Approximation of Brownian Motion on Simple Graphs

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Abstract

The first author recently derived several approximation results by neural network operators see his new monograph [19]. There, the approximation methods derived from the parametrized and deformed neural networks induced by the parametrized error and q-deformed and β-parametrized half hyperbolic tangent activation functions. The results we apply here are univariate on a compact interval, regular and fractional. The outcome is the quantitative approximation of Brownian motion on simple graphs, in particular over a system S of semiaxes emanating from a common origin radially arranged and a particle moving randomly on S. We derive several Jackson type inequalities estimating the degree of convergence of our neural network operators to a general expectation function of Brownian motion. We give a detailed list of approximation applications regarding the expectation of well known functions of this Brownian motion. Smoothness of our functions is taken into account producing higher speeds of approximation.

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1 Introduction

The first author in [1] and [2], see chapters $2-5$, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and 'Squasing' types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining

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these operators 'bell-shaped' and 'squashing' functions are assumed to be compact support. Also the first author inspired by [23], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [4], [5], [6], [7] and [9], by treating both the univariate and multivariate cases. He did also the corresponding fractional cases [8], [10] and [14].
In [16], [19] the first author continued similar studies for Banach space valued functions for activation functions deriving from the parametrized error and $q-$deformed and $\beta-$ parametrized half hyperbolic tangent sigmoid functions. The authors based and inspired by [27] perform here neural network quantitative approximations to Brownian motion over a simple graph of a system of semiaxes.
They present a collection of Jackson type inequalities estimating the error of approximation to a general expectation function of this Brownian motion and its derivative. They produce regular and fractional calculus results. They finish with a lot of important applications.

2 Basics

2.1 About the parametrized (Gauss) error special activation function

Here we follow [20].
We consider here the parametrized (Gauss) error special activation function

$$erf\lambda z = \frac{2}{\sqrt{\pi}} \int_0^{\lambda z} e^{-t^2} dt, \quad \lambda > 0, \; z \in \mathbb{R}, \tag{1}$$

which is a sigmoidal type function and a strictly increasing function.

Of special interest in neural network theory is when $0 < \lambda < 1$, see [1] - Introduction.

It has the basic properties

$$\begin{gathered} erf\,(0) = 0, \quad erf\,(-\lambda x) = -erf\,(\lambda x)\,, \text{for every } 0 < \lambda < 1 \\ erf\,(+\infty) = 1, \;\; erf\,(-\infty) = -1, \end{gathered} \tag{2}$$

and

$$(erf\,(\lambda x))' = \frac{2\lambda}{\sqrt{\pi}} e^{-(\lambda x)^2}, \;\; x \in \mathbb{R}. \tag{3}$$

We consider the function

$$\chi\,(x) = \frac{1}{4}\left(erf\,(\lambda\,(x+1)) - erf\,(\lambda\,(x-1))\right), \;\; x \in \mathbb{R}, \tag{4}$$

and we notice that

$$\chi\,(-x) = \chi\,(x)\,. \tag{5}$$

Thus χ is an even function.

Since $x + 1 > x - 1$, then $erf\,(\lambda\,(x+1)) > erf\,(\lambda\,(x-1))$, and $\chi\,(x) > 0$, all $x \in \mathbb{R}$.

We see

$$\chi\,(0) = \frac{erf\lambda}{2}. \tag{6}$$

Let $x > 0$, we have that

$$\chi'(x) = \frac{\lambda}{2\sqrt{\pi}} \left(\frac{e^{\lambda^2(x-1)^2} - e^{\lambda^2(x+1)^2}}{e^{\lambda^2(x+1)^2} e^{\lambda^2(x-1)^2}} \right) < 0, \tag{7}$$

proving $\chi'(x) < 0$, for $x > 0$. That is χ is strictly decreasing on $[0, \infty)$ and it is strictly increasing on $(-\infty, 0]$, and $\chi'(0) = 0$.

Clearly, the x-axis is the horizontal asymptote of χ.

Conclusion, χ is a bell symmetric function with maximum

$$\chi(0) = \frac{erf\lambda}{2}.$$

Theorem 1. *It holds*

$$\sum_{i=-\infty}^{\infty} \chi(x - i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{8}$$

We have

Theorem 2. *We have that*

$$\int_{-\infty}^{\infty} \chi(x)\, dx = 1. \tag{9}$$

Hence $\chi(x)$ is a density function on $\mathbb{R}$. We need

Theorem 3. *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$, $\lambda > 0$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \chi(nx - k) < \frac{\left(1 - erf\left(\lambda\left(n^{1-\alpha} - 2\right)\right)\right)}{2}, \tag{10}$$

with

$$\lim_{n \to +\infty} \frac{\left(1 - erf\left(\lambda\left(n^{1-\alpha} - 2\right)\right)\right)}{2} = 0.$$

Denote by $\lfloor \cdot \rfloor$ the integral part and by $\lceil \cdot \rceil$ the ceiling of a number.

Furthermore we need

Theorem 4. *Let $x \in [a, b] \subset \mathbb{R}$, $\lambda > 0$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k)} < \frac{1}{\chi(1)} = \frac{4}{erf(2\lambda)}. \tag{11}$$

Remark 5. *As in [17], we have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k) \neq 1. \tag{12}$$

Note 6. *For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. As in [17], we obtain that*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k) \leq 1. \tag{13}$$

Definition 7. *Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the X-valued linear neural network operators*

$$A_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \chi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)}, \quad x \in [a,b]. \tag{14}$$

Clearly here $A_n(f,x) \in C([a,b])$. We study here the pointwise and uniform convergence of $A_n(f,x)$ to $f(x)$ with rates.

For convenience also we call

$$A_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \chi(nx-k), \tag{15}$$

that is

$$A_n(f,x) = \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)}. \tag{16}$$

So that

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)} - f(x)$$

$$= \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)}. \tag{17}$$

Consequently we derive

$$|A_n(f,x) - f(x)| \leq \frac{4}{erf(2\lambda)} \left| A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)\right) \right|. \tag{18}$$

That is

$$|A_n(f,x) - f(x)| \leq \frac{4}{erf(2\lambda)} \left| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \chi(nx-k) \right|. \tag{19}$$

We will estimate the right hand side of (19).

For that we need, for $f \in C([a,b])$ the first modulus of continuity

$$\omega_1(f,\delta)_{[a,b]} := \omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{20}$$

The fact $f \in C([a,b])$ is equivalent to $\lim_{\delta \to 0} \omega_1(f,\delta) = 0$, see [13].

We present a series of real valued neural network approximations to a function given with rates.

We first give

Theorem 8. *Let $f \in C([a,b])$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $\lambda > 0$, $x \in [a,b]$. Then*

i)

$$|A_n(f,x) - f(x)| \leq \frac{4}{erf(2\lambda)} \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + \left(1 - erf\left(\lambda\left(n^{1-\alpha} - 2\right)\right)\right) \|f\|_\infty \right] =: \rho, \tag{21}$$

and

ii)

$$\|A_n(f) - f\|_\infty \leq \rho. \tag{22}$$

We notice $\lim_{n\to\infty} A_n(f) = f$*, pointwise and uniformly.*

The speed of convergence is $\max\left\{\frac{1}{n^\alpha}, 1 - erf\left(\lambda\left(n^{1-\alpha} - 2\right)\right)\right\}$.

We need

Definition 9. *([11]) Let* $[a,b] \subset \mathbb{R}$*,* $\alpha > 0$*;* $m = \lceil \alpha \rceil \in \mathbb{N}$*, (*$\lceil \cdot \rceil$ *is the ceiling of the number),* $f : [a,b] \to \mathbb{R}$*. We assume that* $f^{(m)} \in L_1([a,b])$*. We call the left Caputo fractional derivative of order* α*:*

$$\left(D_{*a}^{\alpha} f\right)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \quad \forall\ x \in [a,b]. \tag{23}$$

If $\alpha \in \mathbb{N}$*, we set* $D_{*a}^{\alpha} f := f^{(m)}$ *the ordinary real valued derivative (defined similar to numerical one, see [29], p. 83), and also set* $D_{*a}^{0} f := f$*.*

By [11], $\left(D_{*a}^{\alpha} f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b])$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b])} < \infty$, then by [11], $D_{*a}^{\alpha} f \in C([a,b])$, hence $\left|D_{*a}^{\alpha} f\right| \in C([a,b])$.

We mention

Lemma 10. *([13]) Let* $\alpha > 0$*,* $\alpha \notin \mathbb{N}$*,* $m = \lceil \alpha \rceil$*,* $f \in C^{m-1}([a,b])$ *and* $f^{(m)} \in L_\infty([a,b])$*. Then* $D_{*a}^{\alpha} f(a) = 0$*.*

We mention

Definition 11. *([12]) Let* $[a,b] \subset \mathbb{R}$*,* $\alpha > 0$*,* $m := \lceil \alpha \rceil$*. We assume that* $f^{(m)} \in L_1([a,b])$*, where* $f : [a,b] \to \mathbb{R}$*. We call the right Caputo fractional derivative of order* α*:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\ x \in [a,b]. \tag{24}$$

We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$*, for* $m \in \mathbb{N}$*, and* $\left(D_{b-}^{0} f\right)(x) = f(x)$.

