SEVERAL NUMERICAL RADIUS AND p-SCHATTEN NORM INEQUALITIES FOR POWER SERIES OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. Consider the power series with complex coefficients $f(z) := \sum_{k=1}^{\infty} a_k z^k$ convergent on $D(0,R)$ and A, B, $C \in B(H)$ with $0 \neq \|A\| < R$, then we have the Schwarz type inequality

$$|\langle C^* f(A) Bx, y\rangle| \leq \frac{f_a(\|A\|)}{\|A\|} \left\langle \left|\left|A\right|^{\alpha} B\right|^2 x, x\right\rangle^{1/2} \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 y, y\right\rangle^{1/2}$$

for $\alpha \in [0,1]$ and x, $y \in H$. Some natural applications for *numerical radius* and *p-Schatten norm* are also provided.

1. Introduction

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leq w(T)\|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T + V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{1.3}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [12], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [13] improved the inequality (1.3) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{1.5}$$

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for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [9]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \leq \frac{1}{2}\left\| |T|^{2\alpha r} + |T^*|^{2(1-\alpha)r} \right\| \tag{1.6}$$

and

$$\omega^{2r}(T) \leq \left\| \alpha |T|^{2r} + (1-\alpha)|T^*|^{2r} \right\|, \tag{1.7}$$

where $\alpha \in (0,1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (1.6) that

$$\omega(T) \leq \frac{1}{2}\| |T| + |T^*| \| \tag{1.8}$$

and from (1.7) that

$$\omega^2(T) \leq \frac{1}{2}\left\| |T|^2 + |T^*|^2 \right\|. \tag{1.9}$$

For more related results, see the recent books on inequalities for numerical radii [7] and [4].

Let $(H; \langle .,. \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.10}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{1.11}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.11) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.12}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.13}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ then $AB, BA \in \mathcal{B}_1(H)$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, the space of operators of finite rank, is a dense subspace of $\mathcal{B}_1(H)$.

For a large number of results concerning trace inequalities, see the recent survey paper [8].

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\leq p<\infty$ if the *p-Schatten norm* is finite [17, p. 60-64]

$$\|A\|_p:=\left[\operatorname{tr}\left(|A|^p\right)\right]^{1/p}=\left(\sum_{i\in I}\langle |A|^p e_i,e_i\rangle\right)^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.14}$$

and

$$\|A\|_1\geq\|A\|_p\geq\|A\|_q\geq\|A\|. \tag{1.15}$$

For $p\geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [17, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.16}$$

$$\|AB\|_p\leq\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.17}$$

and

$$\|AB\|_p\leq\|A\|_p\|B\|,\ \ \|BA\|_p\leq\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.18}$$

This implies that

$$\|CAB\|_p\leq\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.19}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$\left(|\operatorname{tr}(AB)|\leq\right)\|AB\|_1\leq\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{1.20}$$

For the theory of trace functionals and their applications the reader is referred to [15] and [17].

For $\mathcal{E}:=\{e_i\}_{i\in I}$ an orthonormal basis of H we define for $A\in\mathcal{B}_p(H)$, $p\geq 1$

$$\|A\|_{\mathcal{E},p}:=\left(\sum_{i\in I}|\langle Ae_i,e_i\rangle|^p\right)^{1/p}.$$

We observe that $\|\cdot\|_{\mathcal{E},p}$ is a norm on $\mathcal{B}_p(H)$ and

$$\|A\|_{\mathcal{E},p}\leq\|A\|_p \text{ for } A\in\mathcal{B}_p(H).$$

Further, we can take the supremum over all orthonormal basis in H we can also define, for $A\in\mathcal{B}_p(H)$, that

$$\omega_p(A):=\sup_{\mathcal{E}}\|A\|_{\mathcal{E},p}\leq\|A\|_p,$$

which is a *norm* on $\mathcal{B}_p(H)$.

It is also known that, if $\mathcal{E}=\{e_i\}_{i\in I}$ and $\mathcal{F}=\{f_i\}_{i\in I}$ are orthonormal basis, then [16]

$$\sup_{\mathcal{E},\mathcal{F}}\sum_{i\in I}|\langle Te_i,f_i\rangle|^s=\|T\|_s^s \text{ for } s\geq 1. \tag{1.21}$$

2. Vector Inequalities

In 1988, F. Kittaneh [11, Corollary 7], obtained the following Schwarz type inequality for powers of operators:

Lemma 1. *Let $A \in B(H)$ and $\alpha \in [0,1]$. Then for $n \geq 1$ we have*

$$|\langle A^n x, y\rangle|^2 \leq \|A\|^{2n-2} \left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle \tag{2.1}$$

for all x, $y \in H$.

We can state the following result as well:

Corollary 1. *Let A, B, $C \in B(H)$ and $\alpha \in [0,1]$. Then for $n \geq 1$ we have*

$$|\langle C^* A^n Bx, y\rangle|^2 \leq \|A\|^{2n-2} \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{2.2}$$

for all x, $y \in H$.

Proof. If we replace x by Bx and y by Cy in (2.1), then we get

$$|\langle C^* A^n Bx, y\rangle|^2 \leq \|A\|^{2n-2} \left\langle B^* |A|^{2\alpha} Bx, x\right\rangle \left\langle C^* |A^*|^{2(1-\alpha)} Cy, y\right\rangle. \tag{2.3}$$

Observe that $B^* |A|^{2\alpha} B = \left||A|^{\alpha} B\right|^2$ and $C^* |A^*|^{2(1-\alpha)} C = \left||A^*|^{1-\alpha} C\right|^2$, then by (2.3) we get (2.2). □

We consider the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ with $a_k \in \mathbb{C}$ for $k \in \mathbb{N} := \{0, 1, ...\}$. We assume that this power series in convergent on the open disk $D(0,R) := \{z \in \mathbb{C} | \ z < R\}$. If $R = \infty$ then $D(0,R) = \mathbb{C}$. We define $f_a(z) := \sum_{k=0}^{\infty} |a_k| z^k$ which has the same radius of convergence R.

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \ \lambda \in D(0,1); \\ g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \ \lambda \in \mathbb{C}; \\ h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \ \lambda \in \mathbb{C}; \\ l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \ \lambda \in D(0,1); \end{aligned} \tag{2.4}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n}\lambda^n = \ln \frac{1}{1-\lambda},\ \lambda \in D(0,1); \tag{2.5}$$

$$g_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\lambda^{2n} = \cosh \lambda,\ \lambda \in \mathbb{C};$$

$$h_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\lambda^{2n+1} = \sinh \lambda,\ \lambda \in \mathbb{C};$$

$$l_a(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda},\ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \qquad \lambda \in \mathbb{C}, \tag{2.6}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

The following result is of interest:

Theorem 2. *Consider the power series with complex coefficients $f(z) := \sum_{k=1}^{\infty} a_k z^k$ (observe that $f(0) = 0$) convergent on $D(0,R)$ and A, B, $C \in B(H)$ with $\|A\| < R$, then*

$$|\langle C^* f(A) Bx, y\rangle|^2 \le \left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{2.7}$$

for $\alpha \in [0,1]$ and x, $y \in H$.

Proof. From (2.2) we have for $k \ge 1$ that

$$\left|\langle C^* A^k Bx, y\rangle\right| \le \frac{\|A\|^k}{\|A\|} \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle$$

for all x, $y \in H$.

Further, if we multiply by $|a_k| \geq 0$, $k \in \{1, ...\}$ and sum over k from 1 to m, then we get

$$\begin{aligned}(2.8)\quad & \left|\left\langle C^* \sum_{k=1}^{m} a_k A^k Bx, y\right\rangle\right| \\ & = \left|\sum_{k=1}^{m} a_k \left\langle C^* A^k Bx, y\right\rangle\right| \leq \sum_{k=1}^{m} |a_k| \left|\left\langle C^* A^k Bx, y\right\rangle\right| \\ & \leq \frac{1}{\|A\|} \sum_{k=1}^{m} |a_k| \|A\|^k \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle^{1/2}\end{aligned}$$

for all x, $y \in H$.

Since $\|A\| < R$ then the series $\sum_{k=1}^{\infty} a_k A^k$ and $\sum_{k=1}^{\infty} |a_k| \|A\|^k$ are convergent,

$$\sum_{k=1}^{\infty} a_k A^k = f(A) \text{ and } \sum_{k=1}^{\infty} |a_k| \|A\|^k = f_a(\|A\|).$$

By taking now the limit over $m \to \infty$ in (2.8) we deduce the desired result (2.7). □

The following particular inequalities are of interest:

Remark 1. *If A, B, $C \in B(H)$ with $\|A\| < 1$, then for $\alpha \in [0, 1]$ we have*

$$\begin{aligned}(2.9)\quad & \left|\left\langle C^* \ln(I \pm A) Bx, y\right\rangle\right|^2 \\ & \leq \left[\frac{\ln(1 - \|A\|)}{\|A\|}\right]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle\end{aligned}$$

and

$$\begin{aligned}(2.10)\quad & \left|\left\langle C^* \left[(I \pm A)^{-1} - I\right] Bx, y\right\rangle\right|^2 \\ & \leq (1 - \|A\|)^{-2} \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle\end{aligned}$$

for $\alpha \in [0, 1]$ and x, $y \in H$.

Since

$$(I - A)^{-1} - I = (I - A)^{-1} A,$$

hence from (2.10) we get

$$\begin{aligned}(2.11)\quad & \left|\left\langle C^* (I - A)^{-1} ABx, y\right\rangle\right|^2 \\ & \leq (1 - \|A\|)^{-2} \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle,\end{aligned}$$

for $\alpha \in [0, 1]$ and x, $y \in H$.

Also, due to the fact that

$$(I + A)^{-1} - I = -(I + A)^{-1} A,$$

then by (2.10) we have

$$\begin{aligned}(2.12)\quad & \left|\left\langle C^* (I + A)^{-1} ABx, y\right\rangle\right|^2 \\ & \leq (1 - \|A\|)^{-2} \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle\end{aligned}$$

for $\alpha \in [0,1]$ *and* $x, y \in H$.

