\right\|^{2\lambda}\left\|\left|A^*\right|^{1-\alpha}C\right\|^{2(1-\lambda)} \tag{3.25}$$

and

$$\omega^2\left(C^*Af\left(A\right)B\right) \leq f_a^2\left(\|A\|\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^\alpha B\right|^4 + \lambda\left|\left|A^*\right|^{1-\alpha}C\right|^4\right\|^{1/2} \times \left\|\lambda\left|\left|A\right|^\alpha B\right|^4 + \left(1-\lambda\right)\left|\left|A^*\right|^{1-\alpha}C\right|^4\right\|^{1/2} \tag{3.26}$$

for all α, $\lambda \in [0,1]$.

If we take $\lambda = 1/2$ in (3.25), then we obtain

$$\omega^2\left(C^*Af\left(A\right)B\right) \leq \frac{\sqrt{2}}{2}f_a^2\left(\|A\|\right)\left\|\left|\left|A\right|^\alpha B\right|^4 + \left|\left|A^*\right|^{1-\alpha}C\right|^4\right\|^{1/2} \left\|\left|A\right|^\alpha B\right\| \left\|\left|A^*\right|^{1-\alpha}C\right\| \tag{3.27}$$

4. Inequalities for Trace of Operators

We have the following result for trace of operatorss:

Theorem 6. *Let $r \geq 1/2$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$. Assume that the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on $D(0,R)$ and A, B, $C \in B(H)$ with $\|A\| < R$. If $\left|A\right|^\alpha B \in \mathcal{B}_{2pr}(H)$ and $\left|A^*\right|^{1-\alpha}C \in \mathcal{B}_{2qr}(H)$ for $\alpha \in [0,1]$, then $C^*Af(A)B \in \mathcal{B}_{2r}(H)$ and*

$$\left\|C^*Af\left(A\right)B\right\|_{2r} \leq f_a\left(\|A\|\right)\left\|\left|A\right|^\alpha B\right\|_{2pr}\left\|\left|A^*\right|^{1-\alpha}C\right\|_{2qr}. \tag{4.1}$$

In particular,

$$\left\|C^*Af\left(A\right)B\right\|_{2r} \leq f_a\left(\|A\|\right)\left\|\left|A\right|^{1/2} B\right\|_{2pr}\left\|\left|A^*\right|^{1/2}C\right\|_{2qr} \tag{4.2}$$

for $\left|A\right|^{1/2}B \in \mathcal{B}_{2pr}(H)$ and $\left|A^*\right|^{1/2}C \in \mathcal{B}_{2qr}(H)$.

Proof. If we take in (2.7) the power $r > 0$ and $x = e_i$, $y = f_i$ where $\mathcal{E} = \{e_i\}_{i\in I}$ and $\mathcal{F} = \{f_i\}_{i\in I}$ are orthonormal basis and sum, then we get

$$\sum_{i\in I}\left|\left\langle C^*Af\left(A\right)Be_i, f_i\right\rangle\right|^{2r} \leq f_a^{2r}\left(\|A\|\right)\sum_{i\in I}\left\langle\left|\left|A\right|^\alpha B\right|^2 e_i, e_i\right\rangle^r\left\langle\left|\left|A^*\right|^{1-\alpha}C\right|^2 f_i, f_i\right\rangle^r. \tag{4.3}$$

If we use the Hölder's inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we get

$$\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{r} \tag{4.4}$$
$$\leq\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{pr}\right)^{1/p}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\right)^{1/q}$$

By the McCarthy inequality for pr, $qr\geq1$, we have

$$\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{pr}\leq\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2pr}e_{i},e_{i}\right\rangle$$

and

$$\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\leq\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}f_{i},f_{i}\right\rangle,$$

therefore

$$\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{pr}\right)^{1/p}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\right)^{1/q}$$
$$\leq\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2pr}e_{i},e_{i}\right\rangle\right)^{1/p}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\right)^{1/q}$$
$$=\left(\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2pr}\right)^{1/p}\left(\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2qr}\right)^{1/q}=\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}.$$