By [12], $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b])$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b])} < \infty$, and $\alpha \notin \mathbb{N}$, by [12], $D_{b-}^{\alpha} f \in C([a,b])$, hence $\left|D_{b-}^{\alpha} f\right| \in C([a,b])$.

We need

Lemma 12. *([13]) Let* $f \in C^{m-1}([a,b])$*,* $f^{(m)} \in L_\infty([a,b])$*,* $m = \lceil \alpha \rceil$*,* $\alpha > 0$*,* $\alpha \notin \mathbb{N}$*. Then* $D_{b-}^{\alpha} f(b) = 0$*.*

We present the following real valued fractional approximation result by $erf\lambda$ based neural networks.

Theorem 13. *Let* $0 < \alpha, \beta^* < 1$*,* $f \in C^1([a,b])$*,* $x \in [a,b]$*,* $n \in \mathbb{N} : n^{1-\beta^*} > 2$*,* $\lambda > 0$*. Then*

i)

$$|A_n(f,x) - f(x)| \leq$$

$$\frac{4}{erf(2\lambda)\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta^*}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\left(\frac{1-erf\left(\lambda\left(n^{1-\beta^*}-2\right)\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \quad (25)$$

and

ii)

$$\|A_nf-f\|_{\infty}\leq\frac{4}{\Gamma(\alpha+1)\,erf(2\lambda)}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta^*}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\left(\frac{1-erf\left(\lambda\left(n^{1-\beta^*}-2\right)\right)}{2}\right)(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}. \quad (26)$$

When $\alpha=\frac{1}{2}$ we derive

Corollary 14. *Let* $0<\beta^*<1$, $f\in C^1([a,b])$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta^*}>2$, $\lambda>0$. *Then*

i)

$$|A_n(f,x)-f(x)|\leq$$

$$\frac{8}{erf(2\lambda)\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\left(\frac{1-erf\left(\lambda\left(n^{1-\beta^*}-2\right)\right)}{2}\right)\left(\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}\sqrt{(x-a)}+\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\sqrt{(b-x)}\right)\right\}, \quad (27)$$

and

ii)

$$\|A_nf-f\|_{\infty}\leq\frac{8}{erf(2\lambda)\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}}+\right.$$

$$\left.\left(\frac{1-erf\left(\lambda\left(n^{1-\beta^*}-2\right)\right)}{2}\right)\sqrt{(b-a)}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\right)\right\}<\infty. \quad (28)$$

2.2 About q-deformed and β-parametrized half hyperbolic tangent function φ_q

All the next background comes from [21].

Here we describe the properties of the activation function

$$\varphi_q(t) := \frac{1 - qe^{-\beta t}}{1 + qe^{-\beta t}}, \quad \forall\, t \in \mathbb{R}, \tag{29}$$

where $q, \beta > 0$.

We have that

$$\varphi_q(0) = \frac{1-q}{1+q}$$

and

$$\varphi_q(-t) = -\varphi_{\frac{1}{q}}(t), \ \ \forall\, t \in \mathbb{R}, \tag{30}$$

hence

$$\varphi'_{\frac{1}{q}}(t) = \varphi'_q(-t). \tag{31}$$

It is

$$\lim_{t \to +\infty} \varphi_q(t) = \varphi_q(+\infty) = 1, \tag{32}$$

and

$$\lim_{t \to -\infty} \varphi_q(t) = \varphi_q(-\infty) = -1. \tag{33}$$

Furthermore

$$\varphi'_q(t) = \frac{2\beta q e^{\beta t}}{\left(e^{\beta t} + q\right)^2} > 0, \quad \forall\, t \in \mathbb{R}, \tag{34}$$

therefore φ_q is striclty increasing. Moreover, in case of $t < \frac{\ln q}{\beta}$, we have that φ_q is strictly concave up, with $\varphi''_q\left(\frac{\ln q}{\beta}\right) = 0$.

And in case of $t > \frac{\ln q}{\beta}$, we have that φ_q is strictly concave down.

Clearly, φ_q is a shifted sigmoid function with $\varphi_q(0) = \frac{1-q}{1+q}$, and $\varphi_q(-x) = -\varphi_{q^{-1}}(x)$, $\forall$ $x \in \mathbb{R}$, (a semi-odd function), see also [21].

We consider the function

$$\phi_q(x) := \frac{1}{4}\left(\varphi_q(x+1) - \varphi_q(x-1)\right) > 0, \tag{35}$$

$\forall x \in \mathbb{R}$; $\beta, q > 0$. Notice that $\phi_q(\pm\infty) = 0$, so the x-axis is horizontal asymptote. We have that

$$\phi_q(-x) = \phi_{\frac{1}{q}}(x), \ \ \forall\, x \in \mathbb{R}, \tag{36}$$

a deformed symmetry.

Next we have that

φ_q is striclty increasing over $\left(-\infty, \frac{\ln q}{\beta} - 1\right)$ and strictly decreasing over $\left(\frac{\ln q}{\beta} + 1, +\infty\right)$.

Moreover, ϕ_q is concave down over $\left[\frac{\ln q}{\beta} - 1, \frac{\ln q}{\beta} + 1\right]$, and strictly concave down over $\left(\frac{\ln q}{\beta} - 1, \frac{\ln q}{\beta} + 1\right)$.

Consequently ϕ_q has a bell-type shape over $\mathbb{R}$.

Of course it holds $\phi_q''\left(\frac{\ln q}{\beta}\right) < 0$. Thus at $x = \frac{\ln q}{\beta}$, we have the maximum value of ϕ_q which is

$$\phi_q\left(\frac{\ln q}{\beta}\right) = \frac{\left(1 - e^{-\beta}\right)}{2\left(1 + e^{-\beta}\right)} = \frac{\varphi_1(1)}{2}. \tag{37}$$

We mention

Theorem 15. *([18]) We have that*

$$\sum_{i=-\infty}^{\infty} \phi_q(x - i) = 1, \ \ \forall\ x \in \mathbb{R}, \forall\ q, \beta > 0. \tag{38}$$

It follows

Theorem 16. *([18]) It holds*

$$\int_{-\infty}^{\infty} \phi_q(x)\, dx = 1, \quad q, \beta > 0. \tag{39}$$

So that ϕ_q is a density function on $\mathbb{R}$; $q, \beta > 0$.

We need the following result,

Theorem 17. *([18]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$; $q, \beta > 0$. Then*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \phi_q(nx - k) < \max\left\{q, \frac{1}{q}\right\} e^{2\beta} e^{-\beta n^{(1-\alpha)}} = K e^{-\beta n^{(1-\alpha)}}, \tag{40}$$

where $K := \max\left\{q, \frac{1}{q}\right\} e^{2\beta}$.

Let $\lceil \cdot \rceil$ the ceiling of the number, and $\lfloor \cdot \rfloor$ the integral part of the number.

We mention the following result:

Theorem 18. *([18]) Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q > 0$, we consider the number $\lambda_q > z_0 > 0$ with $\phi_q(z_0) = \phi_q(0)$ and $\beta, \lambda_q > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_q(nx - k)} < max\left\{\frac{1}{\phi_q(\lambda_q)}, \frac{1}{\phi_{\frac{1}{q}}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \theta(q). \tag{41}$$

We also mention

Remark 19. *([18]) (i) We have that*

$$\lim_{n \to +\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_q(nx - k) \neq 1, \quad \text{for at least some } x \in [a, b], \tag{42}$$

where $\beta, q > 0$.

(ii) Let $[a, b] \subset \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_q(nx - k) \leq 1. \tag{43}$$

We need

Definition 20. *Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the real valued linear neural network operators*

$$H_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi_q(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)}, \quad x \in [a,b]\,;\, q,\beta > 0. \tag{44}$$

Clearly $H_n(f) \in C([a,b])$.
We study here the pointwise and uniform convergence of $H_n(f,x)$ to $f(x)$ with rates.

For convenience, also we call

$$H_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi_q(nx-k),\,. \tag{45}$$

That is

$$H_n(f,x) := \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)}. \tag{46}$$

So that

$$H_n(f,x) - f(x) = \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)} - f(x) = \tag{47}$$

$$\frac{H_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)}.$$

Consequently, we derive that

$$|H_n(f,x) - f(x)| \leq \theta(q) \left| H_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)\right)\right| =$$

$$\theta(q) \left| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x)\right) \Phi_q(nx-k) \right|, \tag{48}$$

where $\theta(q)$ as in (41). We will estimate the right hand side of the last quantity.
We present a set of real valued neural network approximations to a function given with rates.

