VECTOR INEQUALITIES FOR ANALYTIC FUNCTIONS OF OPERATORS IN HILBERT SPACES AND APPLICATIONS FOR NUMERICAL RADIUS AND p-SCHATTEN NORM

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a complex Hilbert space, f be an analytic function on the domain G with $0 \in G$, A an operator with $\operatorname{Sp}(A) \subset G$ and γ a closed path in $G \setminus \{0\}$ with $\operatorname{Sp}(A) \subset \operatorname{ins}(\gamma)$, then for B, $C \in B(H)$ we have

$$\begin{aligned} & |\langle C^* [f(A) - f(0) I] Bx, y\rangle| \\ & \leq M(f, A; \gamma) \left\langle \left| |A|^\alpha B \right|^2 x, x \right\rangle^{1/2} \left\langle \left| |A^*|^{1-\alpha} C \right|^2 y, y \right\rangle^{1/2} \end{aligned}$$

for x, $y \in H$, where

$$M(f, A; \gamma) := \frac{1}{2\pi} \int_\gamma \frac{|f(\xi)| |d\xi|}{|\xi| (|\xi| - \|A\|)}.$$

Some natural applications for *numerical radius* and *p-Schatten norm* are also provided.

1. Introduction

In 1988, F. Kittaneh obtained the following generalization of Schwarz inequality [17]:

Theorem 1. *Assume that h and g are non-negative functions on $[0, \infty)$ which are continuous and satisfying the relation $h(t)g(t) = t$ for all $t \in [0, \infty)$. For any $T \in \mathcal{B}(H)$*

$$|\langle Tx, y\rangle| \leq \|h(|T|) x\| \|g(|T^*|) y\| \tag{1.1}$$

for all x, $y \in H$.

If we take $h(t) = t^\alpha$, $g(t) = t^{1-\alpha}$ with $\alpha \in [0,1]$, then we obtain *Kato's inequality*

$$|\langle Tx, y\rangle| \leq \| |T|^\alpha x\| \left\| |T^*|^{1-\alpha} y \right\| \text{ for all } x, y \in H. \tag{1.2}$$

The *numerical radius* $w(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.3}$$

Obviously, by (1.3), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leq w(T) \|x\|^2. \tag{1.4}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;

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(ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $\omega(T + V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{1.5}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [18], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.5):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{1.6}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [19] improved the inequality (1.5) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{1.7}$$

for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [13]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{1.8}$$

and

$$\omega^{2r}(T) \leq \left\|\alpha |T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|, \tag{1.9}$$

where $\alpha \in (0,1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (1.8) that

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{1.10}$$

and from (1.9) that

$$\omega^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{1.11}$$

For more related results, see the recent books on inequalities for numerical radii [10] and [5].

Let $(H; \langle ., . \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.12}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{1.13}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.13) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.14}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.15}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

For a large number of results concerning trace inequalities, see the recent survey paper [11].

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [25, p. 60-64]

$$\|A\|_p:=[\operatorname{tr}(|A|^p)]^{1/p}=\left(\sum_{i\in I}\langle|A|^p e_i,e_i\rangle\right)^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.16}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{1.17}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [25, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.18}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.19}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \|BA\|_p\le\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.20}$$

This implies that

$$\|CAB\|_p\le\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.21}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\le)\,\|AB\|_1\le\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{1.22}$$

For the theory of trace functionals and their applications the reader is referred to [24] and [25].

For $\mathcal{E} := \{e_i\}_{i\in I}$ an orthonormal basis of H we define for $A \in \mathcal{B}_p(H)$, $p \geq 1$

$$\|A\|_{\mathcal{E},p} := \left(\sum_{i\in I} |\langle Ae_i, e_i\rangle|^p\right)^{1/p}.$$

We observe that $\|\cdot\|_{\mathcal{E},p}$ is a norm on $\mathcal{B}_p(H)$ and

$$\|A\|_{\mathcal{E},p} \leq \|A\|_p \text{ for } A \in \mathcal{B}_p(H).$$

Further, we can take the supremum over all orthonormal basis in H we can also define, for $A \in \mathcal{B}_p(H)$, that

$$\omega_p(A) := \sup_{\mathcal{E}} \|A\|_{\mathcal{E},p} \leq \|A\|_p,$$

which is a *norm* on $\mathcal{B}_p(H)$.