By (4.3) and (4.4) we derive

$$\sum_{i\in I}\left|\left\langle C^{*}Af\left(A\right)Be_{i},f_{i}\right\rangle\right|^{2r}\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}. \tag{4.5}$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (2.9), then by (1.21) we get

$$\left\|C^{*}Af\left(A\right)B\right\|_{2r}^{2r}\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}$$

and the inequality (4.1) is obtained. □

Remark 5. *If we take $r=1/2$ and $p=q=2$, then by (4.1) we get*

$$\left\|C^{*}Af\left(A\right)B\right\|_{1}\leq f_{a}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2} \tag{4.6}$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_{2}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2}\left(H\right)$ for $\alpha\in\left[0,1\right]$.

Also, if $r=1$ and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (4.1) we get

$$\left\|C^{*}Af\left(A\right)B\right\|_{2}\leq f_{a}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2p}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2q} \tag{4.7}$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_{2p}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2q}\left(H\right)$ for $\alpha\in\left[0,1\right]$.

We also have:

Theorem 7. *Let $r \geq 1/2$, p, $q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Assume that the power series with complex coefficients $f(z):=\sum_{k=0}^{\infty} a_k z^k$ is convergent on $D(0,R)$ and A, B, $C \in B(H)$ with $\|A\|<R$. If $|A|^{\alpha} B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2q}(H)$ for $\alpha \in [0,1]$, then $C^* A f(A) B \in \mathcal{B}_{2r}(H)$ and*

$$\|C^* A f(A) B\|_{2r} \leq f_a(\|A\|) \left\||A|^{\alpha} B\right\|_{2p} \left\||A^*|^{1-\alpha} C\right\|_{2q}. \tag{4.8}$$

In particular,

$$\|C^* A f(A) B\|_{2r} \leq f_a(\|A\|) \left\||A|^{1/2} B\right\|_{2p} \left\||A^*|^{1/2} C\right\|_{2q} \tag{4.9}$$

for $|A|^{1/2} B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2q}(H)$.

Proof. Assume that $\mathcal{E}=\{e_i\}_{i\in I}$ and $\mathcal{F}=\{f_i\}_{i\in I}$ are orthonormal basis in H. Observe that we have $\frac{1}{\frac{p}{r}}+\frac{1}{\frac{q}{r}}=1$ and by Hölder's inequality for $\frac{p}{r}$ and $\frac{q}{r}$ we have

$$\begin{aligned} &\sum_{i\in I} \left\langle \left||A|^{\alpha} B\right|^2 e_i, e_i\right\rangle^r \left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i\right\rangle^r \\ &= \sum_{i\in I} \left[\left\langle \left||A|^{\alpha} B\right|^2 e_i, e_i\right\rangle^p\right]^{\frac{r}{p}} \left[\left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i\right\rangle^q\right]^{\frac{r}{q}} \\ &\leq \left(\sum_{i\in I} \left\langle \left||A|^{\alpha} B\right|^2 e_i, e_i\right\rangle^p\right)^{r/p} \left(\sum_{i\in I} \left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i\right\rangle^q\right)^{r/q}. \end{aligned} \tag{4.10}$$

By McCarthy inequality for p, $q>1$ we get

$$\sum_{i\in I} \left\langle \left||A|^{\alpha} B\right|^2 e_i, e_i\right\rangle^p \leq \sum_{i\in I} \left\langle \left||A|^{\alpha} B\right|^{2p} e_i, e_i\right\rangle$$

and

$$\sum_{i\in I} \left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i\right\rangle^q \leq \sum_{i\in I} \left\langle \left||A^*|^{1-\alpha} C\right|^{2q} f_i, f_i\right\rangle$$

and by (4.10)