Theorem 21. *Let $f \in C([a,b])$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $q,\beta > 0$, $x \in [a,b]$. Then*

i)

$$|H_n(f,x) - f(x)| \leq \theta(q) \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty K e^{-\beta n^{(1-\alpha)}}\right] =: \tau, \tag{49}$$

where K as in (40),

and

ii)

$$\|H_n(f) - f\|_\infty \leq \tau. \tag{50}$$

We get that $\lim_{n\to\infty} H_n(f) = f$, *pointwise and uniformly.*

Next we present

Theorem 22. *Let* $0 < \alpha, \beta^* < 1$, $q, \beta > 0$, $f \in C^1([a,b])$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then*

i)

$$|H_n(f,x) - f(x)| \leq$$

$$\frac{\theta(q)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^{\beta^*}}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. Ke^{-\beta n^{(1-\beta^*)}} \left(\left\|D_{x-}^\alpha f\right\|_{\infty,[a,x]} (x-a)^\alpha + \left\|D_{*x}^\alpha f\right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \tag{51}$$

and

ii)

$$\|H_n f - f\|_\infty \leq \frac{\theta(q)}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1\left(D_{x-}^\alpha f, \frac{1}{n^{\beta^*}}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. (b-a)^\alpha Ke^{-\beta n^{(1-\beta^*)}} \left(\sup_{x\in[a,b]} \left\|D_{x-}^\alpha f\right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\|D_{*x}^\alpha f\right\|_{\infty,[x,b]} \right) \right\}. \tag{52}$$

When $\alpha = \frac{1}{2}$ we derive

Corollary 23. *Let* $0 < \beta^* < 1$, $q, \beta > 0$, $f \in C^1([a,b])$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then*

i)

$$|H_n(f,x) - f(x)| \leq$$

$$\frac{2\theta(q)}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta^*}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. Ke^{-\beta n^{(1-\beta^*)}} \left(\left\|D_{x-}^{\frac{1}{2}} f\right\|_{\infty,[a,x]} \sqrt{(x-a)} + \left\|D_{*x}^{\frac{1}{2}} f\right\|_{\infty,[x,b]} \sqrt{(b-x)} \right) \right\}, \tag{53}$$

and

ii)

$$\|H_n f - f\|_\infty \leq \frac{2\theta(q)}{\sqrt{\pi}}$$

$$\left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta^*}} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta^*}} \right)_{[x,b]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \sqrt{(b-a)} K e^{-\beta n^{(1-\beta^*)}} \left(\sup_{x \in [a,b]} \left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} + \sup_{x \in [a,b]} \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \right) \right\} < \infty. \quad (54)$$

3 Combine 2.1 and 2.2

Let $a, b \in \mathbb{R}$ with $a < b$, $f \in C([a,b])$. Let also $q, \lambda, \beta > 0, \gamma = \max\left\{q, \frac{1}{q}\right\}$.
For the next theorems we call

$$_1L_n(f,x) := A_n(f,x), x \in [a,b]$$

$$_2L_n(f,x) := H_n(f,x), x \in [a,b].$$

Also we set

$$K_1 = K_1(\lambda) = \frac{4}{erf(2\lambda)}$$

$$K_2 = K_2(q) = \theta(q).$$

Furthermore we set

$$\hat{\beta}_{1,n} = \hat{\beta}_{1,n}(\lambda, \beta^*) = 1 - erf\left(\lambda\left(n^{1-\beta^*} - 2\right)\right), n \in \mathbb{N}, \lambda > 0, 0 < \beta^* < 1.$$

$$\hat{\beta}_{2,n} = \hat{\beta}_{2,n}(q, \beta, \beta^*) = 2\gamma e^{2\beta - \beta n^{1-\beta^*}}, n \in \mathbb{N}, q, \beta > 0, 0 < \beta^* < 1$$

We present

Theorem 24. *Let $f \in C([a,b])$, $0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $q, \lambda, \beta > 0$, $x \in [a,b]$. Then for $i = 1, 2$*

i)

$$|_iL_n(f,x) - f(x)| \leq K_i \cdot \left[\omega_1\left(f, \frac{1}{n^{\beta^*}}\right) + \hat{\beta}_{i,n} \|f\|_\infty\right] =: \rho_i, \quad (55)$$

and

ii)

$$\|_iL_n(f) - f\|_\infty \leq \rho_i. \quad (56)$$

We get that $\lim_{n\to\infty} {}_iL_n(f) = f$, pointwise and uniformly.

Proof. From Theorems 8 and 21. □
Next we present

Theorem 25. *Let $0 < \alpha, \beta^* < 1$, $q, \lambda, \beta > 0$, $f \in C^1([a,b])$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$. Then for $i = 1, 2$*

i)

$$|_iL_n(f,x) - f(x)| \leq$$

$$\frac{K_i}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta^*}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \tag{57}$$

and

ii)

$$\|{}_iL_nf-f\|_{\infty}\leq\frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta^*}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\frac{(b-a)^{\alpha}\hat{\beta}_{i,n}}{2}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}. \tag{58}$$

Proof. From Theorems 13 and 22. □

When $\alpha=\frac{1}{2}$ we derive

Corollary 26. *Let* $0<\beta^*<1$, $q,\lambda,\beta>0$, $f\in C^1([a,b])$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta^*}>2$. *Then for* $i=1,2$

i)

$$|{}_iL_n(f,x)-f(x)|\leq$$

$$\frac{2K_i}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}\sqrt{(x-a)}+\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\sqrt{(b-x)}\right)\right\}, \tag{59}$$

and

ii)

$$\|{}_iL_nf-f\|_{\infty}\leq\frac{2K_i}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\sqrt{(b-a)}\hat{\beta}_{i,n}}{2}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\right)\right\}<\infty. \tag{60}$$

4 About Random Motion on Simple Graphs

Here we follow [27].

Suppose we have a system of S semiaxes with a common origin radially arranged and a particle moving randomly on S. Possible applications include spread of toxic particles in a system of channels or vessels or propagation of information in networks.

The mathematical model is the following: Let S be the set consisting of n semiaxes $S_1, \ldots, S_n, n \geq 2$, with a common origin 0 and X_t the Brownian motion process on S, namely the diffusion process on S whose infinitesimal generator L is

$$Lu = \frac{1}{2}u'', \tag{61}$$

where

$$u = (u_1, \ldots, u_n),$$

together with the continuity conditions (a total of $n-1$ equations),

$$u_1(0) = \ldots = u_n(0) \tag{62}$$

and the so-called "Kirchoff condition"

$$u_1'(0) + \ldots + u_n'(0) = 0. \tag{63}$$

This is a Walsh's-type Brownian motion (see [22]).

The process X_t does a standard Brownian motion on each of the semiaxes and, when it hits 0, it continues its motion on the j-th semiaxis, $1 \leq j \leq n$, with probability $\frac{1}{n}$.

For each semiaxes $S_j, 1 \leq j \leq n$ it is convenient to use the coordinate $x_j, 0 \leq x_j \leq \infty$. Notice that, if $u = (u_1, \ldots, u_n)$ is a function on S, then its j-th component, u_j , is a function on S_j, thus $u_j = u_j(x_j)$.

The transition density of X_t is

$$p(t, x_k, y_j) = \frac{2}{n\sqrt{2\pi t}} e^{-\frac{(x_k+y_j)^2}{2t}}, \text{ if } k \neq j,$$

and

$$p(t, x_k, y_k) = \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(x_k-y_k)^2}{2t}} - \frac{n-2}{n}e^{-\frac{(x_k+y_k)^2}{2t}}\right). \tag{64}$$

We need the following result.

Theorem 27. *Let $t \in [t_1, t_2]$, where $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$ are fixed. We Consider the function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $(0, \infty)$, i.e. there exists $M > 0$ such that $|g(x)| \leq M$, for every $x \in (0, \infty)$, and Lebesgue measurable on $\mathbb{R}$. Let also X_t be the standard Brownian motion on each of the semiaxes $j = 1, \ldots, n$ as described above. Here x_k is fixed on S_k semiaxes, $k \in \{1, \ldots, n\}$. We consider the related expected value function*

$$r(t) := E_k(|g(X_t)|) = \int_0^\infty |g(y_k)|\, p(t, x_k, y_k) dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty |g(y_j)|\, p(t, x_k, y_j) dy_j,\ t \in [t_1, t_2].$$

The function $r(t)$ is continuous in t, and differentiable.