If $A, B, C \in B(H)$ *and* $\alpha \in [0,1]$, *then*

$$|\langle C^* [\exp(A) - I] Bx, y\rangle|^2 \leq \left[\frac{\exp(\|A\|) - 1}{\|A\|}\right]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle, \tag{2.13}$$

$$|\langle C^* \sin(A) Bx, y\rangle|^2 \leq \left[\frac{\sinh(\|A\|)}{\|A\|}\right]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{2.14}$$

and

$$|\langle C^* [\cos(A) - I] Bx, y\rangle|^2 \leq \left[\frac{\cosh(\|A\|) - 1}{\|A\|}\right]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{2.15}$$

for all $x, y \in H$.

We also have

$$|\langle C^* \sinh(A) Bx, y\rangle|^2 \leq \left[\frac{\sinh(\|A\|)}{\|A\|}\right]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{2.16}$$

and

$$|\langle C^* [\cosh(A) - I] Bx, y\rangle|^2 \leq \left[\frac{\cosh(\|A\|) - 1}{\|A\|}\right]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{2.17}$$

for all $x, y \in H$.

3. Norm and Numerical Radius Inequalities

The following vector inequality for positive operators $A \geq 0$, obtained by C. A. McCarthy in [14] is well known,

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle, \ p \geq 1$$

for $x \in H$, $\|x\| = 1$.

Buzano's inequality [5],

$$\frac{1}{2}[\|x\| \|y\| + |\langle x, y\rangle|] \geq |\langle x, e\rangle \langle e, y\rangle| \tag{3.1}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$ will also be used in the sequel.

Our first main result is as follows:

Theorem 3. *Assume that the power series with complex coefficients* $f(z) := \sum_{k=1}^{\infty} a_k z^k$ *is convergent on* $D(0,R)$, $\alpha \in [0,1]$ *and* $A, B, C \in B(H)$ *with* $\|A\| < R$, *then we have the norm inequality*

$$\|C^* f(A) B\| \leq \frac{f_a(\|A\|)}{\|A\|} \||A|^\alpha B\| \left\||A^*|^{1-\alpha} C\right\|. \tag{3.2}$$

We also have the numerical radius inequalities

$$\omega\left(C^{*}f\left(A\right)B\right)\leq\frac{1}{2}\frac{f_{a}\left(\left\Vert A\right\Vert\right)}{\left\Vert A\right\Vert}\left\Vert\left\vert\left\vert A\right\vert^{\alpha}B\right\vert^{2}+\left\vert\left\vert A^{*}\right\vert^{1-\alpha}C\right\vert^{2}\right\Vert \tag{3.3}$$

and

$$\begin{aligned}\omega^{2}\left(C^{*}f\left(A\right)B\right)\leq{}&\frac{1}{2}\left[\frac{f_{a}\left(\left\Vert A\right\Vert\right)}{\left\Vert A\right\Vert}\right]^{2}\\&\times\left[\left\Vert\left\vert A\right\vert^{\alpha}B\right\Vert^{2}\left\Vert\left\vert A^{*}\right\vert^{1-\alpha}C\right\Vert^{2}+\omega\left(\left\vert\left\vert A^{*}\right\vert^{1-\alpha}C\right\vert^{2}\left\vert\left\vert A\right\vert^{\alpha}B\right\vert^{2}\right)\right].\end{aligned} \tag{3.4}$$

Proof. We have from (2.7), by taking the supremum over $\|x\|=\|y\|=1$, that

$$\begin{aligned}&\left\Vert C^{*}f\left(A\right)B\right\Vert^{2}\\&=\sup_{\|x\|=\|y\|=1}\left\vert\left\langle C^{*}f\left(A\right)Bx,y\right\rangle\right\vert^{2}\\&\leq\left[\frac{f_{a}\left(\left\Vert A\right\Vert\right)}{\left\Vert A\right\Vert}\right]^{2}\sup_{\|x\|=1}\left\langle\left\vert\left\vert A\right\vert^{\alpha}B\right\vert^{2}x,x\right\rangle\sup_{\|y\|=1}\left\langle\left\vert\left\vert A^{*}\right\vert^{1-\alpha}C\right\vert^{2}y,y\right\rangle\\&=\left[\frac{f_{a}\left(\left\Vert A\right\Vert\right)}{\left\Vert A\right\Vert}\right]^{2}\left\Vert\left\vert\left\vert A\right\vert^{\alpha}B\right\vert^{2}\right\Vert\left\Vert\left\vert\left\vert A^{*}\right\vert^{1-\alpha}C\right\vert^{2}\right\Vert\\&=\left[\frac{f_{a}\left(\left\Vert A\right\Vert\right)}{\left\Vert A\right\Vert}\right]^{2}\left\Vert\left\vert A\right\vert^{\alpha}B\right\Vert^{2}\left\Vert\left\vert A^{*}\right\vert^{1-\alpha}C\right\Vert^{2},\end{aligned}$$

which gives (3.2).

From (2.7) we get, by taking $y=x$, the square root and using the *A-G-mean inequality*, that

$$\begin{aligned}&\left\vert\left\langle C^{*}f\left(A\right)Bx,x\right\rangle\right\vert\\&\leq\frac{f_{a}\left(\left\Vert A\right\Vert\right)}{\left\Vert A\right\Vert}\left\langle\left\vert\left\vert A\right\vert^{\alpha}B\right\vert^{2}x,x\right\rangle^{1/2}\left\langle\left\vert\left\vert A^{*}\right\vert^{1-\alpha}C\right\vert^{2}x,x\right\rangle^{1/2}\\&\leq\frac{1}{2}\frac{f_{a}\left(\left\Vert A\right\Vert\right)}{\left\Vert A\right\Vert}\left(\left\langle\left\vert\left\vert A\right\vert^{\alpha}B\right\vert^{2}x,x\right\rangle+\left\langle\left\vert\left\vert A^{*}\right\vert^{1-\alpha}C\right\vert^{2}x,x\right\rangle\right)\\&=\frac{1}{2}\frac{f_{a}\left(\left\Vert A\right\Vert\right)}{\left\Vert A\right\Vert}\left\langle\left(\left\vert\left\vert A\right\vert^{\alpha}B\right\vert^{2}+\left\vert\left\vert A^{*}\right\vert^{1-\alpha}C\right\vert^{2}\right)x,x\right\rangle\end{aligned} \tag{3.5}$$

for all $x\in H$.

By taking the supremum over $\|x\|=1$ in (3.5) we get that

$$\begin{aligned}&\omega\left(C^{*}f\left(A\right)B\right)\\&=\sup_{\|x\|=1}\left\vert\left\langle C^{*}f\left(A\right)Bx,x\right\rangle\right\vert\\&\leq\frac{1}{2}\frac{f_{a}\left(\left\Vert A\right\Vert\right)}{\left\Vert A\right\Vert}\sup_{\|x\|=1}\left\langle\left(\left\vert\left\vert A\right\vert^{\alpha}B\right\vert^{2}+\left\vert\left\vert A^{*}\right\vert^{1-\alpha}C\right\vert^{2}\right)x,x\right\rangle\\&=\frac{1}{2}\frac{f_{a}\left(\left\Vert A\right\Vert\right)}{\left\Vert A\right\Vert}\left\Vert\left\vert\left\vert A\right\vert^{\alpha}B\right\vert^{2}+\left\vert\left\vert A^{*}\right\vert^{1-\alpha}C\right\vert^{2}\right\Vert,\end{aligned}$$

which proves (3.3).

From (2.7) for $y=x$ and Buzano's inequality we derive that

$$
\begin{aligned}
(3.6)\qquad & \left|\left\langle C^{*}f\left(A\right)Bx,x\right\rangle\right|^{2} \\
& \leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2}\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle \\
& \leq\frac{1}{2}\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2} \\
& \times\left[\left\|\left||A|^{\alpha}B\right|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle\left||A|^{\alpha}B\right|^{2}x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle\right|\right] \\
& =\frac{1}{2}\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2} \\
& \times\left[\left\|\left||A|^{\alpha}B\right|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right]
\end{aligned}
$$

for all $x\in H$.

By taking the supremum over $\|x\|=1$ in (3.6) we get that

$$
\begin{aligned}
& \omega^{2}\left(C^{*}f\left(A\right)B\right) \\
& =\sup_{\|x\|=1}\left|\left\langle C^{*}f\left(A\right)Bx,x\right\rangle\right|^{2} \\
& \leq\frac{1}{2}\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2} \\
& \times\sup_{\|x\|=1}\left[\left\|\left||A|^{\alpha}B\right|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right] \\
& \leq\frac{1}{2}\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2} \\
& \times\left[\sup_{\|x\|=1}\left\{\left\|\left||A|^{\alpha}B\right|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\|\right\}\right. \\
& \left.+\sup_{\|x\|=1}\left|\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right] \\
& \leq\frac{1}{2}\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2} \\
& \times\left[\sup_{\|x\|=1}\left\|\left||A|^{\alpha}B\right|^{2}x\right\|\sup_{\|x\|=1}\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\|\right. \\
& \left.+\sup_{\|x\|=1}\left|\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right] \\
& =\frac{1}{2}\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2}\left[\left\|\left||A|^{\alpha}B\right|^{2}\right\|\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}\right\|+\omega\left(\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right)\right] \\
& =\frac{1}{2}\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2}\left[\left\||A|^{\alpha}B\right\|^{2}\left\||A^{*}|^{1-\alpha}C\right\|^{2}+\omega\left(\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right)\right],
\end{aligned}
$$

which proves (3.4). □

Remark 2. *If we take $\alpha = 1/2$ in Theorem 3, then we get the norm inequality*

$$\|C^* f(A) B\| \le \frac{f_a(\|A\|)}{\|A\|} \left\| |A|^{1/2} B \right\| \left\| |A^*|^{1/2} C \right\| \tag{3.7}$$

and the numerical radius inequalities

$$\omega(C^* f(A) B) \le \frac{1}{2} \frac{f_a(\|A\|)}{\|A\|} \left\| \left| |A|^{1/2} B \right|^2 + \left| |A^*|^{1/2} C \right|^2 \right\| \tag{3.8}$$

and

$$\begin{aligned} &\omega^2(C^* f(A) B) \\ &\le \frac{1}{2} \left[\frac{f_a(\|A\|)}{\|A\|} \right]^2 \\ &\times \left[\left\| |A|^{1/2} B \right\|^2 \left\| |A^*|^{1/2} C \right\|^2 + \omega \left(\left| |A^*|^{1/2} C \right|^2 \left| |A|^{1/2} B \right|^2 \right) \right]. \end{aligned} \tag{3.9}$$