$$\begin{aligned} &\sum_{i\in I} \left\langle \left||A|^{\alpha} B\right|^2 e_i, e_i\right\rangle^r \left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i\right\rangle^r \\ &\leq \left(\sum_{i\in I} \left\langle \left||A|^{\alpha} B\right|^{2p} e_i, e_i\right\rangle\right)^{r/p} \left(\sum_{i\in I} \left\langle \left||A^*|^{1-\alpha} C\right|^{2q} f_i, f_i\right\rangle\right)^{r/q} \\ &= \left\||A|^{\alpha} B\right\|_{2p}^{2r} \left\||A^*|^{1-\alpha} C\right\|_{2q}^{2r}. \end{aligned} \tag{4.11}$$

By (4.3) and (4.11) we get

$$\sum_{i\in I} \left|\langle C^* A f(A) B e_i, f_i\rangle\right|^{2r} \leq f_a^{2r}(\|A\|) \left\||A|^{\alpha} B\right\|_{2p}^{2r} \left\||A^*|^{1-\alpha} C\right\|_{2q}^{2r}. \tag{4.12}$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (4.12) we get

$$\|C^* A f(A) B\|_{2r}^{2r} \leq f_a^{2r}(\|A\|) \left\||A|^{\alpha} B\right\|_{2p}^{2r} \left\||A^*|^{1-\alpha} C\right\|_{2q}^{2r}$$

and the inequality (4.8) is thus proved. □

Remark 6. *If we take $p=q=2r=s\geq 1$, then by (4.8) we get*

$$\left\| C^{*}Af\left(A\right) B\right\| _{s}\leq f_{a}\left(\left\| A\right\| \right) \left\| \left\vert A\right\vert ^{\alpha }B\right\| _{2s}\left\| \left\vert A^{*}\right\vert ^{1-\alpha }C\right\| _{2s} \tag{4.13}$$

provided that $\left\vert A\right\vert ^{\alpha }B\in \mathcal{B}_{2s}\left(H\right)$ and $\left\vert A^{*}\right\vert ^{1-\alpha }C\in \mathcal{B}_{2s}\left(H\right)$ for $\alpha \in \left[0,1\right]$.

For $\alpha =1/2$ we have

$$\left\| C^{*}Af\left(A\right) B\right\| _{s}\leq f_{a}\left(\left\| A\right\| \right) \left\| \left\vert A\right\vert ^{1/2}B\right\| _{2s}\left\| \left\vert A^{*}\right\vert ^{1/2}C\right\| _{2s} \tag{4.14}$$

provided that $\left\vert A\right\vert ^{1/2}B\in \mathcal{B}_{2s}\left(H\right)$ and $\left\vert A^{*}\right\vert ^{1/2}C\in \mathcal{B}_{2s}\left(H\right)$.

If $r=2$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, then

$$\left\| C^{*}Af\left(A\right) B\right\| _{4}\leq f_{a}\left(\left\| A\right\| \right) \left\| \left\vert A\right\vert ^{\alpha }B\right\| _{2p}\left\| \left\vert A^{*}\right\vert ^{1-\alpha }C\right\| _{2q} \tag{4.15}$$

provided that $\left\vert A\right\vert ^{\alpha }B\in \mathcal{B}_{2p}\left(H\right)$ and $\left\vert A^{*}\right\vert ^{1-\alpha }C\in \mathcal{B}_{2q}\left(H\right)$ for $\alpha \in \left[0,1\right]$.

In particular,

$$\left\| C^{*}Af\left(A\right) B\right\| _{4}\leq f_{a}\left(\left\| A\right\| \right) \left\| \left\vert A\right\vert ^{1/2}B\right\| _{2p}\left\| \left\vert A^{*}\right\vert ^{1/2}C\right\| _{2q} \tag{4.16}$$

for $\left\vert A\right\vert ^{1/2}B\in \mathcal{B}_{2p}\left(H\right)$ and $\left\vert A^{*}\right\vert ^{1/2}C\in \mathcal{B}_{2q}\left(H\right)$.