Proof. First we observe that for $t \in [t_1, t_2]$ and $k, j \in \{1, \ldots, n\}$ with $k \neq j$

$$0 < p\left(t, x_k, y_j\right) < \frac{2}{\sqrt{2\pi t_1}}.$$

Also for $t \in [t_1, t_2]$, and $k \in \{1, \ldots, n\}$ it is

$$0 < p\left(t, x_k, y_k\right) < \frac{2}{\sqrt{2\pi t_1}}.$$

It is enough to prove that

$$I(t) := \int_0^\infty |g(y_k)|\, p(t, x_k, y_k) dy_k$$

is continuous in $t \in [t_1, t_2]$.
We have that

$$|g(y_k)| \leq M.$$

Thus,

$$|g(y_k)|\, p\left(t, x_k, y_k\right) \leq M \frac{2}{\sqrt{2\pi t_1}}.$$

Furthermore, as $0 < t_N \to t$, with $N \longrightarrow \infty$, we get

$$|g(y_k)|\, p\left(t_N, x_k, y_k\right) \longrightarrow |g(y_k)|\, p\left(t, x_k, y_k\right), \text{ for every } y_k \geq 0.$$

By the dominating convergence theorem $I\left(t_n\right) \longrightarrow I\left(t\right)$ and thus, $I\left(t\right)$ is continuous in t, consequently the function

$$r\left(t\right) := E_k\left(|g(X_t)|\right)$$

is continuous in t. □
We also need the next theorem.

Theorem 28. *Let $t \in [t_1, t_2]$, where $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$ are fixed. We consider function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $(0, \infty)$ and Lebesgue measurable on $\mathbb{R}$. Let also X_t be the standard Brownian motion on each of the semiaxes $j = 1, \ldots, n$ as described above. Here x_k is fixed on S_k semiaxes, $k \in \{1, \ldots, n\}$. Then the related expected value function*

$$r\left(t\right) := E_k\left(|g(X_t)|\right) = \int_0^\infty |g(y_k)|\, p(t, x_k, y_k) dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty |g(y_j)|\, p(t, x_k, y_j) dy_j,\ t \in [t_1, t_2],$$

is differentiable in t, and

$$\frac{\partial r\left(t\right)}{\partial t} = \int_0^\infty |g(y_k)| \frac{\partial p(t, x_k, y_k)}{\partial t} dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty |g(y_j)| \frac{\partial p(t, x_k, y_j)}{\partial t} dy_j,\ t \in [t_1, t_2], \quad (65)$$

which is continuous in t.

Proof. First we observe that for $t \in [t_1, t_2]$ and $k, j \in \{1, \ldots, n\}$ with $k \neq j$

$$\frac{\partial p\left(t, x_k, y_j\right)}{\partial t} = \frac{1}{nt\sqrt{2\pi t}} e^{-\frac{\left(x_k + y_j\right)^2}{2t}} \left(\frac{\left(x_k + y_j\right)^2}{t} - 1\right).$$

Also for $t \in [t_1, t_2]$, and $k \in \{1, \ldots, n\}$ it is

$$\frac{\partial p(t, x_k, y_k)}{\partial t} = \frac{1}{2t\sqrt{2\pi t}} \left[e^{-\frac{(x_k - y_k)^2}{2t}} \left(\frac{(x_k - y_k)^2}{t} - 1 \right) - \frac{(n-2)}{n} e^{-\frac{(x_k + y_k)^2}{2t}} \left(\frac{(x_k + y_k)^2}{t} - 1 \right) \right].$$

Furthermore for $k \neq j$,

$$\left| \frac{\partial p(t, x_k, y_j)}{\partial t} \right| \leq \frac{1}{n t_1 \sqrt{2\pi t_1}} \left(\frac{(x_k + y_j)^2}{t_1} + 1 \right)$$

for every $y_j \in (0, \infty)$,
and

$$\left| \frac{\partial p(t, x_k, y_k)}{\partial t} \right| \leq \frac{1}{2t_1\sqrt{2\pi t_1}} \left[\left(\frac{(x_k - y_k)^2}{t_1} + 1 \right) + \frac{(n-2)}{n} \left(\frac{(x_k + y_k)^2}{t_1} + 1 \right) \right]$$

for every $y_k \in (0, \infty)$.
So $\frac{\partial p(t, x_k, y_j)}{\partial t}$ and $\frac{\partial p(t, x_k, y_k)}{\partial t}$ is bounded with respect to t. The bounds are integrable with respect to y_j and y_k respectively.

We have

$$r(t) := E_k(|g(X_t)|) = \int_0^\infty |g(y_k)| \, p(t, x_k, y_k) dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty |g(y_j)| \, p(t, x_k, y_j) dy_j, \ t \in [t_1, t_2].$$

We apply differentiation under the integral sign.
We notice that

$$|g(y_k)| \, p(t, x_k, y_k) \leq M \frac{1}{2t_1\sqrt{2\pi t_1}} \left[\left(\frac{(x_k - y_k)^2}{t_1} + 1 \right) + \frac{(n-2)}{n} \left(\frac{(x_k + y_k)^2}{t_1} + 1 \right) \right].$$

and

$$|g(y_k)| \, p(t, x_k, y_j) \leq M \frac{1}{n t_1 \sqrt{2\pi t_1}} \left(\frac{(x_k + y_j)^2}{t_1} + 1 \right).$$

Therefore there exists

$$\frac{\partial r(t)}{\partial t} = \int_0^\infty |g(y_k)| \frac{\partial p(t, x_k, y_k)}{\partial t} dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty |g(y_j)| \frac{\partial p(t, x_k, y_j)}{\partial t} dy_j, \ t \in [t_1, t_2],$$

which is continuous in t (same proof as in Theorem 27). □

5 Main Results

We present the following general approximation results of Brownian Motion on simple graphs.

Theorem 29. *We consider function* $g : \mathbb{R} \to \mathbb{R}$, *which is bounded on* $(0, \infty)$ *and Lebesgue measurable on* $\mathbb{R}$. *Let also* $r(t) := E_k[g(X_t)]$ *the related expected value function.*
If $0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, *where* $t_1, t_2 \in (0, \infty)$ *with* $t_1 < t_2$, *then for* $i = 1, 2$

i)

$$\left|{}_iL_n\left(r\left(t\right)\right)-r\left(t\right)\right|\leq K_i\cdot\left[\omega_1\left(r,\frac{1}{n^{\beta^*}}\right)+\hat{\beta}_{i,n}\left\|r\right\|_{\infty,[t_1,t_2]}\right]=:\rho_i, \tag{66}$$

and

ii)

$$\left\|{}_iL_n\left(r\left(t\right)\right)-r\left(t\right)\right\|_{\infty,[t_1,t_2]}\leq\rho_i. \tag{67}$$

We get that $\lim_{n\to\infty}{}_iL_n\left(r\right)=r$*, pointwise and uniformly.*

Proof. From Theorem 24. □

Next we present

Theorem 30. *We consider function* $g:\mathbb{R}\to\mathbb{R}$*, which is bounded on* $(0,\infty)$ *and Lebesgue measurable on* $\mathbb{R}$*. Let also* $r\left(t\right):=E_k\left[g\left(X_t\right)\right]$ *the related expected value function.*
If $0<\alpha,\beta^*<1$*,* $q,\lambda,\beta>0$*,* $t\in[t_1,t_2]$*, where* $t_1,t_2\in(0,\infty)$ *with* $t_1<t_2$*, and* $n\in\mathbb{N}:n^{1-\beta^*}>2$*. Then for* $i=1,2$

i)

$$\left|{}_iL_n\left(r\left(t\right)\right)-r\left(t\right)\right|\leq$$

$$\frac{K_i}{\Gamma\left(\alpha+1\right)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}r,\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}r,\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{t-}^{\alpha}r\right\|_{\infty,[t_1,t]}\left(t-t_1\right)^{\alpha}+\left\|D_{*t}^{\alpha}r\right\|_{\infty,[t,t_2]}\left(t_2-t\right)^{\alpha}\right)\right\}, \tag{68}$$

and

ii)

$$\left\|{}_iL_nr-r\right\|_{\infty,[t_1,t_2]}\leq\frac{K_i}{\Gamma\left(\alpha+1\right)}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}r,\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}r,\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\frac{\left(t_2-t_1\right)^{\alpha}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}r\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\alpha}r\right\|_{\infty,[t,t_2]}\right)\right\}. \tag{69}$$