The second main result is as follows:

Theorem 4. *Assume that the conditions of Theorem 3 are satisfied. If $\alpha \in [0, 1]$, $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 1$, then*

$$\begin{aligned} \omega^{2r}(C^* f(A) B) \le & \left[\frac{f_a(\|A\|)}{\|A\|} \right]^{2r} \\ & \times \left\| \frac{1}{p} \left| |A|^\alpha B \right|^{2rp} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2rq} \right\|. \end{aligned} \tag{3.10}$$

If $r \ge 1$, then

$$\begin{aligned} \omega^{2r}(C^* f(A) B) \le \frac{1}{2} \left[\frac{f_a(\|A\|)}{\|A\|} \right]^{2r} & \left[\left\| |A|^\alpha B \right\|^{2r} \left\| |A^*|^{1-\alpha} C \right\|^{2r} \right. \\ & \left. + \omega^r \left(\left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^\alpha B \right|^2 \right) \right]. \end{aligned} \tag{3.11}$$

If $r \ge 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 2$, then also

$$\begin{aligned} &\omega^{2r}(C^* f(A) B) \\ &\le \frac{1}{2} \left[\frac{f_a(\|A\|)}{\|A\|} \right]^{2r} \left(\left\| \frac{1}{p} \left| |A|^\alpha B \right|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2qr} \right\| \right. \\ &\left. + \omega^r \left(\left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^\alpha B \right|^2 \right) \right). \end{aligned} \tag{3.12}$$

Proof. If we take the power $r>0$ in (2.7) written for $y=x$ then we get, by Young and McCarthy inequalities that

$$
\begin{aligned}
&\left|\left\langle C^{*}f\left(A\right)Bx,x\right\rangle\right|^{2r}\\
&\leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{r}\\
&\leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\\
&\times\left[\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{rp}+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{rq}\right]\\
&\leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\\
&\times\left[\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2rp}x,x\right\rangle+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}x,x\right\rangle\right]\\
&=\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\\
&\times\left[\left\langle\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}x,x\right\rangle\right]
\end{aligned}
$$

for $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, then we get that

$$
\begin{aligned}
&\omega^{2r}\left(C^{*}f\left(A\right)B\right)\\
&=\sup_{\|x\|=1}\left|\left\langle C^{*}f\left(A\right)Bx,x\right\rangle\right|^{2r}\\
&\leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\sup_{\|x\|=1}\left[\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}\right)x,x\right\rangle\right]\\
&=\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\left\|\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}\right\|,
\end{aligned}
$$

which proves (3.10).

If we take the power $r \geq 1$ in (3.6) and by using the convexity of the power function, we get

$$
\begin{aligned}
(3.13)\quad & |\langle C^* f(A) Bx, x\rangle|^{2r} \\
&= \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \\
&\times \left[\frac{\left\| \left||A|^{\alpha} B\right|^2 x\right\| \left\| \left||A^*|^{1-\alpha} C\right|^2 x\right\| + \left|\left\langle \left||A^*|^{1-\alpha} C\right|^2 \left||A|^{\alpha} B\right|^2 x, x\right\rangle\right|}{2}\right]^r \\
&\leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \\
&\times \frac{\left\| \left||A|^{\alpha} B\right|^2 x\right\|^r \left\| \left||A^*|^{1-\alpha} C\right|^2 x\right\|^r + \left|\left\langle \left||A^*|^{1-\alpha} C\right|^2 \left||A|^{\alpha} B\right|^2 x, x\right\rangle\right|^r}{2}
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$
\begin{aligned}
& \omega^{2r}(C^* f(A) B) \\
&\leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \\
&\times \frac{\left\| \left||A|^{\alpha} B\right|^2\right\|^r \left\| \left||A^*|^{1-\alpha} C\right|^2\right\|^r + \omega^r\left(\left||A^*|^{1-\alpha} C\right|^2 \left||A|^{\alpha} B\right|^2\right)}{2} \\
&= \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \\
&\times \frac{\left\||A|^{\alpha} B\right\|^{2r} \left\||A^*|^{1-\alpha} C\right\|^{2r} + \omega^r\left(\left||A^*|^{1-\alpha} C\right|^2 \left||A|^{\alpha} B\right|^2\right)}{2},
\end{aligned}
$$

which proves (3.11).

Also, observe that

$$
\begin{aligned}
& \left\| \left||A|^{\alpha} B\right|^2 x\right\|^r \left\| \left||A^*|^{1-\alpha} C\right|^2 x\right\|^r \\
&\leq \frac{1}{p} \left\| \left||A|^{\alpha} B\right|^2 x\right\|^{pr} + \frac{1}{q} \left\| \left||A^*|^{1-\alpha} C\right|^2 x\right\|^{qr} \\
&= \frac{1}{p} \left\| \left||A|^{\alpha} B\right|^2 x\right\|^{2\frac{pr}{2}} + \frac{1}{q} \left\| \left||A^*|^{1-\alpha} C\right|^2 x\right\|^{2\frac{qr}{2}} \\
&= \frac{1}{p} \left\langle \left||A|^{\alpha} B\right|^4 x, x\right\rangle^{\frac{pr}{2}} + \frac{1}{q} \left\langle \left||A^*|^{1-\alpha} C\right|^4 x, x\right\rangle^{\frac{qr}{2}} \\
&\leq \frac{1}{p} \left\langle \left||A|^{\alpha} B\right|^{2pr} x, x\right\rangle + \frac{1}{q} \left\langle \left||A^*|^{1-\alpha} C\right|^{2qr} x, x\right\rangle \\
&= \left\langle \left(\frac{1}{p} \left||A|^{\alpha} B\right|^{2pr} + \frac{1}{q} \left||A^*|^{1-\alpha} C\right|^{2qr}\right) x, x\right\rangle,
\end{aligned}
$$

for $x \in H$ with $\|x\| = 1$. Then

$$\frac{\left\| \left| |A|^{\alpha} B \right|^{2} x \right\|^{r} \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\|^{r} + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right|^{r}}{2}$$
$$\leq \frac{1}{2} \left[\left\langle \left(\frac{1}{p} \left| |A|^{\alpha} B \right|^{2pr} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} C \right|^{2qr} \right) x, x \right\rangle \right.$$
$$\left. + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right|^{r} \right]$$

and by (3.13)

$$\left| \langle C^{*} f(A) Bx, x \rangle \right|^{2r}$$
$$\leq \frac{1}{2} \left[\frac{f_{a}(\|A\|)}{\|A\|} \right]^{2r} \left[\left\langle \left(\frac{1}{p} \left| |A|^{\alpha} B \right|^{2pr} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} C \right|^{2qr} \right) x, x \right\rangle \right.$$
$$\left. + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right|^{r} \right]$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, we derive (3.12). □

Remark 3. *If we take $r = 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ in (3.10), then we obtain*

$$\omega^{2}(C^{*} f(A) B) \leq \left[\frac{f_{a}(\|A\|)}{\|A\|} \right]^{2} \left\| \frac{1}{p} \left| |A|^{\alpha} B \right|^{2p} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} C \right|^{2q} \right\|, \tag{3.14}$$

which for $p = q = 2$ gives

$$\omega^{2}(C^{*} f(A) B) \leq \frac{1}{2} \left[\frac{f_{a}(\|A\|)}{\|A\|} \right]^{2} \left\| \left| |A|^{\alpha} B \right|^{4} + \left| |A^{*}|^{1-\alpha} C \right|^{4} \right\|. \tag{3.15}$$

If we take $r = 1$ and $p = q = 2$ in (3.12), then we get

$$\omega^{2}(C^{*} f(A) B) \tag{3.16}$$
$$\leq \frac{1}{2} \left[\frac{f_{a}(\|A\|)}{\|A\|} \right]^{2} \left(\frac{1}{2} \left\| \left| |A|^{\alpha} B \right|^{4} + \left| |A^{*}|^{1-\alpha} C \right|^{4} \right\| \right.$$
$$\left. + \omega \left(\left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} \right) \right).$$

If we take $r = 2$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ in (3.12), then we get

$$\omega^{4}(C^{*} f(A) B) \tag{3.17}$$
$$\leq \frac{1}{2} \left[\frac{f_{a}(\|A\|)}{\|A\|} \right]^{4} \left(\left\| \frac{1}{p} \left| |A|^{\alpha} B \right|^{4p} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} C \right|^{4q} \right\| \right.$$
$$\left. + \omega^{2} \left(\left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} \right) \right).$$

We also have:

Theorem 5. *With the assumptions of Theorem 3, we have for $r \geq 1$, $\lambda \in [0,1]$ that*

$$\begin{aligned}(3.18)\qquad & \omega^2 (C^* f(A) B) \\ & \leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \left\|(1-\lambda)\left||A|^\alpha B\right|^{2r} + \lambda \left||A^*|^{1-\alpha} C\right|^{2r}\right\|^{1/r} \\ & \times \left\||A|^\alpha B\right\|^{2\lambda} \left\||A^*|^{1-\alpha} C\right\|^{2(1-\lambda)}\end{aligned}$$

for all $\alpha \in [0,1]$.