It is also known that, if $\mathcal{E} = \{e_i\}_{i\in I}$ and $\mathcal{F} = \{f_i\}_{i\in I}$ are orthonormal basis, then [22]

$$\sup_{\mathcal{E},\mathcal{F}} \sum_{i\in I} |\langle Te_i, f_i\rangle|^s = \|T\|_s^s \text{ for } s \geq 1. \tag{1.23}$$

Let $\mathcal{B}$ be a unital Banach algebra, $A \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\operatorname{Sp}(A) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(A)$ in $\mathcal{B}$ by

$$f(A) := \frac{1}{2\pi i} \int_\delta f(\xi)(\xi - A)^{-1} d\xi, \tag{1.24}$$

where $\delta \subset G$ is taken to be closed rectifiable curve in G and such that $\operatorname{Sp}(A) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [7, pp. 201-204]) that $f(A)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\operatorname{Sp}(f(A)) = f(\operatorname{Sp}(A)) \tag{1.25}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [9] and [23].

2. Vector Inequalities

In 1988, F. Kittaneh [17, Corollary 7] obtained the following Schwarz type inequality for natural powers of operators:

Lemma 1. *Let $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$. Then for natural number $n \geq 1$ we have*

$$|\langle T^n x, y\rangle|^2 \leq \|T\|^{2n-2} \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle \tag{2.1}$$

for all x, $y \in H$.

Our first main result is as follows:

Theorem 3. *Let f be an analytic function on the domain G with $0 \in G$, A an operator with $\operatorname{Sp}(A) \subset G$ and γ a closed path in $G \setminus \{0\}$ with $\operatorname{Sp}(A) \subset \operatorname{ins}(\gamma)$, then for B, $C \in B(H)$ we have*

$$\begin{aligned} &|\langle C^*[f(A) - f(0)I]Bx, y\rangle| \\ &\leq M(f, A; \gamma) \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle^{1/2} \end{aligned} \tag{2.2}$$

for $x,\, y\in H$, *where* $\alpha \in [0,1]$ *and*

$$M\left(f,A;\gamma\right) := \frac{1}{2\pi}\int_{\gamma}\frac{\left|f\left(\xi\right)\right|\left|d\xi\right|}{\left|\xi\right|\left(\left|\xi\right|-\left\|A\right\|\right)}.$$

In particular, we have

$$\begin{aligned}(2.3)\qquad & \left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Bx,y\right\rangle\right| \\ & \leq M\left(f,A;\gamma\right)\left\langle \left|\left|A\right|^{1/2}B\right|^{2}x,x\right\rangle^{1/2}\left\langle \left|\left|A^{*}\right|^{1/2}C\right|^{2}y,y\right\rangle^{1/2}\end{aligned}$$

for $x,\, y\in H$.

Proof. From Kittaneh's result (2.1) we have that

$$(2.4)\qquad \left|\left\langle T^{n}x,y\right\rangle\right| \leq \left\|T\right\|^{n-1}\left\langle \left|T\right|^{2\alpha}x,x\right\rangle^{1/2}\left\langle \left|T^{*}\right|^{2(1-\alpha)}y,y\right\rangle^{1/2}$$

for all $x,\, y\in H$.

If we sum from $n=1$ to $n=m+1$, then we get

$$\begin{aligned}(2.5)\qquad & \left|\left\langle T\left(1+T+...+T^{m}\right)x,y\right\rangle\right| \\ & \leq \left(1+\left\|T\right\|+...+\left\|T\right\|^{m}\right)\left\langle \left|T\right|^{2\alpha}x,x\right\rangle^{1/2}\left\langle \left|T^{*}\right|^{2(1-\alpha)}y,y\right\rangle^{1/2}\end{aligned}$$

for all $x,\, y\in H$.

If we take $m\to\infty$ in (2.5) and take into account that $\sum_{m=0}^{\infty}T^{m}=\left(I-T\right)^{-1}$ and $\sum_{m=0}^{\infty}\left\|T\right\|^{m}=\left(I-\left\|T\right\|\right)^{-1}$ for $\left\|T\right\|<1$, then we get

$$(2.6)\qquad \left|\left\langle T\left(I-T\right)^{-1}x,y\right\rangle\right| \leq \left(I-\left\|T\right\|\right)^{-1}\left\langle \left|T\right|^{2\alpha}x,x\right\rangle^{1/2}\left\langle \left|T^{*}\right|^{2(1-\alpha)}y,y\right\rangle^{1/2}$$

for all $x,\, y\in H$.