Proof. From Theorem 25. □

When $\alpha=\frac{1}{2}$ we derive

Corollary 31. *We consider function* $g:\mathbb{R}\to\mathbb{R}$*, which is bounded on* $(0,\infty)$ *and Lebesgue measurable on* $\mathbb{R}$*. Let also* $r\left(t\right):=E_k\left[g\left(X_t\right)\right]$ *the related expected value function.*
If $0<\beta^*<1$*,* $q,\lambda,\beta>0$*,* $t\in[t_1,t_2]$ *and* $n\in\mathbb{N}:n^{1-\beta^*}>2$*. Then for* $i=1,2$

i)

$$\left|{}_iL_n\left(r\left(t\right)\right)-r\left(t\right)\right|\leq$$

$$\frac{2K_i}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{t-}^{\frac{1}{2}}r,\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\frac{1}{2}}r,\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{t-}^{\frac{1}{2}}r\right\|_{\infty,[t_1,t]}\sqrt{(t-t_1)}+\left\|D_{*t}^{\frac{1}{2}}r\right\|_{\infty,[t,t_2]}\sqrt{(t_2-t)}\right)\right\}, \tag{70}$$

and

ii)

$$\|{}_iL_nr-r\|_{\infty,[t_1,t_2]}\leq\frac{2K_i}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\frac{1}{2}}r,\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\frac{1}{2}}r,\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\sqrt{(t_2-t_1)}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\frac{1}{2}}r\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\frac{1}{2}}r\right\|_{\infty,[t,t_2]}\right)\right\}<\infty. \tag{71}$$

Proof. From Corollary 26. □

We continue with

Theorem 32. *We consider function $g:\mathbb{R}\to\mathbb{R}$, which is bounded on $(0,\infty)$ and Lebesgue measurable on $\mathbb{R}$. Let also $r(t):=E_k[g(X_t)]$ the related expected value function.*
If $0<\beta^*<1$, $n\in\mathbb{N}:n^{1-\beta^*}>2$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, where $t_1,t_2\in(0,\infty)$ with $t_1<t_2$, then for $i=1,2$

i)

$$\left|{}_iL_n\left(\frac{\partial r(t)}{\partial t}\right)-\frac{\partial r(t)}{\partial t}\right|\leq K_i\cdot\left[\omega_1\left(\frac{\partial r}{\partial t},\frac{1}{n^{\beta^*}}\right)+\hat{\beta}_{i,n}\left\|\frac{\partial r}{\partial t}\right\|_{\infty,[t_1,t_2]}\right]=:\rho_i, \tag{72}$$

and

ii)

$$\left\|{}_iL_n\left(\frac{\partial r}{\partial t}\right)-\frac{\partial r}{\partial t}\right\|_{\infty,[t_1,t_2]}\leq\rho_i. \tag{73}$$

We get that $\lim_{n\to\infty}{}_iL_n\left(\frac{\partial r}{\partial t}\right)=\frac{\partial r}{\partial t}$, pointwise and uniformly.

Proof. From Theorem 24. □

6 Applications

Let a function $g:\mathbb{R}\to\mathbb{R}$, which is bounded on $[t_1,t_2]$, where $t_1,t_2>0$ with $t_1<t_2$ and Lebesgue measurable on $\mathbb{R}$. Fot the Brownian Motion on simple graphs X_t We will use the following notations

$$r(t):=E_k(|g(X_t)|)=$$

$$\int_0^\infty|g(y_k)|\,p(t,x_k,y_k)dy_k+\sum_{j=1,j\neq k}^{n}\int_0^\infty|g(y_j)|\,p(t,x_k,y_j)dy_j:=E_k(|g(X_t)|)^{(0)}. \tag{74}$$

and

$$\frac{\partial r(t)}{\partial t}=\int_0^\infty |g(y_k)|\frac{\partial p(t,x_k,y_k)}{\partial t}dy_k+\sum_{j=1,j\neq k}^{n}\int_0^\infty |g(y_j)|\frac{\partial p(t,x_k,y_j)}{\partial t}dy_j:=E_k\left(|g(X_t)|\right)^{(1)}. \tag{75}$$

We can apply our main results to the function $g(W)=W$. Consider the function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=x$ for every $x\in\mathbb{R}$. Let also $W=X_t$ be the Brownian Motion on simple graphs. Then the expectation

$$E_k\left(|W|\right)(t)=\int_0^\infty |y_k|\,p(t,x_k,y_k)dy_k+\sum_{j=1,j\neq k}^{n}\int_0^\infty |y_j|\,p(t,x_k,y_j)dy_j$$

is continuous in t.

Moreover

Corollary 33. *Let $0<\beta^*<1$, $n\in\mathbb{N}:n^{1-\beta^*}>2$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, where $t_1,t_2\in(0,\infty)$ with $t_1<t_2$, then for $i=1,2$ and $j=0,1$*

i)

$$\left|{}_iL_n\left(E_k\left(|W|\right)^{(j)}\right)(t)-E_k\left(|W|\right)^{(j)}(t)\right|\leq K_i\cdot\left[\omega_1\left(E_k\left(|W|\right)^{(j)},\frac{1}{n^{\beta^*}}\right)+\hat{\beta}_{i,n}\left\|E_k\left(|W|\right)^{(j)}\right\|_{\infty,[t_1,t_2]}\right]=:\rho_i, \tag{76}$$

and

ii)

$$\left\|{}_iL_n\left(E_k\left(|W|\right)^{(j)}\right)(t)-E_k\left(|W|\right)^{(j)}(t)\right\|_{\infty,[t_1,t_2]}\leq\rho_i. \tag{77}$$

We get that $\lim_{n\to\infty}{}_iL_nE_k\left(|W|\right)^{(j)}(t)=E_k\left(|W|\right)^{(j)}$, pointwise and uniformly.

Proof. From Theorems 29 and 32. □

Next we present

Corollary 34. *Let $0<\alpha,\beta^*<1$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, where $t_1,t_2\in(0,\infty)$ with $t_1<t_2$, and $n\in\mathbb{N}:n^{1-\beta^*}>2$. Then for $i=1,2$*

i)

$$\left|{}_iL_n\left(E_k\left(|W|\right)(t)\right)-E_k\left(|W|\right)(t)\right|\leq$$

$$\frac{K_i}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}E_k\left(|W|\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}E_k\left(|W|\right),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{t-}^{\alpha}E_k\left(|W|\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}E_k\left(|W|\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}, \tag{78}$$

and

ii)

$$\left\|{}_iL_nE_k\left(|W|\right)-E_k\left(|W|\right)\right\|_{\infty,[t_1,t_2]}\leq\frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\sup_{t\in[t_1,t_2]} \omega_1 \left(D_{t-}^{\alpha} E_k\left(|W|\right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left(D_{*t}^{\alpha} E_k\left(|W|\right) \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \frac{(t_2-t_1)^{\alpha}\,\hat{\beta}_{i,n}}{2} \left(\sup_{t\in[t_1,t_2]} \left\| D_{t-}^{\alpha} E_k\left(|W|\right) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D_{*t}^{\alpha} E_k\left(|W|\right) \right\|_{\infty,[t,t_2]} \right) \right\}. \tag{79}$$

Proof. From Theorem 30. □

When $\alpha = \frac{1}{2}$ we derive

Corollary 35. *Let $0 < \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$ and $n \in \mathbb{N} : n^{1-\beta^*} > 2$. Then for $i = 1, 2$*

i)

$$\left| {}_iL_n\left(E_k\left(|W|\right)(t)\right) - E_k\left(|W|\right)(t) \right| \leq$$

$$\frac{2K_i}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D_{t-}^{\frac{1}{2}} E_k\left(|W|\right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \omega_1 \left(D_{*t}^{\frac{1}{2}} E_k\left(|W|\right), \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\hat{\beta}_{i,n}}{2} \left(\left\| D_{t-}^{\frac{1}{2}} E_k\left(|W|\right) \right\|_{\infty,[t_1,t]} \sqrt{(t-t_1)} + \left\| D_{*t}^{\frac{1}{2}} E_k\left(|W|\right) \right\|_{\infty,[t,t_2]} \sqrt{(t_2-t)} \right) \right\}, \tag{80}$$

and

ii)

$$\left\| {}_iL_nE_k\left(|W|\right) - E_k\left(|W|\right) \right\|_{\infty,[t_1,t_2]} \leq \frac{2K_i}{\sqrt{\pi}}$$

$$\left\{ \frac{\left(\sup_{t\in[t_1,t_2]} \omega_1 \left(D_{t-}^{\frac{1}{2}} E_k\left(|W|\right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left(D_{*t}^{\frac{1}{2}} E_k\left(|W|\right), \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\sqrt{(t_2-t_1)}\hat{\beta}_{i,n}}{2} \left(\sup_{t\in[t_1,t_2]} \left\| D_{t-}^{\frac{1}{2}} E_k\left(|W|\right) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D_{*t}^{\frac{1}{2}} E_k\left(|W|\right) \right\|_{\infty,[t,t_2]} \right) \right\} < \infty. \tag{81}$$

Proof. From Corollary 31. □

For the next application we consider the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = \cos x$ for every $x \in \mathbb{R}$. Let also $W = X_t$ be the Brownian Motion on simple graphs. Then the expectation

$$E_k\left(|\cos W|\right)(t) = \int_0^{\infty} |\cos(y_k)|\, p(t, x_k, y_k) dy_k + \sum_{j=1, j\neq k}^{n} \int_0^{\infty} |\cos(y_j)|\, p(t, x_k, y_j) dy_j$$

is continuous in t.