Also, we have

$$\begin{aligned}(3.19)\qquad & \omega^2 (C^* f(A) B) \\ & \leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \left\|(1-\lambda)\left||A|^\alpha B\right|^{2r} + \lambda \left||A^*|^{1-\alpha} C\right|^{2r}\right\|^{1/r} \\ & \times \left\|\lambda \left||A|^\alpha B\right|^{2r} + (1-\lambda)\left||A^*|^{1-\alpha} C\right|^{2r}\right\|^{1/r}\end{aligned}$$

for all $\alpha \in [0,1]$ and $r \geq 1$.

Proof. From the first part of (3.6) we have

$$\begin{aligned}& |\langle C^* f(A) Bx, x\rangle|^2 \\ & \leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle \\ & = \left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle^{1-\lambda} \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle^{\lambda} \\ & \times \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle^{\alpha} \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle^{1-\lambda} \\ & \leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \\ & \times \left[(1-\lambda)\left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle + \lambda \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle\right] \\ & \times \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle^{\lambda} \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle^{1-\lambda}\end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

If we take the power $r \geq 1$, then we get by the convexity of power r that

$$\begin{aligned}
&|\langle C^* f(A) Bx, x\rangle|^{2r} \\
&\leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \\
&\times \left[(1-\lambda)\left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle + \lambda \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle\right]^r \\
&\times \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle^{r\lambda} \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle^{r(1-\lambda)} \\
&\leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \\
&\times \left[(1-\lambda)\left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle^r + \lambda \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle^r\right] \\
&\times \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle^{r\lambda} \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle^{r(1-\lambda)}
\end{aligned} \tag{3.20}$$

for all $x \in H$, $\|x\| = 1$.

If we use McCarthy inequality for power $r \geq 1$, then we get

$$\begin{aligned}
&(1-\lambda)\left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle^r + \lambda \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle^r \\
&\leq (1-\lambda)\left\langle \left||A|^{\alpha} B\right|^{2r} x, x\right\rangle + \lambda \left\langle x, \left||A^*|^{1-\alpha} C\right|^{2r} x\right\rangle \\
&= \left\langle \left[(1-\lambda)\left||A|^{\alpha} B\right|^{2r} + \lambda \left||A^*|^{1-\alpha} C\right|^{2r}\right] x, x\right\rangle
\end{aligned}$$

and by (3.20)

$$\begin{aligned}
&|\langle C^* f(A) Bx, x\rangle|^{2r} \\
&\leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \\
&\times \left[\left\langle \left[(1-\lambda)\left||A|^{\alpha} B\right|^{2r} + \lambda \left||A^*|^{1-\alpha} C\right|^{2r}\right] x, x\right\rangle\right] \\
&\times \left\langle \left||A|^{\alpha} B\right|^2 x, x\right\rangle^{r\lambda} \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x\right\rangle^{r(1-\lambda)}
\end{aligned} \tag{3.21}$$

for all $x \in H$, $\|x\| = 1$.

If we take the supremum over $\|x\|=1$, then we get

$$\begin{aligned}
&\omega^{2r}\left(C^{*}f\left(A\right)B\right)\\
&=\sup_{\|x\|=1}\left|\left\langle C^{*}f\left(A\right)Bx,x\right\rangle\right|^{2r}\\
&\leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\\
&\times\sup_{\|x\|=1}\left[\left\langle\left[\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\\
&\times\sup_{\|x\|=1}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r\lambda}\sup_{\|x\|=1}\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{r(1-\lambda)}\\
&=\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\\
&\times\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\\
&\times\left\|\left|A\right|^{\alpha}B\right\|^{2r\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2r(1-\lambda)},
\end{aligned}$$

which gives (3.18).

We also have

$$\begin{aligned}
&\left|\left\langle C^{*}f\left(A\right)Bx,x\right\rangle\right|^{2r}\\
&\leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\\
&\times\left[\left\langle\left[\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\\
&\times\left[\left\langle\left[\lambda\left|\left|A\right|^{\alpha}B\right|^{2r}+\left(1-\lambda\right)\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]
\end{aligned}$$

for all $x\in H$, $\|x\|=1$, which proves (3.19). □

Remark 4. *If we take $r=1$ in Theorem 5, then we get*

$$\begin{aligned}
&\omega^{2}\left(C^{*}f\left(A\right)B\right)\\
&\leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2}\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\\
&\times\left\|\left|A\right|^{\alpha}B\right\|^{2\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2(1-\lambda)}
\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}
&\omega^{2}\left(C^{*}f\left(A\right)B\right)\\
&\leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2}\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|\\
&\times\left\|\lambda\left|\left|A\right|^{\alpha}B\right|^{2}+\left(1-\lambda\right)\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|
\end{aligned}\tag{3.23}$$

for all α, $\lambda\in[0,1]$.

If we take $\lambda = 1/2$ in (3.22), then we obtain

$$\omega^2 \left(C^* f(A) B\right) \leq \frac{1}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \||A|^\alpha B\| \left\||A^*|^{1-\alpha} C\right\| \times \left\|\left||A|^\alpha B\right|^2 + \left||A^*|^{1-\alpha} C\right|^{2r}\right\|. \tag{3.24}$$

If we take $r = 2$ in Theorem 5, then we get

$$\omega^2 \left(C^* f(A) B\right) \leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \||A|^\alpha B\|^{2\lambda} \left\||A^*|^{1-\alpha} C\right\|^{2(1-\lambda)} \times \left\|(1-\lambda)\left||A|^\alpha B\right|^4 + \lambda\left||A^*|^{1-\alpha} C\right|^4\right\|^{1/2} \tag{3.25}$$

and

$$\begin{aligned} &\omega^2 \left(C^* f(A) B\right) \\ &\leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \left\|(1-\lambda)\left||A|^\alpha B\right|^4 + \lambda\left||A^*|^{1-\alpha} C\right|^4\right\|^{1/2} \\ &\times \left\|\lambda\left||A|^\alpha B\right|^4 + (1-\lambda)\left||A^*|^{1-\alpha} C\right|^4\right\|^{1/2} \end{aligned} \tag{3.26}$$

for all $\alpha, \lambda \in [0,1]$.

If we take $\lambda = 1/2$ in (3.25), then we obtain

$$\omega^2 \left(C^* f(A) B\right) \leq \frac{\sqrt{2}}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \||A|^\alpha B\| \left\||A^*|^{1-\alpha} C\right\| \times \left\|\left||A|^\alpha B\right|^4 + \left||A^*|^{1-\alpha} C\right|^4\right\|^{1/2}. \tag{3.27}$$

4. Inequalities for Trace of Operators

We have the following result for trace of operators:

Theorem 6. *Let $r \geq 1/2$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$. Assume that the power series with complex coefficients $f(z) := \sum_{k=1}^\infty a_k z^k$ is convergent on $D(0,R)$ and A, B, $C \in B(H)$ with $\|A\| < R$. If $|A|^\alpha B \in \mathcal{B}_{2pr}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2qr}(H)$ for $\alpha \in [0,1]$, then $C^* f(A) B \in \mathcal{B}_{2r}(H)$ and*

$$\|C^* f(A) B\|_{2r} \leq \frac{f_a(\|A\|)}{\|A\|} \||A|^\alpha B\|_{2pr} \left\||A^*|^{1-\alpha} C\right\|_{2qr}. \tag{4.1}$$

In particular,

$$\|C^* f(A) B\|_{2r} \leq \frac{f_a(\|A\|)}{\|A\|} \left\||A|^{1/2} B\right\|_{2pr} \left\||A^*|^{1/2} C\right\|_{2qr} \tag{4.2}$$

for $|A|^{1/2} B \in \mathcal{B}_{2pr}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2qr}(H)$.

Proof. If we take in (2.7) the power $r>0$ and $x=e_i$, $y=f_i$ where $\mathcal{E}=\{e_i\}_{i\in I}$ and $\mathcal{F}=\{f_i\}_{i\in I}$ are orthonormal basis and sum, then we get

$$\sum_{i\in I}\left|\left\langle C^{*}f\left(A\right)Be_i,f_i\right\rangle\right|^{2r}\le\left[\frac{f_a\left(\|A\|\right)}{\|A\|}\right]^{2r}\times\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_i,f_i\right\rangle^{r}. \tag{4.3}$$

If we use the Hölder's inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we get

$$\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_i,f_i\right\rangle^{r}\le\left(\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{pr}\right)^{1/p}\left(\sum_{i\in I}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_i,f_i\right\rangle^{qr}\right)^{1/q}. \tag{4.4}$$

By the McCarthy inequality for pr, $qr\ge1$, we have

$$\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{pr}\le\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2pr}e_i,e_i\right\rangle$$

and

$$\sum_{i\in I}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_i,f_i\right\rangle^{qr}\le\sum_{i\in I}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}f_i,f_i\right\rangle,$$

therefore

$$\begin{aligned}&\left(\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{pr}\right)^{1/p}\left(\sum_{i\in I}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_i,f_i\right\rangle^{qr}\right)^{1/q}\\&\le\left(\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2pr}e_i,e_i\right\rangle\right)^{1/p}\left(\sum_{i\in I}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_i,f_i\right\rangle^{qr}\right)^{1/q}\\&=\left(\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2pr}\right)^{1/p}\left(\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2qr}\right)^{1/q}=\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}.\end{aligned}$$

By (4.3) and (4.4) we derive

$$\sum_{i\in I}\left|\left\langle C^{*}f\left(A\right)Be_i,f_i\right\rangle\right|^{2r}\le\left[\frac{f_a\left(\|A\|\right)}{\|A\|}\right]^{2r}\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}. \tag{4.5}$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (2.8), then by (1.21) we get

$$\left\|C^{*}f\left(A\right)B\right\|_{2r}^{2r}\le\left[\frac{f_a\left(\|A\|\right)}{\|A\|}\right]^{2r}\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}$$

and the inequality (4.1) is obtained. □

Remark 5. *If we take $r=1/2$ and $p=q=2$, then by (4.1) we get*

$$\left\|C^{*}f\left(A\right)B\right\|_{1}\le\frac{f_a\left(\|A\|\right)}{\|A\|}\left\|\left|A\right|^{\alpha}B\right\|_{2}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2} \tag{4.6}$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_2\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_2\left(H\right)$ for $\alpha\in[0,1]$.