Theorem 8. *Assume that the power series with complex coefficients $f\left(z\right) :=\sum_{k=0}^{\infty }a_{k}z^{k}$ is convergent on $D(0,R)$, $A,\ B,\ C\in B\left(H\right)$ with $\left\| A\right\| <R$.*

If $r\geq 1/2$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $pr,\ qr\geq 1$ and $\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2pr},\ \left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2qr}\in \mathcal{B}_{1}\left(H\right)$, then $C^{*}Af\left(A\right) B\in \mathcal{B}_{2r}\left(H\right)$ and

$$\omega _{2r}^{2r}\left(C^{*}Af\left(A\right) B\right) \leq f_{a}^{2r}\left(\left\| A\right\| \right) \operatorname{tr}\left(\frac{1}{p}\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2pr}+\frac{1}{q}\left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2qr}\right) . \tag{4.17}$$

If $r\geq 1$ and $\left\vert A\right\vert ^{\alpha }B,\left\vert A^{*}\right\vert ^{1-\alpha }C\in \mathcal{B}_{4r}\left(H\right)$, then $C^{*}Af\left(A\right) B\in \mathcal{B}_{2r}\left(H\right)$ and

$$\begin{aligned} &\omega _{2r}^{2r}\left(C^{*}Af\left(A\right) B\right) \\ &\leq \frac{1}{2}f_{a}^{2r}\left(\left\| A\right\| \right) \left(\left\| \left\vert A\right\vert ^{\alpha }B\right\| _{4r}^{2r}\left\| \left\vert A^{*}\right\vert ^{1-\alpha }C\right\| _{4r}^{2r}+\omega _{r}^{r}\left(\left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2}\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2}\right) \right) \\ &\leq \frac{1}{2}f_{a}^{2r}\left(\left\| A\right\| \right) \left(\left\| \left\vert A\right\vert ^{\alpha }B\right\| _{4r}^{2r}\left\| \left\vert A^{*}\right\vert ^{1-\alpha }C\right\| _{4r}^{2r}+\left\| \left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2}\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2}\right\| _{r}^{r}\right) . \end{aligned} \tag{4.18}$$

If $r\geq 1$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $pr,qr\geq 2$, then

$$\begin{aligned} \omega _{2r}^{2r}\left(C^{*}Af\left(A\right) B\right) &\leq \frac{1}{2}f_{a}^{2r}\left(\left\| A\right\| \right) \left[\operatorname{tr}\left(\frac{1}{p}\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2pr}+\frac{1}{q}\left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2qr}\right) \right. \\ &\left. +\omega _{r}^{r}\left(\left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2}\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2}\right) \right] \\ &\leq \frac{1}{2}f_{a}^{2r}\left(\left\| A\right\| \right) \left[\operatorname{tr}\left(\frac{1}{p}\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2pr}+\frac{1}{q}\left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2qr}\right) \right. \\ &\left. +\left\| \left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2}\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2}\right\| _{r}^{r}\right] . \end{aligned} \tag{4.19}$$

Proof. From (2.7) for $y=x$ we have that

$$\left\vert \left\langle C^{*}Af\left(A\right) Bx,x\right\rangle \right\vert ^{2}\leq f_{a}^{2}\left(\left\| A\right\| \right) \left\langle \left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2}x,x\right\rangle \left\langle \left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2}x,x\right\rangle \tag{4.20}$$

for $x \in H$ with $\|x\| = 1$.