Moreover

Corollary 36. *Let* $0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, *where* $t_1, t_2 \in (0, \infty)$ *with* $t_1 < t_2$, *then for* $i = 1, 2$ *and* $j = 0, 1$

i)

$$\left| {}_iL_n \left(E_k (|\cos W|)^{(j)} \right) (t) - E_k (|\cos W|)^{(j)} (t) \right| \leq$$

$$K_i \cdot \left[\omega_1 \left(E_k (|\cos W|)^{(j)}, \frac{1}{n^{\beta^*}} \right) + \hat{\beta}_{i,n} \left\| E_k (|\cos W|)^{(j)} \right\|_{\infty, [t_1, t_2]} \right] =: \rho_i, \tag{82}$$

and

ii)

$$\left\| {}_iL_n \left(E_k (|\cos W|)^{(j)} \right) (t) - E_k (|\cos W|)^{(j)} (t) \right\|_{\infty, [t_1, t_2]} \leq \rho_i. \tag{83}$$

We get that $\lim_{n \to \infty} {}_iL_n E_k (|\cos W|)^{(j)} (t) = E_k (|\cos W|)^{(j)}$, *pointwise and uniformly.*

Proof. From Theorems 29 and 32. □

Next we present

Corollary 37. *Let* $0 < \alpha, \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, *where* $t_1, t_2 \in (0, \infty)$ *with* $t_1 < t_2$, *and* $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then for* $i = 1, 2$

i)

$$|{}_iL_n (E_k (|\cos W|) (t)) - E_k (|\cos W|) (t)| \leq$$

$$\frac{K_i}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D^{\alpha}_{t-} E_k (|\cos W|), \frac{1}{n^{\beta^*}} \right)_{[t_1, t]} + \omega_1 \left(D^{\alpha}_{*t} E_k (|\cos W|), \frac{1}{n^{\beta^*}} \right)_{[t, t_2]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \frac{\hat{\beta}_{i,n}}{2} \left(\left\| D^{\alpha}_{t-} E_k (|\cos W|) \right\|_{\infty, [t_1, t]} (t - t_1)^{\alpha} + \left\| D^{\alpha}_{*t} E_k (|\cos W|) \right\|_{\infty, [t, t_2]} (t_2 - t)^{\alpha} \right) \right\}, \tag{84}$$

and

ii)

$$\|{}_iL_n E_k (|\cos W|) - E_k (|\cos W|)\|_{\infty, [t_1, t_2]} \leq \frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\sup_{t \in [t_1, t_2]} \omega_1 \left(D^{\alpha}_{t-} E_k (|\cos W|), \frac{1}{n^{\beta^*}} \right)_{[t_1, t]} + \sup_{t \in [t_1, t_2]} \omega_1 \left(D^{\alpha}_{*t} E_k (|\cos W|) \frac{1}{n^{\beta^*}} \right)_{[t, t_2]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \frac{(t_2 - t_1)^{\alpha} \hat{\beta}_{i,n}}{2} \left(\sup_{t \in [t_1, t_2]} \left\| D^{\alpha}_{t-} E_k (|\cos W|) \right\|_{\infty, [t_1, t]} + \sup_{t \in [t_1, t_2]} \left\| D^{\alpha}_{*t} E_k (|\cos W|) \right\|_{\infty, [t, t_2]} \right) \right\}. \tag{85}$$

Proof. From Theorem 30. □

When $\alpha = \frac{1}{2}$ we derive

Corollary 38. *Let* $0 < \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$ *and* $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then for* $i = 1, 2$

i)

$$|{}_iL_n (E_k (|\cos W|) (t)) - E_k (|\cos W|) (t)| \leq$$

$$\frac{2K_i}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{t-}^{\frac{1}{2}}E_k\left(|\cos W|\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\frac{1}{2}}E_k\left(|\cos W|\right),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{t-}^{\frac{1}{2}}E_k\left(|\cos W|\right)\right\|_{\infty,[t_1,t]}\sqrt{(t-t_1)}+\left\|D_{*t}^{\frac{1}{2}}E_k\left(|\cos W|\right)\right\|_{\infty,[t,t_2]}\sqrt{(t_2-t)}\right)\right\}, \quad (86)$$

and

ii)

$$\left\|{}_iL_nE_k\left(|\cos W|\right)-E_k\left(|\cos W|\right)\right\|_{\infty,[t_1,t_2]}\leq\frac{2K_i}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\frac{1}{2}}E_k\left(|\cos W|\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\frac{1}{2}}E_k\left(|\cos W|\right),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\sqrt{(t_2-t_1)}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\frac{1}{2}}E_k\left(|\cos W|\right)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\frac{1}{2}}E_k\left(|\cos W|\right)\right\|_{\infty,[t,t_2]}\right)\right\}<\infty. \quad (87)$$

Proof. From Corollary 31. □

Let us consider now the function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=\tanh x$ for every $x\in\mathbb{R}$. Let also $W=X_t$ be the Brownian Motion on simple graphs. Then the expectation

$$E_k\left(|\tanh W|\right)(t)=\int_0^\infty|\tanh(y_k)|\,p(t,x_k,y_k)dy_k+\sum_{j=1,j\neq k}^{n}\int_0^\infty|\tanh(y_j)|\,p(t,x_k,y_j)dy_j$$

is continuous in t.

Moreover

Corollary 39. *Let* $0<\beta^*<1$, $n\in\mathbb{N}:n^{1-\beta^*}>2$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, *where* $t_1,t_2\in(0,\infty)$ *with* $t_1<t_2$, *then for* $i=1,2$ *and* $j=0,1$

i)

$$\left|{}_iL_n\left(E_k\left(|\tanh W|\right)^{(j)}\right)(t)-E_k\left(|\tanh W|\right)^{(j)}(t)\right|\leq$$

$$K_i\cdot\left[\omega_1\left(E_k\left(|\tanh W|\right)^{(j)},\frac{1}{n^{\beta^*}}\right)+\hat{\beta}_{i,n}\left\|E_k\left(|\tanh W|\right)^{(j)}\right\|_{\infty,[t_1,t_2]}\right]=:\rho_i, \quad (88)$$

and

ii)

$$\left\|{}_iL_n\left(E_k\left(|\tanh W|\right)^{(j)}\right)(t)-E_k\left(|\tanh W|\right)^{(j)}(t)\right\|_{\infty,[t_1,t_2]}\leq\rho_i. \quad (89)$$

We get that $\lim_{n\to\infty}{}_iL_nE_k\left(|\tanh W|\right)^{(j)}(t)=E_k\left(|\tanh W|\right)^{(j)}$, *pointwise and uniformly.*

Proof. From Theorems 29 and 32. □

Next we present

Corollary 40. *Let $0<\alpha,\beta^*<1$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, where $t_1,t_2\in(0,\infty)$ with $t_1<t_2$, and $n\in\mathbb{N}: n^{1-\beta^*}>2$. Then for $i=1,2$*

i)

$$\left|{}_iL_n\left(E_k\left(\left|\tanh W\right|\right)(t)\right)-E_k\left(\left|\tanh W\right|\right)(t)\right|\leq$$

$$\frac{K_i}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}E_k\left(\left|\tanh W\right|\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}E_k\left(\left|\tanh W\right|\right),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{t-}^{\alpha}E_k\left(\left|\tanh W\right|\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}E_k\left(\left|\tanh W\right|\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\},\quad(90)$$

and

ii)

$$\left\|{}_iL_nE_k\left(\left|\tanh W\right|\right)-E_k\left(\left|\tanh W\right|\right)\right\|_{\infty,[t_1,t_2]}\leq\frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}E_k\left(\left|\tanh W\right|\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}E_k\left(\left|\tanh W\right|\right)\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\frac{(t_2-t_1)^{\alpha}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}E_k\left(\left|\tanh W\right|\right)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\alpha}E_k\left(\left|\tanh W\right|\right)\right\|_{\infty,[t,t_2]}\right)\right\}.\quad(91)$$