Also, if $r=1$ *and* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*, then by (4.1) we get*

$$\|C^{*}f(A)B\|_{2}\leq\frac{f_{a}(\|A\|)}{\|A\|}\||A|^{\alpha}B\|_{2p}\left\||A^{*}|^{1-\alpha}C\right\|_{2q} \tag{4.7}$$

provided that $|A|^{\alpha}B\in\mathcal{B}_{2p}(H)$ *and* $|A^{*}|^{1-\alpha}C\in\mathcal{B}_{2q}(H)$ *for* $\alpha\in[0,1]$.

We also have:

Theorem 7. *Let* $r\geq 1/2$, $p,\ q\geq 1$ *with* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. *Assume that the power series with complex coefficients* $f(z):=\sum_{k=1}^{\infty}a_{k}z^{k}$ *is convergent on* $D(0,R)$ *and* $A,\ B,\ C\in B(H)$ *with* $\|A\|<R$. *If* $|A|^{\alpha}B\in\mathcal{B}_{2p}(H)$ *and* $|A^{*}|^{1-\alpha}C\in\mathcal{B}_{2q}(H)$ *for* $\alpha\in[0,1]$, *then* $C^{*}f(A)B\in\mathcal{B}_{2r}(H)$ *and*

$$\|C^{*}f(A)B\|_{2r}\leq\frac{f_{a}(\|A\|)}{\|A\|}\||A|^{\alpha}B\|_{2p}\left\||A^{*}|^{1-\alpha}C\right\|_{2q}. \tag{4.8}$$

In particular,

$$\|C^{*}f(A)B\|_{2r}\leq\frac{f_{a}(\|A\|)}{\|A\|}\left\||A|^{1/2}B\right\|_{2p}\left\||A^{*}|^{1/2}C\right\|_{2q} \tag{4.9}$$

for $|A|^{1/2}B\in\mathcal{B}_{2p}(H)$ *and* $|A^{*}|^{1/2}C\in\mathcal{B}_{2q}(H)$.

Proof. Assume that $\mathcal{E}=\{e_{i}\}_{i\in I}$ and $\mathcal{F}=\{f_{i}\}_{i\in I}$ are orthonormal basis in H. Observe that we have $\frac{1}{\frac{p}{r}}+\frac{1}{\frac{q}{r}}=1$ and by Hölder's inequality for $\frac{p}{r}$ and $\frac{q}{r}$ we have

$$\begin{aligned}&\sum_{i\in I}\left\langle||A|^{\alpha}B|^{2}e_{i},e_{i}\right\rangle^{r}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{r}\\&=\sum_{i\in I}\left[\left\langle||A|^{\alpha}B|^{2}e_{i},e_{i}\right\rangle^{p}\right]^{\frac{r}{p}}\left[\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{q}\right]^{\frac{r}{q}}\\&\leq\left(\sum_{i\in I}\left\langle||A|^{\alpha}B|^{2}e_{i},e_{i}\right\rangle^{p}\right)^{r/p}\left(\sum_{i\in I}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{q}\right)^{r/q}.\end{aligned} \tag{4.10}$$

By McCarthy inequality for $p,\ q>1$ we get

$$\sum_{i\in I}\left\langle||A|^{\alpha}B|^{2}e_{i},e_{i}\right\rangle^{p}\leq\sum_{i\in I}\left\langle||A|^{\alpha}B|^{2p}e_{i},e_{i}\right\rangle$$

and

$$\sum_{i\in I}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{q}\leq\sum_{i\in I}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2q}f_{i},f_{i}\right\rangle$$

and by (4.10) we derive

$$\begin{aligned}&\sum_{i\in I}\left\langle||A|^{\alpha}B|^{2}e_{i},e_{i}\right\rangle^{r}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{r}\\&\leq\left(\sum_{i\in I}\left\langle||A|^{\alpha}B|^{2p}e_{i},e_{i}\right\rangle\right)^{r/p}\left(\sum_{i\in I}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2q}f_{i},f_{i}\right\rangle\right)^{r/q}\\&=\||A|^{\alpha}B\|_{2p}^{2r}\left\||A^{*}|^{1-\alpha}C\right\|_{2q}^{2r}.\end{aligned} \tag{4.11}$$

By (4.3) and (4.11) we get

$$\sum_{i\in I}\left|\left\langle C^{*}f\left(A\right)Be_{i},f_{i}\right\rangle\right|^{2r}\leq\left[\frac{f_{a}\left(\left\|A\right\|\right)}{\left\|A\right\|}\right]^{2r}\left\|\left|A\right|^{\alpha}B\right\|_{2p}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2q}^{2r}. \tag{4.12}$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (4.12) we get

$$\left\|C^{*}f\left(A\right)B\right\|_{2r}^{2r}\leq\left[\frac{f_{a}\left(\left\|A\right\|\right)}{\left\|A\right\|}\right]^{2r}\left\|\left|A\right|^{\alpha}B\right\|_{2p}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2q}^{2r}$$

and the inequality (4.8) is thus proved. □

Remark 6. *If we take $p=q=2r=s\geq1$, then by (4.8) we get*

$$\left\|C^{*}f\left(A\right)B\right\|_{s}\leq\frac{f_{a}\left(\left\|A\right\|\right)}{\left\|A\right\|}\left\|\left|A\right|^{\alpha}B\right\|_{2s}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2s} \tag{4.13}$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_{2s}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2s}\left(H\right)$ for $\alpha\in\left[0,1\right]$.

For $\alpha=1/2$ we have

$$\left\|C^{*}f\left(A\right)B\right\|_{s}\leq\frac{f_{a}\left(\left\|A\right\|\right)}{\left\|A\right\|}\left\|\left|A\right|^{1/2}B\right\|_{2s}\left\|\left|A^{*}\right|^{1/2}C\right\|_{2s} \tag{4.14}$$

provided that $\left|A\right|^{1/2}B\in\mathcal{B}_{2s}\left(H\right)$ and $\left|A^{*}\right|^{1/2}C\in\mathcal{B}_{2s}\left(H\right)$.

If $r=2$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, then

$$\left\|C^{*}f\left(A\right)B\right\|_{4}\leq\frac{f_{a}\left(\left\|A\right\|\right)}{\left\|A\right\|}\left\|\left|A\right|^{\alpha}B\right\|_{2p}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2q} \tag{4.15}$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_{2p}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2q}\left(H\right)$ for $\alpha\in\left[0,1\right]$.

In particular,

$$\left\|C^{*}f\left(A\right)B\right\|_{4}\leq\frac{f_{a}\left(\left\|A\right\|\right)}{\left\|A\right\|}\left\|\left|A\right|^{1/2}B\right\|_{2p}\left\|\left|A^{*}\right|^{1/2}C\right\|_{2q} \tag{4.16}$$

for $\left|A\right|^{1/2}B\in\mathcal{B}_{2p}\left(H\right)$ and $\left|A^{*}\right|^{1/2}C\in\mathcal{B}_{2q}\left(H\right)$.

Theorem 8. *Assume that the power series with complex coefficients $f\left(z\right):=\sum_{k=1}^{\infty}a_{k}z^{k}$ is convergent on $D(0,R)$, $A,\ B,\ C\in B\left(H\right)$ with $\left\|A\right\|<R$.*

If $r\geq1/2$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $pr,\ qr\geq1$ and $\left|\left|A\right|^{\alpha}B\right|^{2pr},\ \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\in\mathcal{B}_{1}\left(H\right)$, then $C^{*}f\left(A\right)B\in\mathcal{B}_{2r}\left(H\right)$ and

$$\omega_{2r}^{2r}\left(C^{*}f\left(A\right)B\right)\leq\left[\frac{f_{a}\left(\left\|A\right\|\right)}{\left\|A\right\|}\right]^{2r}\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right). \tag{4.17}$$

If $r \geq 1$ *and* $|A|^{\alpha} B, |A^*|^{1-\alpha} C \in \mathcal{B}_{4r}(H)$, *then* $C^* f(A) B \in \mathcal{B}_{2r}(H)$ *and*

$$\begin{aligned}(4.18)\qquad & \omega_{2r}^{2r}(C^* f(A) B) \\ & \leq \frac{1}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \\ & \times \left(\||A|^{\alpha} B\|_{4r}^{2r} \left\||A^*|^{1-\alpha} C\right\|_{4r}^{2r} + \omega_r^r\left(\left|A^*|^{1-\alpha} C\right|^2 ||A|^{\alpha} B|^2\right)\right) \\ & \leq \frac{1}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \\ & \times \left(\||A|^{\alpha} B\|_{4r}^{2r} \left\||A^*|^{1-\alpha} C\right\|_{4r}^{2r} + \left\|\left|A^*|^{1-\alpha} C\right|^2 ||A|^{\alpha} B|^2\right\|_r^r\right).\end{aligned}$$

If $r \geq 1$, p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, $pr, qr \geq 2$, *then*

$$\begin{aligned}(4.19)\qquad & \omega_{2r}^{2r}(C^* f(A) B) \\ & \leq \frac{1}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r}\left[\operatorname{tr}\left(\frac{1}{p}||A|^{\alpha} B|^{2pr} + \frac{1}{q}\left|A^*|^{1-\alpha} C\right|^{2qr}\right)\right. \\ & \left. + \omega_r^r\left(\left|A^*|^{1-\alpha} C\right|^2 ||A|^{\alpha} B|^2\right)\right] \\ & \leq \frac{1}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r}\left[\operatorname{tr}\left(\frac{1}{p}||A|^{\alpha} B|^{2pr} + \frac{1}{q}\left|A^*|^{1-\alpha} C\right|^{2qr}\right)\right. \\ & \left. + \left\|\left|A^*|^{1-\alpha} C\right|^2 ||A|^{\alpha} B|^2\right\|_r^r\right].\end{aligned}$$

Proof. From (2.7) for $y = x$ we have that

$$(4.20)\qquad |\langle C^* f(A) Bx, x\rangle|^2 \leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \left\langle ||A|^{\alpha} B|^2 x, x\right\rangle \left\langle \left|A^*|^{1-\alpha} C\right|^2 x, x\right\rangle$$

for $x \in H$ with $\|x\| = 1$.