If we take the power $r > 0$, we get, by Young and McCarthy inequalities, that

$$
\begin{aligned}
& |\langle C^* A f(A) Bx, x\rangle|^{2r} \\
& \le f_a^{2r}(\|A\|) \left\langle \left||A|^{\alpha} B\right|^2 x, x \right\rangle^r \left\langle \left||A^*|^{1-\alpha} C\right|^2 x, x \right\rangle^r \\
& \le f_a^{2r}(\|A\|) \left[\frac{1}{p} \left\langle \left||A|^{\alpha} B\right|^2 x, x \right\rangle^{pr} + \frac{1}{q} \left\langle \left||A^*|^{1-\alpha} C\right|^2 x, x \right\rangle^{qr} \right] \\
& \le f_a^{2r}(\|A\|) \left[\frac{1}{p} \left\langle \left||A|^{\alpha} B\right|^{2pr} x, x \right\rangle + \frac{1}{q} \left\langle \left||A^*|^{1-\alpha} C\right|^{2qr} x, x \right\rangle \right] \\
& = f_a^{2r}(\|A\|) \left\langle \left(\frac{1}{p} \left||A|^{\alpha} B\right|^{2pr} + \frac{1}{q} \left||A^*|^{1-\alpha} C\right|^{2qr} \right) x, x \right\rangle
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$.

If $\mathcal{E} = \{e_i\}_{i \in I}$ is an orthonormal basis, then by taking $x = e_i$ and summing over $i \in I$ we get

$$
\begin{aligned}
& \|C^* A f(A) B\|_{\mathcal{E},2r}^{2r} \\
& = \sum_{i \in I} |\langle C^* A f(A) B e_i, e_i\rangle|^{2r} \\
& \le f_a^{2r}(\|A\|) \sum_{i \in I} \left\langle \left(\frac{1}{p} \left||A|^{\alpha} B\right|^{2pr} + \frac{1}{q} \left||A^*|^{1-\alpha} C\right|^{2qr} \right) e_i, e_i \right\rangle \\
& = f_a^{2r}(\|A\|) \operatorname{tr} \left(\frac{1}{p} \left||A|^{\alpha} B\right|^{2pr} + \frac{1}{q} \left||A^*|^{1-\alpha} C\right|^{2qr} \right),
\end{aligned}
$$

which, by taking the supremum over $\mathcal{E}$, proves (4.17).

By Buzano's inequality we have

$$
\begin{aligned}
& \left\langle \left||A|^{\alpha} B\right|^2 x, x \right\rangle \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x \right\rangle \\
& \le \frac{1}{2} \left[\left\| \left||A|^{\alpha} B\right|^2 x \right\| \left\| \left||A^*|^{1-\alpha} C\right|^2 x \right\| + \left| \left\langle \left||A|^{\alpha} B\right|^2 x, \left||A^*|^{1-\alpha} C\right|^2 x \right\rangle \right| \right] \\
& = \frac{1}{2} \left[\left\| \left||A|^{\alpha} B\right|^2 x \right\| \left\| \left||A^*|^{1-\alpha} C\right|^2 x \right\| + \left| \left\langle \left||A^*|^{1-\alpha} C\right|^2 \left||A|^{\alpha} B\right|^2 x, x \right\rangle \right| \right]
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$.

If we take the power $r \geq 1$ and use the convexity of power function, then we get

$$\begin{aligned}
&\left\langle \left| |A|^{\alpha} B \right|^2 x, x \right\rangle^r \left\langle x, \left| |A^*|^{1-\alpha} C \right|^2 x \right\rangle^r \\
&\leq \left[\frac{\left\| \left| |A|^{\alpha} B \right|^2 x \right\| \left\| \left| |A^*|^{1-\alpha} C \right|^2 x \right\| + \left| \left\langle \left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^{\alpha} B \right|^2 x, x \right\rangle \right|}{2} \right]^r \\
&\leq \frac{\left\| \left| |A|^{\alpha} B \right|^2 x \right\|^r \left\| \left| |A^*|^{1-\alpha} C \right|^2 x \right\|^r + \left| \left\langle \left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^{\alpha} B \right|^2 x, x \right\rangle \right|^r}{2} \\
&= \frac{\left\| \left| |A|^{\alpha} B \right|^2 x \right\|^{2\frac{r}{2}} \left\| \left| |A^*|^{1-\alpha} C \right|^2 x \right\|^{2\frac{r}{2}} + \left| \left\langle \left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^{\alpha} B \right|^2 x, x \right\rangle \right|^r}{2} \\
&= \frac{\left\langle \left| |A|^{\alpha} B \right|^4 x, x \right\rangle^{\frac{r}{2}} \left\langle \left| |A^*|^{1-\alpha} C \right|^4 x, x \right\rangle^{\frac{r}{2}} + \left| \left\langle \left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^{\alpha} B \right|^2 x, x \right\rangle \right|^r}{2}
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