Proof. From Theorem 30. □

When $\alpha=\frac{1}{2}$ we derive

Corollary 41. *Let $0<\beta^*<1$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$ and $n\in\mathbb{N}: n^{1-\beta^*}>2$. Then for $i=1,2$*

i)

$$\left|{}_iL_n\left(E_k\left(\left|\tanh W\right|\right)(t)\right)-E_k\left(\left|\tanh W\right|\right)(t)\right|\leq$$

$$\frac{2K_i}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{t-}^{\frac{1}{2}}E_k\left(\left|\tanh W\right|\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\frac{1}{2}}E_k\left(\left|\tanh W\right|\right),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{t-}^{\frac{1}{2}}E_k\left(\left|\tanh W\right|\right)\right\|_{\infty,[t_1,t]}\sqrt{(t-t_1)}+\left\|D_{*t}^{\frac{1}{2}}E_k\left(\left|\tanh W\right|\right)\right\|_{\infty,[t,t_2]}\sqrt{(t_2-t)}\right)\right\},\quad(92)$$

and

ii)

$$\left\|{}_iL_nE_k\left(\left|\tanh W\right|\right)-E_k\left(\left|\tanh W\right|\right)\right\|_{\infty,[t_1,t_2]}\leq\frac{2K_i}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\frac{1}{2}}E_k\left(\left|\tanh W\right|\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\frac{1}{2}}E_k\left(\left|\tanh W\right|\right),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\frac{\sqrt{(t_2-t_1)}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\frac{1}{2}}E_k\left(|\tanh W|\right)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\frac{1}{2}}E_k\left(|\tanh W|\right)\right\|_{\infty,[t,t_2]}\right)\Bigg\}<\infty. \tag{93}$$

Proof. From Corollary 31. □

In the following we consider the function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=e^{-\ell x}, \ell>0$ for every $x\in\mathbb{R}$. Let also $W=X_t$ be the Brownian Motion on simple graphs. Then the expectation

$$E_k\left(e^{-\ell W}\right)(t)=\int_0^\infty e^{-\ell y_k}p(t,x_k,y_k)dy_k+\sum_{j=1,j\neq k}^{n}\int_0^\infty e^{-\ell y_j}p(t,x_k,y_j)dy_j$$

is continuous in t.

Moreover

Corollary 42. *Let* $0<\beta^*<1$, $n\in\mathbb{N}:n^{1-\beta^*}>2$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, *where* $t_1,t_2\in(0,\infty)$ *with* $t_1<t_2$, *then for* $i=1,2$ *and* $j=0,1$

i)

$$\left|{}_iL_n\left(E_k\left(e^{-\ell W}\right)^{(j)}\right)(t)-E_k\left(e^{-\ell W}\right)^{(j)}(t)\right|\leq$$

$$K_i\cdot\left[\omega_1\left(E_k\left(e^{-\ell W}\right)^{(j)},\frac{1}{n^{\beta^*}}\right)+\hat{\beta}_{i,n}\left\|E_k\left(e^{-\ell W}\right)^{(j)}\right\|_{\infty,[t_1,t_2]}\right]=:\rho_i, \tag{94}$$

and

ii)

$$\left\|{}_iL_n\left(E_k\left(e^{-\ell W}\right)^{(j)}\right)(t)-E_k\left(e^{-\ell W}\right)^{(j)}(t)\right\|_{\infty,[t_1,t_2]}\leq\rho_i. \tag{95}$$

We get that $\lim_{n\to\infty}{}_iL_nE_k\left(e^{-\ell W}\right)^{(j)}(t)=E_k\left(e^{-\ell W}\right)^{(j)}$, *pointwise and uniformly.*

Proof. From Theorems 29 and 32. □

Next we present

Corollary 43. *Let* $0<\alpha,\beta^*<1$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, *where* $t_1,t_2\in(0,\infty)$ *with* $t_1<t_2$, *and* $n\in\mathbb{N}:n^{1-\beta^*}>2$. *Then for* $i=1,2$

i)

$$\left|{}_iL_n\left(E_k\left(e^{-\ell W}\right)(t)\right)-E_k\left(e^{-\ell W}\right)(t)\right|\leq$$

$$\frac{K_i}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}E_k\left(e^{-\ell W}\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}E_k\left(e^{-\ell W}\right),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{t-}^{\alpha}E_k\left(e^{-\ell W}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^\alpha+\left\|D_{*t}^{\alpha}E_k\left(e^{-\ell W}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^\alpha\right)\right\}, \tag{96}$$

and

ii)

$$\left\|{}_iL_nE_k\left(e^{-\ell W}\right)-E_k\left(e^{-\ell W}\right)\right\|_{\infty,[t_1,t_2]}\leq\frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\sup_{t\in[t_1,t_2]} \omega_1 \left(D^{\alpha}_{t-} E_k \left(e^{-\ell W}\right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left(D^{\alpha}_{*t} E_k \left(e^{-\ell W}\right) \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \frac{(t_2-t_1)^{\alpha}\hat{\beta}_{i,n}}{2} \left(\sup_{t\in[t_1,t_2]} \left\| D^{\alpha}_{t-} E_k \left(e^{-\ell W}\right) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D^{\alpha}_{*t} E_k \left(e^{-\ell W}\right) \right\|_{\infty,[t,t_2]} \right) \right\}. \quad (97)$$

Proof. From Theorem 30. □

When $\alpha = \frac{1}{2}$ we derive

Corollary 44. *Let $0 < \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$ and $n \in \mathbb{N} : n^{1-\beta^*} > 2$. Then for $i = 1, 2$*

i)

$$\left| {}_iL_n \left(E_k \left(e^{-\ell W}\right)(t) \right) - E_k \left(e^{-\ell W}\right)(t) \right| \leq$$

$$\frac{2K_i}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D^{\frac{1}{2}}_{t-} E_k \left(e^{-\ell W}\right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \omega_1 \left(D^{\frac{1}{2}}_{*t} E_k \left(e^{-\ell W}\right), \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\hat{\beta}_{i,n}}{2} \left(\left\| D^{\frac{1}{2}}_{t-} E_k \left(e^{-\ell W}\right) \right\|_{\infty,[t_1,t]} \sqrt{(t-t_1)} + \left\| D^{\frac{1}{2}}_{*t} E_k \left(e^{-\ell W}\right) \right\|_{\infty,[t,t_2]} \sqrt{(t_2-t)} \right) \right\}, \quad (98)$$

and

ii)

$$\left\| {}_iL_n E_k \left(e^{-\ell W}\right) - E_k \left(e^{-\ell W}\right) \right\|_{\infty,[t_1,t_2]} \leq \frac{2K_i}{\sqrt{\pi}}$$

$$\left\{ \frac{\left(\sup_{t\in[t_1,t_2]} \omega_1 \left(D^{\frac{1}{2}}_{t-} E_k \left(e^{-\ell W}\right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left(D^{\frac{1}{2}}_{*t} E_k \left(e^{-\ell W}\right), \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\sqrt{(t_2-t_1)}\hat{\beta}_{i,n}}{2} \left(\sup_{t\in[t_1,t_2]} \left\| D^{\frac{1}{2}}_{t-} E_k \left(e^{-\ell W}\right) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D^{\frac{1}{2}}_{*t} E_k \left(e^{-\ell W}\right) \right\|_{\infty,[t,t_2]} \right) \right\} < \infty. \quad (99)$$

Proof. From Corollary 31. □

Let the generalized logistic sigmoid function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = \left(1 + e^{-x}\right)^{\delta}, \delta > 0$ for every $x \in \mathbb{R}$. Let also $W = X_t$ be the Brownian Motion on simple graphs. Then the expectation

$$E_k \left(\left(1 + e^{-W}\right)^{\delta} \right)(t) = \int_0^{\infty} \left(1 + e^{-y_k}\right)^{\delta} p(t, x_k, y_k) dy_k + \sum_{j=1, j\neq k}^{n} \int_0^{\infty} \left(1 + e^{-y_j}\right)^{\delta} p(t, x_k, y_j) dy_j$$

is continuous in t.

Moreover

Corollary 45. *Let* $0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, *where* $t_1, t_2 \in (0, \infty)$ *with* $t_1 < t_2$, *then for* $i = 1, 2$ *and* $j = 0, 1$

i)

$$\left| {}_iL_n \left(E_k \left(\left(1 + e^{-W}\right)^{\delta} \right)^{(j)} \right)(t) - E_k \left(\left(1 + e^{-W}\right)^{\delta} \right)^{(j)}(t) \right| \leq$$

$$K_i \cdot \left[\omega_1 \left(E_k \left(\left(1 + e^{-W}\right)^{\delta} \right)^{(j)}, \frac{1}{n^{\beta^*}} \right) + \hat{\beta}_{i,n} \left\| E_k \left(\left(1 + e^{-W}\right)^{\delta} \right)^{(j)} \right\|_{\infty,[t_1,t_2]} \right] =: \rho_i, \quad (100)$$

and

ii)

$$\left\| {}_iL_n \left(E_k \left(\left(1 + e^{-W}\right)^{\delta} \right)^{(j)} \right)(t) - E_k \left(\left(1 + e^{-W}\right)^{\delta} \right)^{(j)}(t) \right\|_{\infty,[t_1,t_2]} \leq \rho_i. \quad (101)$$

We get that $\lim_{n\to\infty} {}_iL_n E_k \left(\left(1 + e^{-W}\right)^{\delta} \right)^{(j)}(t) = E_k \left(\left(1 + e^{-W}\right)^{\delta} \right)^{(j)}$, *pointwise and uniformly.*