If we take the power $r > 0$, we get, by Young and McCarthy inequalities, that

$$\begin{aligned} & |\langle C^* f(A) Bx, x\rangle|^{2r} \\ & \leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \left\langle ||A|^{\alpha} B|^2 x, x\right\rangle^r \left\langle \left|A^*|^{1-\alpha} C\right|^2 x, x\right\rangle^r \\ & \leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \left[\frac{1}{p}\left\langle ||A|^{\alpha} B|^2 x, x\right\rangle^{pr} + \frac{1}{q}\left\langle \left|A^*|^{1-\alpha} C\right|^2 x, x\right\rangle^{qr}\right] \\ & \leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \left[\frac{1}{p}\left\langle ||A|^{\alpha} B|^{2pr} x, x\right\rangle + \frac{1}{q}\left\langle \left|A^*|^{1-\alpha} C\right|^{2qr} x, x\right\rangle\right] \\ & = \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \left\langle \left(\frac{1}{p}||A|^{\alpha} B|^{2pr} + \frac{1}{q}\left|A^*|^{1-\alpha} C\right|^{2qr}\right) x, x\right\rangle\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

If $\mathcal{E}=\{e_i\}_{i\in I}$ is an orthonormal basis, then by taking $x=e_i$ and summing over $i\in I$ we get

$$\begin{aligned}
&\|C^*f(A)B\|_{\mathcal{E},2r}^{2r} \\
&=\sum_{i\in I}|\langle C^*f(A)Be_i,e_i\rangle|^{2r} \\
&\le\left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r}\sum_{i\in I}\left\langle\left(\frac{1}{p}\left||A|^\alpha B\right|^{2pr}+\frac{1}{q}\left||A^*|^{1-\alpha}C\right|^{2qr}\right)e_i,e_i\right\rangle \\
&=\left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r}\operatorname{tr}\left(\frac{1}{p}\left||A|^\alpha B\right|^{2pr}+\frac{1}{q}\left||A^*|^{1-\alpha}C\right|^{2qr}\right),
\end{aligned}$$

which, by taking the supremum over $\mathcal{E}$, proves (4.17).

By Buzano's inequality we have

$$\begin{aligned}
&\left\langle\left||A|^\alpha B\right|^2x,x\right\rangle\left\langle x,\left||A^*|^{1-\alpha}C\right|^2x\right\rangle \\
&\le\frac{1}{2}\left[\left\|\left||A|^\alpha B\right|^2x\right\|\left\|\left||A^*|^{1-\alpha}C\right|^2x\right\|+\left|\left\langle\left||A|^\alpha B\right|^2x,\left||A^*|^{1-\alpha}C\right|^2x\right\rangle\right|\right] \\
&=\frac{1}{2}\left[\left\|\left||A|^\alpha B\right|^2x\right\|\left\|\left||A^*|^{1-\alpha}C\right|^2x\right\|+\left|\left\langle\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2x,x\right\rangle\right|\right]
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

If we take the power $r\ge1$ and use the convexity of power function, then we get

$$\begin{aligned}
&\left\langle\left||A|^\alpha B\right|^2x,x\right\rangle^r\left\langle x,\left||A^*|^{1-\alpha}C\right|^2x\right\rangle^r \\
&\le\left[\frac{\left\|\left||A|^\alpha B\right|^2x\right\|\left\|\left||A^*|^{1-\alpha}C\right|^2x\right\|+\left|\left\langle\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2x,x\right\rangle\right|}{2}\right]^r \\
&\le\frac{\left\|\left||A|^\alpha B\right|^2x\right\|^r\left\|\left||A^*|^{1-\alpha}C\right|^2x\right\|^r+\left|\left\langle\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2x,x\right\rangle\right|^r}{2} \\
&=\frac{\left\|\left||A|^\alpha B\right|^2x\right\|^{2\frac{r}{2}}\left\|\left||A^*|^{1-\alpha}C\right|^2x\right\|^{2\frac{r}{2}}+\left|\left\langle\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2x,x\right\rangle\right|^r}{2} \\
&=\frac{\left\langle\left||A|^\alpha B\right|^4x,x\right\rangle^{\frac{r}{2}}\left\langle\left||A^*|^{1-\alpha}C\right|^4x,x\right\rangle^{\frac{r}{2}}+\left|\left\langle\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2x,x\right\rangle\right|^r}{2}
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

Therefore

$$
\begin{aligned}
&\|C^*f(A)B\|_{\mathcal{E},2r}^{2r} \\
&= \sum_{i\in I} |\langle C^*f(A)Be_i, e_i\rangle|^{2r} \\
&\le \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \sum_{i\in I} \left\langle ||A|^\alpha B|^2 e_i, e_i\right\rangle^r \left\langle e_i, \left||A^*|^{1-\alpha} C\right|^2 e_i\right\rangle^r \\
&\le \frac{1}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r} \\
&\times \left[\sum_{i\in I} \left\langle ||A|^\alpha B|^4 e_i, e_i\right\rangle^{\frac{r}{2}} \left\langle \left||A^*|^{1-\alpha} C\right|^4 e_i, e_i\right\rangle^{\frac{r}{2}}\right. \\
&\left. + \sum_{i\in I} \left|\left\langle \left||A^*|^{1-\alpha} C\right|^2 ||A|^\alpha B|^2 e_i, e_i\right\rangle\right|^r\right].
\end{aligned}
\tag{4.21}
$$

Using Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\sum_{i\in I} \left\langle ||A|^\alpha B|^4 e_i, e_i\right\rangle^{\frac{r}{2}} \left\langle \left||A^*|^{1-\alpha} C\right|^4 e_i, e_i\right\rangle^{\frac{r}{2}} \\
&\le \left(\sum_{i\in I} \left\langle ||A|^\alpha B|^4 e_i, e_i\right\rangle^r\right)^{1/2} \left(\sum_{i\in I} \left\langle \left||A^*|^{1-\alpha} C\right|^4 e_i, e_i\right\rangle^r\right)^{1/2} \\
&\le \left(\sum_{i\in I} \left\langle ||A|^\alpha B|^{4r} e_i, e_i\right\rangle\right)^{1/2} \left(\sum_{i\in I} \left\langle \left||A^*|^{1-\alpha} C\right|^{4r} e_i, e_i\right\rangle\right)^{1/2} \\
&= \||A|^\alpha B\|_{4r}^{2r} \left\||A^*|^{1-\alpha} C\right\|_{4r}^{2r},
\end{aligned}
$$

where for the last inequality we used McCarthy's result for $r \ge 1$. This proves (4.18).

Further, if we use Young's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$
ab \le \frac{1}{p}a^p + \frac{1}{q}b^q, \ a, b \ge 0,
$$

then we get

$$\begin{aligned}
&\left\| \left| |A|^{\alpha} B \right|^{2} x \right\|^{r} \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\|^{r} \\
&\leq \frac{1}{p} \left\| \left| |A|^{\alpha} B \right|^{2} x \right\|^{pr} + \frac{1}{q} \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\|^{qr} \\
&= \frac{1}{p} \left\| \left| |A|^{\alpha} B \right|^{2} x \right\|^{2\frac{pr}{2}} + \frac{1}{q} \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\|^{2\frac{qr}{2}} \\
&= \frac{1}{p} \left\langle \left| |A|^{\alpha} B \right|^{4} x, x \right\rangle^{\frac{pr}{2}} + \frac{1}{q} \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{4} x, x \right\rangle^{\frac{qr}{2}} \\
&\leq \frac{1}{p} \left\langle \left| |A|^{\alpha} B \right|^{2pr} x, x \right\rangle + \frac{1}{q} \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2qr} x, x \right\rangle \\
&= \left\langle \left(\frac{1}{p} \left| |A|^{\alpha} B \right|^{2pr} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} C \right|^{2qr} \right) x, x \right\rangle
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

Therefore

$$\begin{aligned}
&\|C^{*} f(A) B\|_{\mathcal{E},2r}^{2r} \\
&= \sum_{i \in I} \left| \langle C^{*} f(A) B e_{i}, e_{i} \rangle \right|^{2r} \\
&\leq \left[\frac{f_{a}(\|A\|)}{\|A\|} \right]^{2r} \sum_{i \in I} \left\langle \left| |A|^{\alpha} B \right|^{2} e_{i}, e_{i} \right\rangle^{r} \left\langle e_{i}, \left| |A^{*}|^{1-\alpha} C \right|^{2} e_{i} \right\rangle^{r} \\
&\leq \frac{1}{2} \left[\frac{f_{a}(\|A\|)}{\|A\|} \right]^{2r} \\
&\times \left[\sum_{i \in I} \left\langle \left(\frac{1}{p} \left| |A|^{\alpha} B \right|^{2pr} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} C \right|^{2qr} \right) e_{i}, e_{i} \right\rangle \right. \\
&\left. + \sum_{i \in I} \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} e_{i}, e_{i} \right\rangle \right|^{r} \right] \\
&= \frac{1}{2} \left[\frac{f_{a}(\|A\|)}{\|A\|} \right]^{2r} \left[\operatorname{tr} \left(\frac{1}{p} \left| |A|^{\alpha} B \right|^{2pr} + \frac{1}{q} \left| |A^{*}|^{1-\alpha} C \right|^{2qr} \right) \right. \\
&\left. + \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} \right\|_{\mathcal{E},r}^{r} \right],
\end{aligned}$$

which proves, by taking the supremum over $\mathcal{E}$, the desired inequality (4.19). □

Remark 7. *Let* $\alpha \in [0,1]$. *If* $r = 1/2$, p, $q = 2$ *and* $\left| |A|^{\alpha} B \right|^{2}$, $\left| |A^{*}|^{1-\alpha} C \right|^{2} \in \mathcal{B}_{1}(H)$, *then* $C^{*} f(A) B \in \mathcal{B}_{1}(H)$ *and by (4.17) we get*

$$\omega_{1}(C^{*} f(A) B) \leq \frac{1}{2} \frac{f_{a}(\|A\|)}{\|A\|} \operatorname{tr} \left(\left| |A|^{\alpha} B \right|^{2} + \left| |A^{*}|^{1-\alpha} C \right|^{2} \right). \tag{4.22}$$

If $r=1$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (4.17) we obtain

$$\begin{aligned}\omega_2^2\left(C^*f(A)B\right) & \\ \leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \operatorname{tr}\left(\frac{1}{p}\left||A|^\alpha B\right|^{2p}+\frac{1}{q}\left||A^*|^{1-\alpha}C\right|^{2q}\right),\end{aligned} \tag{4.23}$$

provided that $\left||A|^\alpha B\right|^{2p},\ \left||A^*|^{1-\alpha}C\right|^{2q}\in\mathcal{B}_1(H)$.