Therefore

$$\begin{aligned}
(4.21) \qquad &\|C^* A f(A) B\|_{\mathcal{E},2r}^{2r} \\
&= \sum_{i \in I} \left| \langle C^* A f(A) B e_i, e_i \rangle \right|^{2r} \\
&\leq f_a^{2r}(\|A\|) \sum_{i \in I} \left\langle \left| |A|^{\alpha} B \right|^2 e_i, e_i \right\rangle^r \left\langle e_i, \left| |A^*|^{1-\alpha} C \right|^2 e_i \right\rangle^r \\
&\leq \frac{1}{2} f_a^{2r}(\|A\|) \left[\sum_{i \in I} \left\langle \left| |A|^{\alpha} B \right|^4 e_i, e_i \right\rangle^{\frac{r}{2}} \left\langle \left| |A^*|^{1-\alpha} C \right|^4 e_i, e_i \right\rangle^{\frac{r}{2}} \right. \\
&\quad \left. + \sum_{i \in I} \left| \left\langle \left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^{\alpha} B \right|^2 e_i, e_i \right\rangle \right|^r \right]
\end{aligned}$$

Using Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\sum_{i \in I} \left\langle \left| |A|^{\alpha} B \right|^4 e_i, e_i \right\rangle^{\frac{r}{2}} \left\langle \left| |A^*|^{1-\alpha} C \right|^4 e_i, e_i \right\rangle^{\frac{r}{2}} \\
&\leq \left(\sum_{i \in I} \left\langle \left| |A|^{\alpha} B \right|^4 e_i, e_i \right\rangle^r \right)^{1/2} \left(\sum_{i \in I} \left\langle \left| |A^*|^{1-\alpha} C \right|^4 e_i, e_i \right\rangle^r \right)^{1/2} \\
&\leq \left(\sum_{i \in I} \left\langle \left| |A|^{\alpha} B \right|^{4r} e_i, e_i \right\rangle \right)^{1/2} \left(\sum_{i \in I} \left\langle \left| |A^*|^{1-\alpha} C \right|^{4r} e_i, e_i \right\rangle \right)^{1/2} \\
&= \left\| |A|^{\alpha} B \right\|_{4r}^{2r} \left\| |A^*|^{1-\alpha} C \right\|_{4r}^{2r},
\end{aligned}$$

where for the last inequality we used McCarthy's result for $r \geq 1$. This proves (4.18).

Further, if we use Young's inequality for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$ab\leq\frac{1}{p}a^{p}+\frac{1}{q}b^{q},\ a,b\geq 0,$$

then we get

$$\begin{aligned}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r}&\leq\frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{pr}+\frac{1}{q}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{qr}\\&=\frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{2\frac{pr}{2}}+\frac{1}{q}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{2\frac{qr}{2}}\\&=\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{4}x,x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}x,x\right\rangle^{\frac{qr}{2}}\\&\leq\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}x,x\right\rangle\\&=\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