Moreover, if we replace x by Bx and y by Cy in (2.6), then we obtain

$$\begin{aligned}(2.7)\qquad & \left|\left\langle C^{*}T\left(I-T\right)^{-1}Bx,y\right\rangle\right| \\ & \leq \left(I-\left\|T\right\|\right)^{-1}\left\langle \left|\left|T\right|^{\alpha}B\right|^{2}x,x\right\rangle^{1/2}\left\langle \left|\left|T^{*}\right|^{1-\alpha}C\right|^{2}y,y\right\rangle^{1/2}\end{aligned}$$

for all $x,\, y\in H$.

By the analytic functional calculus, we have

$$\begin{aligned}f\left(A\right)-f\left(0\right)I &= \frac{1}{2\pi i}\int_{\gamma}f\left(\xi\right)\left(\xi I-A\right)^{-1}d\xi-\frac{1}{2\pi i}\int_{\gamma}f\left(\xi\right)\xi^{-1}Id\xi \\ &= \frac{1}{2\pi i}\int_{\gamma}f\left(\xi\right)\left[\left(\xi I-A\right)^{-1}-\xi^{-1}I\right]d\xi \\ &= \frac{1}{2\pi i}\int_{\gamma}\xi^{-1}f\left(\xi\right)A\left(\xi I-A\right)^{-1}d\xi.\end{aligned}$$

Therefore

$$(2.8)\qquad \left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Bx,y\right\rangle = \frac{1}{2\pi i}\int_{\gamma}\xi^{-1}f\left(\xi\right)\left\langle C^{*}A\left(\xi I-A\right)^{-1}Bx,y\right\rangle d\xi$$

for $x,\, y\in H$.

By taking the modulus and using the properties of the integral, we obtain by (2.8) that

$$\begin{aligned} &|\langle C^{*}[f(A)-f(0)I]Bx,y\rangle| \\ &\leq \frac{1}{2\pi}\int_{\gamma}|\xi|^{-1}|f(\xi)|\left|\left\langle C^{*}A(\xi I-A)^{-1}Bx,y\right\rangle\right||d\xi| \\ &=\frac{1}{2\pi}\int_{\gamma}|\xi|^{-1}|f(\xi)|\left|\left\langle C^{*}\frac{A}{\xi}\left(I-\frac{A}{\xi}\right)^{-1}Bx,y\right\rangle\right||d\xi| . \end{aligned} \tag{2.9}$$

If $\xi\in\gamma$, then $\left\|\frac{A}{\xi}\right\|<1$ and by (2.7) for $T=\frac{A}{\xi}$ we get

$$\begin{aligned} &\left|\left\langle C^{*}\frac{A}{\xi}\left(I-\frac{A}{\xi}\right)^{-1}Bx,y\right\rangle\right| \\ &\leq\left(I-\left\|\frac{A}{\xi}\right\|\right)^{-1}\left\langle\left|\left|\frac{A}{\xi}\right|^{\alpha}B\right|^{2}x,x\right\rangle^{1/2}\left\langle\left|\left|\left(\frac{A}{\xi}\right)^{*}\right|^{1-\alpha}C\right|^{2}y,y\right\rangle^{1/2} \\ &=\left(\frac{|\xi|-\|A\|}{|\xi|}\right)^{-1}\frac{1}{|\xi|^{\alpha}|\xi|^{1-\alpha}}\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{1/2}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}y,y\right\rangle^{1/2} \\ &=(|\xi|-\|A\|)^{-1}\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{1/2}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}y,y\right\rangle^{1/2} \end{aligned} \tag{2.10}$$

for x, $y\in H$.