If we take $r=1$ in (4.18), then we get

$$\begin{aligned}&\omega_2^2\left(C^*f(A)B\right)\\ &\leq \frac{1}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^2\\ &\times\left(\left\||A|^\alpha B\right\|_4^2\left\||A^*|^{1-\alpha}C\right\|_4^2+\omega_1\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right)\right)\\ &\leq \frac{1}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^2\\ &\times\left(\left\||A|^\alpha B\right\|_4^2\left\||A^*|^{1-\alpha}C\right\|_4^2+\left\|\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right\|_1\right),\end{aligned} \tag{4.24}$$

provided that $|A|^\alpha B,\ |A^*|^{1-\alpha}C\in\mathcal{B}_4(H)$.

If $r=1$ and $p=q=2$ in (4.19), then we get for $\left||A|^\alpha B\right|^{2p},\ \left||A^*|^{1-\alpha}C\right|^{2q}\in\mathcal{B}_1(H)$ that

$$\begin{aligned}&\omega_2^2\left(C^*f(A)B\right)\\ &\leq \frac{1}{4}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^2\left[\operatorname{tr}\left(\left||A|^\alpha B\right|^{2p}+\left||A^*|^{1-\alpha}C\right|^{2q}\right)\right.\\ &+\frac{1}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^2\omega_1\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right)\\ &\leq \frac{1}{4}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^2\operatorname{tr}\left(\left||A|^\alpha B\right|^{2p}+\left||A^*|^{1-\alpha}C\right|^{2q}\right)\\ &+\frac{1}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^2\left\|\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right\|_1.\end{aligned} \tag{4.25}$$

We also have:

Theorem 9. *With the assumptions of Theorem 8, we have for $r\geq 1,\ \lambda\in[0,1]$ that*

$$\begin{aligned}&\omega_{2r}^{2r}\left(C^*f(A)B\right)\\ &\leq \left[\frac{f_a(\|A\|)}{\|A\|}\right]^{2r}\left\|(1-\lambda)\left||A|^\alpha B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right\|\\ &\times\left\||A|^\alpha B\right\|_{2r}^{2r\lambda}\left\||A^*|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)},\end{aligned} \tag{4.26}$$

provided that $|A|^\alpha B,\ |A^*|^{1-\alpha}C\in\mathcal{B}_{2r}(H)$.

In particular,

$$\omega_{2r}^{2r}\left(C^{*}f\left(A\right)B\right)\leq\frac{1}{2}\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\left\|\left|\left|A\right|^{\alpha}B\right|^{2r}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\| \times\left\|\left|A\right|^{\alpha}B\right\|_{2r}^{r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2r}^{r}. \tag{4.27}$$

Proof. If $\mathcal{E}=\{e_i\}_{i\in I}$ is an orthonormal basis, then by taking $x=e_i$ in (3.21) and summing over $i\in I$ we get

$$\begin{aligned}&\sum_{i\in I}\left|\left\langle C^{*}f\left(A\right)Be_{i},e_{i}\right\rangle\right|^{2r}\\&\leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\\&\times\sum_{i\in I}\left[\left\langle\left[\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]e_{i},e_{i}\right\rangle\right]\\&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r\lambda}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i},e_{i}\right\rangle^{r(1-\lambda)}\\&\leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2r}\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\\&\times\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r\lambda}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i},e_{i}\right\rangle^{r(1-\lambda)}.\end{aligned} \tag{4.28}$$

If we use Hölder's inequality for $p=\frac{1}{\lambda}$, $q=\frac{1}{1-\lambda}$, then we have

$$\begin{aligned}&\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r\lambda}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i},e_{i}\right\rangle^{r(1-\lambda)}\\&\leq\left(\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r}\right)^{\lambda}\left(\sum_{i\in I}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i},e_{i}\right\rangle^{r}\right)^{1-\lambda}\\&\leq\left(\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2r}e_{i},e_{i}\right\rangle\right)^{\lambda}\left(\sum_{i\in I}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}e_{i},e_{i}\right\rangle\right)^{1-\lambda}\\&=\left\|\left|A\right|^{\alpha}B\right\|_{2r}^{2r\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)},\end{aligned}$$

which proves (4.26). □

Remark 8. *If we take $r=1$ in Theorem 9, then we get for $\alpha\in[0,1]$ that*

$$\begin{aligned}&\omega_{2}^{2}\left(C^{*}f\left(A\right)B\right)\\&\leq\left[\frac{f_{a}\left(\|A\|\right)}{\|A\|}\right]^{2}\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|\\&\times\left\|\left|A\right|^{\alpha}B\right\|_{2}^{2\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2}^{2(1-\lambda)},\end{aligned} \tag{4.29}$$

provided that $\left|A\right|^{\alpha}B$, $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2}\left(H\right)$.

In particular,

$$\omega_2^2\left(C^* f(A) B\right) \le \frac{1}{2}\left[\frac{f_a(\|A\|)}{\|A\|}\right]^2 \left\| \left| |A|^\alpha B\right|^2 + \left| |A^*|^{1-\alpha} C\right|^2 \right\| \times \left\| |A|^\alpha B\right\|_2 \left\| |A^*|^{1-\alpha} C\right\|_2 . \tag{4.30}$$

5. Applications for Perspectives

Let f be a continuous function defined on the interval I of real numbers, Q a self-adjoint operator on the Hilbert space H and P a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(P^{-1/2} Q P^{-1/2}\right) \subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(Q, P)$ by setting

$$\mathcal{P}_f(Q, P) := P^{1/2} f\left(P^{-1/2} Q P^{-1/2}\right) P^{1/2}.$$

If P and B are commutative, then

$$\mathcal{P}_\Phi(Q, P) = P f\left(Q P^{-1}\right)$$

provided $\operatorname{Sp}\left(Q P^{-1}\right) \subset \mathring{I}$.

Assume that $P > 0$ and $T \in B(H)$ with $\left\|P^{-1/2} T P^{-1/2}\right\| < R$ while $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is as in Theorem 2, we can define the *complex operator valued perspective* by,

$$\mathcal{P}_f(T, P) := P^{1/2} f\left(P^{-1/2} T P^{-1/2}\right) P^{1/2}.$$

By the inequality (2.7) we have for $\alpha \in [0,1]$ that

$$\begin{aligned} &\left|\left\langle \mathcal{P}_f(T, P) x, y\right\rangle\right|^2 \\ &\le \left[\frac{f_a\left(\left\|P^{-1/2} T P^{-1/2}\right\|\right)}{\left\|P^{-1/2} T P^{-1/2}\right\|}\right]^2 \\ &\times \left\langle \left|\left|P^{-1/2} T P^{-1/2}\right|^\alpha P^{1/2}\right|^2 x, x\right\rangle \left\langle \left|\left|P^{-1/2} T^* P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^2 y, y\right\rangle \end{aligned} \tag{5.1}$$

for all x, $y \in H$.

Observe that

$$\begin{aligned} \left\langle \left|\left|P^{-1/2} T P^{-1/2}\right|^\alpha P^{1/2}\right|^2 x, x\right\rangle &= \left\|\left|P^{-1/2} T P^{-1/2}\right|^\alpha P^{1/2} x\right\|^2 \\ &\le \left\|\left|P^{-1/2} T P^{-1/2}\right|^\alpha\right\|^2 \left\|P^{1/2} x\right\|^2 \\ &= \left\|P^{-1/2} T P^{-1/2}\right\|^{2\alpha} \langle P x, x\rangle \end{aligned}$$

and

$$\left\langle \left| \left| P^{-1/2}T^{*}P^{-1/2} \right|^{1-\alpha} P^{1/2} \right|^{2} y, y \right\rangle = \left\| \left| P^{-1/2}T^{*}P^{-1/2} \right|^{1-\alpha} P^{1/2}y \right\|^{2}$$
$$\leq \left\| \left| P^{-1/2}T^{*}P^{-1/2} \right|^{1-\alpha} \right\|^{2} \left\| P^{1/2}y \right\|^{2}$$
$$= \left\| P^{-1/2}TP^{-1/2} \right\|^{2(1-\alpha)} \langle Py, y \rangle$$

and by (5.1) we obtain

$$\left| \langle \mathcal{P}_{f}(T,P)x,y \rangle \right|^{2} \leq \left[\frac{f_{a}\left(\left\| P^{-1/2}TP^{-1/2} \right\| \right)}{\left\| P^{-1/2}TP^{-1/2} \right\|} \right]^{2}$$
$$\times \left\| P^{-1/2}TP^{-1/2} \right\|^{2\alpha} \langle Px,x \rangle \left\| P^{-1/2}TP^{-1/2} \right\|^{2(1-\alpha)} \langle Py,y \rangle,$$

which gives the following simpler inequality

$$(5.2) \qquad \left| \langle \mathcal{P}_{f}(T,P)x,y \rangle \right| \leq f_{a}\left(\left\| P^{-1/2}TP^{-1/2} \right\| \right) \langle Px,x \rangle^{1/2} \langle Py,y \rangle^{1/2}$$

for all x, $y \in H$.