Therefore

$$\begin{aligned}\left\|C^{*}Af\left(A\right)B\right\|_{\mathcal{E},2r}^{2r}&=\sum_{i\in I}\left|\left\langle C^{*}Af\left(A\right)Be_{i},e_{i}\right\rangle\right|^{2r}\\&\leq f_{a}^{2r}\left(\|A\|\right)\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r}\left\langle e_{i},\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i}\right\rangle^{r}\\&\leq\frac{1}{2}f_{a}^{2r}\left(\|A\|\right)\left[\sum_{i\in I}\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)e_{i},e_{i}\right\rangle\right.\\&\left.+\sum_{i\in I}\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle\right|^{r}\right]\\&=\frac{1}{2}f_{a}^{2r}\left(\|A\|\right)\left[\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)\right.\\&\left.+\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right\|_{\mathcal{E},r}^{r}\right],\end{aligned}$$

which proves, by taking the supremum over $\mathcal{E}$, the desired inequality (4.19). □

Remark 7. *Let $\alpha\in[0,1]$. If $r=1/2$, p, $q=2$ and $\left|\left|A\right|^{\alpha}B\right|^{2}$, $\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\in\mathcal{B}_{1}\left(H\right)$, then $C^{*}Af\left(A\right)B\in\mathcal{B}_{1}\left(H\right)$ and by (4.17) we get*

$$\omega_{1}\left(C^{*}Af\left(A\right)B\right)\leq\frac{1}{2}f_{a}\left(\|A\|\right)\operatorname{tr}\left(\left|\left|A\right|^{\alpha}B\right|^{2}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right).\tag{4.22}$$

If $r=1$ and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (4.17) we obtain

$$\omega_{2}^{2}\left(C^{*}Af\left(A\right)B\right)\leq f_{a}^{2}\left(\|A\|\right)\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2p}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2q}\right),\tag{4.23}$$

provided that $\left|\left|A\right|^{\alpha}B\right|^{2p}$, $\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2q}\in\mathcal{B}_{1}\left(H\right)$.

If we take $r=1$ in (4.18), then we get

$$\begin{aligned}(4.24)\quad & \omega_2^2\left(C^*Af\left(A\right)B\right)\\ & \leq \frac{1}{2}f_a^2\left(\|A\|\right)\left(\||A|^\alpha B\|_4^2\left\||A^*|^{1-\alpha}C\right\|_4^2+\omega_1\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right)\right)\\ & \leq \frac{1}{2}f_a^2\left(\|A\|\right)\left(\||A|^\alpha B\|_4^2\left\||A^*|^{1-\alpha}C\right\|_4^2+\left\|\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right\|_1\right),\end{aligned}$$

provided that $|A|^\alpha B,\ |A^*|^{1-\alpha}C\in\mathcal{B}_4\left(H\right).$

If $r=1$ and $p=q=2$ in (4.19), then we get for $\left||A|^\alpha B\right|^{2p},\ \left||A^*|^{1-\alpha}C\right|^{2q}\in\mathcal{B}_1\left(H\right)$ that

$$\begin{aligned}(4.25)\quad \omega_2^2\left(C^*Af\left(A\right)B\right) & \leq \frac{1}{4}f_a^2\left(\|A\|\right)\left[\operatorname{tr}\left(\left||A|^\alpha B\right|^{2p}+\left||A^*|^{1-\alpha}C\right|^{2q}\right)\right.\\ & \quad+\frac{1}{2}f_a^2\left(\|A\|\right)\omega_1\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right)\\ & \leq \frac{1}{4}f_a^2\left(\|A\|\right)\operatorname{tr}\left(\left||A|^\alpha B\right|^{2p}+\left||A^*|^{1-\alpha}C\right|^{2q}\right)\\ & \quad+\frac{1}{2}f_a^2\left(\|A\|\right)\left\|\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right\|_1.\end{aligned}$$

We also have:

Theorem 9. *With the assumptions of Theorem 8, we have for $r\geq 1,\ \lambda\in[0,1]$ that*

$$\begin{aligned}(4.26)\quad \omega_{2r}^{2r}\left(C^*Af\left(A\right)B\right) & \leq f_a^{2r}\left(\|A\|\right)\left\|\left(1-\lambda\right)\left||A|^\alpha B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right\|\\ & \quad\times\||A|^\alpha B\|_{2r}^{2r\lambda}\left\||A^*|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)},\end{aligned}$$

provided that $|A|^\alpha B,\ |A^*|^{1-\alpha}C\in\mathcal{B}_{2r}\left(H\right).$

In particular,

$$\begin{aligned}(4.27)\quad \omega_{2r}^{2r}\left(C^*Af\left(A\right)B\right) & \leq \frac{1}{2}f_a^{2r}\left(\|A\|\right)\left\|\left||A|^\alpha B\right|^{2r}+\left||A^*|^{1-\alpha}C\right|^{2r}\right\|\\ & \quad\times\||A|^\alpha B\|_{2r}^{r}\left\||A^*|^{1-\alpha}C\right\|_{2r}^{r}.\end{aligned}$$

Proof. If $\mathcal{E}=\{e_i\}_{i\in I}$ is an orthonormal basis, then by taking $x=e_i$ in (3.21) and summing over $i\in I$ we get

$$\begin{aligned}(4.28)\qquad &\sum_{i\in I}\left|\langle C^{*}Af(A)Be_i,e_i\rangle\right|^{2r}\\ &\le f_a^{2r}(\|A\|)\sum_{i\in I}\left[\left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right]e_i,e_i\right\rangle\right]\\ &\times\left\langle\left||A|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r\lambda}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}e_i,e_i\right\rangle^{r(1-\lambda)}\\ &\le f_a^{2r}(\|A\|)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2r}\right\|\\ &\times\sum_{i\in I}\left\langle\left||A|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r\lambda}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}e_i,e_i\right\rangle^{r(1-\lambda)}.\end{aligned}$$

If we use Hölder's inequality for $p=\frac{1}{\lambda}$, $q=\frac{1}{1-\lambda}$, then we have

$$\begin{aligned}&\sum_{i\in I}\left\langle\left||A|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r\lambda}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}e_i,e_i\right\rangle^{r(1-\lambda)}\\ &\le\left(\sum_{i\in I}\left\langle\left||A|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r}\right)^{\lambda}\left(\sum_{i\in I}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}e_i,e_i\right\rangle^{r}\right)^{1-\lambda}\\ &\le\left(\sum_{i\in I}\left\langle\left||A|^{\alpha}B\right|^{2r}e_i,e_i\right\rangle\right)^{\lambda}\left(\sum_{i\in I}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2r}e_i,e_i\right\rangle\right)^{1-\lambda}\\ &=\left\||A|^{\alpha}B\right\|_{2r}^{2r\lambda}\left\||A^{*}|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)},\end{aligned}$$

which proves (4.26). □

Remark 8. *If we take $r=1$ in Theorem 9, then we get for $\alpha\in[0,1]$ that*

$$\begin{aligned}(4.29)\qquad \omega_2^2(C^{*}Af(A)B)&\le f_a^2(\|A\|)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2}+\lambda\left||A^{*}|^{1-\alpha}C\right|^{2}\right\|\\ &\times\left\||A|^{\alpha}B\right\|_{2}^{2\lambda}\left\||A^{*}|^{1-\alpha}C\right\|_{2}^{2(1-\lambda)},\end{aligned}$$

provided that $|A|^{\alpha}B$, $|A^{*}|^{1-\alpha}C\in\mathcal{B}_2(H)$.

In particular,

$$\begin{aligned}(4.30)\qquad \omega_2^2(C^{*}Af(A)B)&\le\frac{1}{2}f_a^2(\|A\|)\left\|\left||A|^{\alpha}B\right|^{2}+\left||A^{*}|^{1-\alpha}C\right|^{2}\right\|\\ &\times\left\||A|^{\alpha}B\right\|_{2}\left\||A^{*}|^{1-\alpha}C\right\|_{2}.\end{aligned}$$

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