Proof. From Theorems 29 and 32. □

Next we present

Corollary 46. *Let* $0 < \alpha, \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, *where* $t_1, t_2 \in (0, \infty)$ *with* $t_1 < t_2$, *and* $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then for* $i = 1, 2$

i)

$$\left| {}_iL_n \left(E_k \left(\left(1 + e^{-W}\right)^{\delta} \right)(t) \right) - E_k \left(\left(1 + e^{-W}\right)^{\delta} \right)(t) \right| \leq$$

$$\frac{K_i}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{t-}^{\alpha} E_k \left(\left(1 + e^{-W}\right)^{\delta} \right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \omega_1 \left(D_{*t}^{\alpha} E_k \left(\left(1 + e^{-W}\right)^{\delta} \right), \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \frac{\hat{\beta}_{i,n}}{2} \left(\left\| D_{t-}^{\alpha} E_k \left(\left(1 + e^{-W}\right)^{\delta} \right) \right\|_{\infty,[t_1,t]} (t - t_1)^{\alpha} + \left\| D_{*t}^{\alpha} E_k \left(\left(1 + e^{-W}\right)^{\delta} \right) \right\|_{\infty,[t,t_2]} (t_2 - t)^{\alpha} \right) \right\}, \quad (102)$$

and

ii)

$$\left\| {}_iL_n E_k \left(\left(1 + e^{-W}\right)^{\delta} \right) - E_k \left(\left(1 + e^{-W}\right)^{\delta} \right) \right\|_{\infty,[t_1,t_2]} \leq \frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\sup_{t\in[t_1,t_2]} \omega_1 \left(D_{t-}^{\alpha} E_k \left(\left(1 + e^{-W}\right)^{\delta} \right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left(D_{*t}^{\alpha} E_k \left(\left(1 + e^{-W}\right)^{\delta} \right) \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \frac{(t_2 - t_1)^{\alpha} \hat{\beta}_{i,n}}{2} \left(\sup_{t\in[t_1,t_2]} \left\| D_{t-}^{\alpha} E_k \left(\left(1 + e^{-W}\right)^{\delta} \right) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D_{*t}^{\alpha} E_k \left(\left(1 + e^{-W}\right)^{\delta} \right) \right\|_{\infty,[t,t_2]} \right) \right\}. \quad (103)$$

Proof. From Theorem 30. □

When $\alpha = \frac{1}{2}$ we derive

Corollary 47. *Let* $0 < \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$ *and* $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then for* $i = 1, 2$

i)

$$\left| {}_iL_n \left(E_k \left(\left(1 + e^{-W}\right)^\delta \right)(t) \right) - E_k \left(\left(1 + e^{-W}\right)^\delta \right)(t) \right| \leq$$

$$\frac{2K_i}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D_{t-}^{\frac{1}{2}} E_k \left(\left(1 + e^{-W}\right)^\delta \right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \omega_1 \left(D_{*t}^{\frac{1}{2}} E_k \left(\left(1 + e^{-W}\right)^\delta \right), \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\hat{\beta}_{i,n}}{2} \left(\left\| D_{t-}^{\frac{1}{2}} E_k \left(\left(1 + e^{-W}\right)^\delta \right) \right\|_{\infty,[t_1,t]} \sqrt{(t - t_1)} + \left\| D_{*t}^{\frac{1}{2}} E_k \left(\left(1 + e^{-W}\right)^\delta \right) \right\|_{\infty,[t,t_2]} \sqrt{(t_2 - t)} \right) \right\}, \tag{104}$$

and

ii)

$$\left\| {}_iL_n E_k \left(\left(1 + e^{-W}\right)^\delta \right) - E_k \left(\left(1 + e^{-W}\right)^\delta \right) \right\|_{\infty,[t_1,t_2]} \leq \frac{2K_i}{\sqrt{\pi}}$$

$$\left\{ \frac{\left(\sup_{t\in[t_1,t_2]} \omega_1 \left(D_{t-}^{\frac{1}{2}} E_k \left(\left(1 + e^{-W}\right)^\delta \right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left(D_{*t}^{\frac{1}{2}} E_k \left(\left(1 + e^{-W}\right)^\delta \right), \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\sqrt{(t_2 - t_1)}\hat{\beta}_{i,n}}{2} \left(\sup_{t\in[t_1,t_2]} \left\| D_{t-}^{\frac{1}{2}} E_k \left(\left(1 + e^{-W}\right)^\delta \right) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D_{*t}^{\frac{1}{2}} E_k \left(\left(1 + e^{-W}\right)^\delta \right) \right\|_{\infty,[t,t_2]} \right) \right\} < \infty. \tag{105}$$

Proof. From Corollary 31. □

When $\delta = 1$ we have the usual logistic sigmoid function.

For the last application we consider the Gompertz function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = e^{\mu e^{-x}} \mu < 0$ for every $x \in \mathbb{R}$. The Gompertz function is also a sigmoid function which describes growth as being slowest at the start and end of a given time period. Let also $W = X_t$ be the Brownian Motion on simple graphs. Then the expectation

$$E_k \left(e^{\mu e^{-W}} \right)(t) = \int_0^\infty e^{\mu e^{-y_k}} p(t, x_k, y_k) dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty e^{\mu e^{-y_j}} p(t, x_k, y_j) dy_j$$

is continuous in t.

Moreover

Corollary 48. *Let* $0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, *where* $t_1, t_2 \in (0, \infty)$ *with* $t_1 < t_2$, *then for* $i = 1, 2$ *and* $j = 0, 1$

i)

$$\left| {}_iL_n \left(E_k \left(e^{\mu e^{-W}} \right)^{(j)} \right)(t) - E_k \left(e^{\mu e^{-W}} \right)^{(j)}(t) \right| \leq$$

$$K_i \cdot \left[\omega_1 \left(E_k \left(e^{\mu e^{-W}} \right)^{(j)}, \frac{1}{n^{\beta^*}} \right) + \hat{\beta}_{i,n} \left\| E_k \left(e^{\mu e^{-W}} \right)^{(j)} \right\|_{\infty,[t_1,t_2]} \right] =: \rho_i, \tag{106}$$

and

ii)

$$\left\| {}_iL_n\left(E_k\left(e^{\mu e^{-W}}\right)^{(j)}\right)(t) - E_k\left(e^{\mu e^{-W}}\right)^{(j)}(t)\right\|_{\infty,[t_1,t_2]} \leq \rho_i. \tag{107}$$

We get that $\lim_{n\to\infty} {}_iL_nE_k\left(e^{\mu e^{-W}}\right)^{(j)}(t) = E_k\left(e^{\mu e^{-W}}\right)^{(j)}$, *pointwise and uniformly.*

Proof. From Theorems 29 and 32. □

Next we present

Corollary 49. *Let* $0 < \alpha, \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, *where* $t_1, t_2 \in (0, \infty)$ *with* $t_1 < t_2$, *and* $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then for* $i = 1, 2$

i)

$$\left| {}_iL_n\left(E_k\left(e^{\mu e^{-W}}\right)(t)\right) - E_k\left(e^{\mu e^{-W}}\right)(t)\right| \leq$$

$$\frac{K_i}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D^{\alpha}_{t-}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \omega_1\left(D^{\alpha}_{*t}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D^{\alpha}_{t-}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha} + \left\|D^{\alpha}_{*t}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}, \tag{108}$$

and

ii)

$$\left\| {}_iL_nE_k\left(e^{\mu e^{-W}}\right) - E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t_2]} \leq \frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D^{\alpha}_{t-}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]}\omega_1\left(D^{\alpha}_{*t}E_k\left(e^{\mu e^{-W}}\right)\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left.\frac{(t_2-t_1)^{\alpha}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D^{\alpha}_{t-}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]}\left\|D^{\alpha}_{*t}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t,t_2]}\right)\right\}. \tag{109}$$

Proof. From Theorem 30. □

When $\alpha = \frac{1}{2}$ we derive

Corollary 50. *Let* $0 < \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$ *and* $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then for* $i = 1, 2$

i)

$$\left| {}_iL_n\left(E_k\left(e^{\mu e^{-W}}\right)(t)\right) - E_k\left(e^{\mu e^{-W}}\right)(t)\right| \leq$$

$$\frac{2K_i}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D^{\frac{1}{2}}_{t-}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \omega_1\left(D^{\frac{1}{2}}_{*t}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{t-}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t]}\sqrt{(t-t_1)}+\left\|D_{*t}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t,t_2]}\sqrt{(t_2-t)}\right)\Bigg\}, \quad (110)$$

and

ii)

$$\left\|{}_iL_nE_k\left(e^{\mu e^{-W}}\right)-E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t_2]}\leq\frac{2K_i}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\sqrt{(t_2-t_1)}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t,t_2]}\right)\right\}<\infty. \quad (111)$$

Proof. From Corollary 31. □

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