If we take $f(z) = \ln(1 \pm z)$, $P > 0$ and $T \in B(H)$ with $\left\| P^{-1/2}TP^{-1/2} \right\| < 1$, then by (5.2) we derive the inequality

$$(5.3) \qquad \left| \left\langle \mathcal{P}_{\ln(1\pm\cdot)}(T,P)x,y \right\rangle \right| \leq \ln\left(1 - \left\| P^{-1/2}TP^{-1/2} \right\| \right) \langle Px,x \rangle^{1/2} \langle Py,y \rangle^{1/2}$$

for all x, $y \in H$.

From (3.2) we get for $C = B = P^{1/2}$ and $A = P^{-1/2}TP^{-1/2}$ that

$$(5.4) \qquad \left\| \mathcal{P}_{f}(T,P) \right\| \leq \frac{f_{a}\left(\left\| P^{-1/2}TP^{-1/2} \right\| \right)}{\left\| P^{-1/2}TP^{-1/2} \right\|}$$
$$\times \left\| \left| P^{-1/2}TP^{-1/2} \right|^{\alpha} P^{1/2} \right\| \left\| \left| P^{-1/2}T^{*}P^{-1/2} \right|^{1-\alpha} P^{1/2} \right\|$$
$$\leq f_{a}\left(\left\| P^{-1/2}TP^{-1/2} \right\| \right) \left\| P \right\|.$$

From (3.3) we also get

$$(5.5) \quad \omega\left(\mathcal{P}_{f}(T,P) \right) \leq \frac{1}{2} \frac{f_{a}\left(\left\| P^{-1/2}TP^{-1/2} \right\| \right)}{\left\| P^{-1/2}TP^{-1/2} \right\|}$$
$$\times \left\| \left| \left| P^{-1/2}TP^{-1/2} \right|^{\alpha} P^{1/2} \right|^{2} + \left| \left| P^{-1/2}T^{*}P^{-1/2} \right|^{1-\alpha} P^{1/2} \right|^{2} \right\|$$

for $\alpha \in [0,1]$.

Observe that

$$\left| \left| P^{-1/2}TP^{-1/2} \right|^{\alpha} P^{1/2} \right|^{2} = P \left| P^{-1/2}TP^{-1/2} \right|^{2\alpha} P = P \left(P^{-1/2}T^{*}P^{-1}TP^{-1/2} \right)^{\alpha} P$$

Since $0 < P^{-1} \leq \left\| P^{-1} \right\| I$, then $0 < T^{*}P^{-1}T \leq \left\| P^{-1} \right\| |T|^{2}$ and

$$P^{-1/2}T^{*}P^{-1}TP^{-1/2} \leq \left\| P^{-1} \right\| P^{-1/2} |T|^{2} P^{-1/2} \leq \left\| P^{-1} \right\|^{2} |T|^{2}.$$

If we use Heinz inequality for $\alpha \in [0,1]$, we get

$$\left(P^{-1/2}T^{*}P^{-1}TP^{-1/2}\right)^{\alpha} \leq \left\|P^{-1}\right\|^{2\alpha} |T|^{2\alpha},$$

which implies that

$$\left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^{2} = P\left(P^{-1/2}T^{*}P^{-1}TP^{-1/2}\right)^{\alpha} P \leq \left\|P^{-1}\right\|^{2\alpha} P|T|^{2\alpha} P.$$

Similarly,

$$\left|\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^{2} \leq \left\|P^{-1}\right\|^{2(1-\alpha)} P|T^{*}|^{2(1-\alpha)} P.$$

Therefore

$$\begin{aligned} 0 &< \left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^{2} + \left|\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^{2} \\ &\leq \left\|P^{-1}\right\|^{2\alpha} P|T|^{2\alpha} P + \left\|P^{-1}\right\|^{2(1-\alpha)} P|T^{*}|^{2(1-\alpha)} P \\ &= P\left(\left\|P^{-1}\right\|^{2\alpha}|T|^{2\alpha} + \left\|P^{-1}\right\|^{2(1-\alpha)}|T^{*}|^{2(1-\alpha)}\right) P, \end{aligned}$$

which implies that

$$\begin{aligned} &\left\|\left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^{2} + \left|\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^{2}\right\| \\ &\leq \left\|P\left(\left\|P^{-1}\right\|^{2\alpha}|T|^{2\alpha} + \left\|P^{-1}\right\|^{2(1-\alpha)}|T^{*}|^{2(1-\alpha)}\right) P\right\| \\ &\leq \left\|\left\|P^{-1}\right\|^{2\alpha}|T|^{2\alpha} + \left\|P^{-1}\right\|^{2(1-\alpha)}|T^{*}|^{2(1-\alpha)}\right\| \|P\|^{2}. \end{aligned}$$

From (5.5) we get

$$\begin{aligned} \omega\left(\mathcal{P}_{f}(T,P)\right) &\leq \frac{1}{2}\frac{f_{a}\left(\left\|P^{-1/2}TP^{-1/2}\right\|\right)}{\left\|P^{-1/2}TP^{-1/2}\right\|} \\ &\quad \times \left\|\left\|P^{-1}\right\|^{2\alpha}|T|^{2\alpha} + \left\|P^{-1}\right\|^{2(1-\alpha)}|T^{*}|^{2(1-\alpha)}\right\| \|P\|^{2} \end{aligned} \tag{5.6}$$

for $\alpha \in [0,1]$.

If we take $\alpha = 1/2$ in (5.6) then we get

$$\omega\left(\mathcal{P}_{f}(T,P)\right) \leq \frac{1}{2}\frac{f_{a}\left(\left\|P^{-1/2}TP^{-1/2}\right\|\right)}{\left\|P^{-1/2}TP^{-1/2}\right\|} \left\||T| + |T^{*}|\right\| \|P\|^{2} \left\|P^{-1}\right\|. \tag{5.7}$$

If we use now the inequality (4.1) for $C = B = P^{1/2}$ and $A = P^{-1/2}TP^{-1/2}$ we get that

$$\begin{aligned} \left\|\mathcal{P}_{f}(T,P)\right\|_{2r} &\leq \frac{f_{a}\left(\left\|P^{-1/2}TP^{-1/2}\right\|\right)}{\left\|P^{-1/2}TP^{-1/2}\right\|} \\ &\quad \times \left\|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right\|_{2pr} \left\|\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha} P^{1/2}\right\|_{2qr} \end{aligned} \tag{5.8}$$

provided that $\alpha \in [0,1]$, $r \geq 1/2$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$ while $\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2} \in \mathcal{B}_{2pr}(H)$ and $\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha} P^{1/2} \in \mathcal{B}_{2qr}(H)$.

Now, we observe that

$$\begin{aligned}\left\|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha}P^{1/2}\right\|_{2pr} &\leq \left\|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha}\right\|\left\|P^{1/2}\right\|_{2pr}\\ &= \left\|P^{-1/2}TP^{-1/2}\right\|^{\alpha}\|P\|_{pr}\end{aligned}$$

and

$$\begin{aligned}\left\|\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha}P^{1/2}\right\|_{2qr} &\leq \left\|\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha}\right\|\left\|P^{1/2}\right\|_{2qr}\\ &= \left\|P^{-1/2}T^{*}P^{-1/2}\right\|^{1-\alpha}\|P\|_{qr}.\end{aligned}$$

These imply that

$$\begin{aligned}&\left\|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha}P^{1/2}\right\|_{2pr}\left\|\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha}P^{1/2}\right\|_{2qr}\\ &\leq \left\|P^{-1/2}TP^{-1/2}\right\|^{\alpha}\|P\|_{pr}\left\|P^{-1/2}T^{*}P^{-1/2}\right\|^{1-\alpha}\|P\|_{qr}\\ &= \left\|P^{-1/2}TP^{-1/2}\right\|^{\alpha}\left\|P^{-1/2}TP^{-1/2}\right\|^{1-\alpha}\|P\|_{pr}\|P\|_{qr}\\ &= \left\|P^{-1/2}TP^{-1/2}\right\|\|P\|_{pr}\|P\|_{qr}\end{aligned}$$

and by (5.8) we obtain the simpler inequality

$$\|\mathcal{P}_f(T,P)\|_{2r} \leq f_a\left(\left\|P^{-1/2}TP^{-1/2}\right\|\right)\|P\|_{pr}\|P\|_{qr} \tag{5.9}$$

provided that $P \in \mathcal{B}_{pr}(H) \cap \mathcal{B}_{qr}(H)$ where $r \geq 1/2$, p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and pr, $qr \geq 1$.

In particular, for $r = 1/2$ and $p = q = 2$ we get

$$\|\mathcal{P}_f(T,P)\|_{1} \leq f_a\left(\left\|P^{-1/2}TP^{-1/2}\right\|\right)\|P\|_{1}^{2} \tag{5.10}$$

provided that $P \in \mathcal{B}_1(H)$.

For $r = 1$ and p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ we obtain

$$\|\mathcal{P}_f(T,P)\|_{2} \leq f_a\left(\left\|P^{-1/2}TP^{-1/2}\right\|\right)\|P\|_{p}\|P\|_{q} \tag{5.11}$$

provided that $P \in \mathcal{B}_p(H) \cap \mathcal{B}_q(H)$. In the case $p = q = 2$, we then derive

$$\|\mathcal{P}_f(T,P)\|_{2} \leq f_a\left(\left\|P^{-1/2}TP^{-1/2}\right\|\right)\|P\|_{2}^{2} \tag{5.12}$$

provided that $P \in \mathcal{B}_2(H)$.

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