Therefore by (2.10) we get

$$\begin{aligned} &\frac{1}{2\pi}\int_{\gamma}|\xi|^{-1}|f(\xi)|\left|\left\langle C^{*}\frac{A}{\xi}\left(I-\frac{A}{\xi}\right)^{-1}Bx,y\right\rangle\right||d\xi| \\ &\leq\frac{1}{2\pi}\int_{\gamma}|\xi|^{-1}|f(\xi)|(|\xi|-\|A\|)^{-1}|d\xi| \\ &\times\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{1/2}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}y,y\right\rangle^{1/2} \\ &=M(f,A;\gamma)\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{1/2}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}y,y\right\rangle^{1/2} \end{aligned}$$

for x, $y\in H$, which by (2.9) gives the desired result (2.2). □

Remark 1. *With the assumptions of Theorem 3 and if* $f(0)=0$, *then we get*

$$\begin{aligned} &|\langle C^{*}f(A)Bx,y\rangle| \\ &\leq M(f,A;\gamma)\left\langle\left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{1/2}\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}y,y\right\rangle^{1/2} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &|\langle C^{*}f(A)Bx,y\rangle| \\ &\leq M(f,A;\gamma)\left\langle\left||A|^{1/2}B\right|^{2}x,x\right\rangle^{1/2}\left\langle\left||A^{*}|^{1/2}C\right|^{2}y,y\right\rangle^{1/2} \end{aligned} \tag{2.12}$$

for x, $y\in H$.

For $B=C=I$ in (2.11) and (2.12) we get the one operator inequalities

$$|\langle f(A)x,y\rangle| \leq M(f,A;\gamma)\left\langle |A|^{2\alpha}x,x\right\rangle^{1/2}\left\langle |A^*|^{2(1-\alpha)}y,y\right\rangle^{1/2} \tag{2.13}$$

and

$$|\langle f(A)x,y\rangle| \leq M(f,A;\gamma)\langle |A|x,x\rangle^{1/2}\langle |A^*|y,y\rangle^{1/2} \tag{2.14}$$

for $x,\ y\in H$.

If $A>0$ and we take $B=A^{-\beta}$, $C=A^{-1+\beta}$, $\beta\in[0,1]$, in (2.11) and (2.12), then we get

$$\left|\left\langle f(A)A^{-1}x,y\right\rangle\right| \leq M(f,A;\gamma)\left\langle A^{2(\alpha-\beta)}x,x\right\rangle^{1/2}\left\langle A^{2(\beta-\alpha)}y,y\right\rangle^{1/2} \tag{2.15}$$

and

$$\left|\left\langle f(A)A^{-1}x,y\right\rangle\right| \leq M(f,A;\gamma)\left\langle A^{2(1/2-\beta)}x,x\right\rangle^{1/2}\left\langle A^{2(\beta-1/2)}y,y\right\rangle^{1/2} \tag{2.16}$$

for $x,\ y\in H$.

Corollary 1. *With the assumptions of Theorem 3 and if*

$$\|f\|_{\gamma,\infty} := \sup_{\xi\in\gamma}|f(\xi)| < \infty,$$

then, by denoting

$$M_\infty(f,A;\gamma) := \frac{1}{2\pi}\|f\|_{\gamma,\infty}\int_\gamma \frac{|f(\xi)|\,|d\xi|}{|\xi|\,(|\xi|-\|A\|)},$$

we have

$$\begin{aligned} &|\langle C^*[f(A)-f(0)I]Bx,y\rangle| \\ &\leq M_\infty(f,A;\gamma)\left\langle \left||A|^\alpha B\right|^2x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1-\alpha}C\right|^2y,y\right\rangle^{1/2} \end{aligned} \tag{2.17}$$

for $x,\ y\in H$.

In particular, we have

$$\begin{aligned} &|\langle C^*[f(A)-f(0)I]Bx,y\rangle| \\ &\leq M_\infty(f,A;\gamma)\left\langle \left||A|^{1/2} B\right|^2x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1/2}C\right|^2y,y\right\rangle^{1/2} \end{aligned} \tag{2.18}$$

for $x,\ y\in H$.

Remark 2. *If we assume that $f:G\subset\mathbb{C}\to\mathbb{C}$ is an analytic function on the domain G and $A\in\mathcal{B}(H)$ with $\operatorname{Sp}(A)\subset D(0,R)\subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then by taking γ parametrized by $\xi(t)=Re^{2\pi it}$ where $t\in[0,1]$, then $d\xi(t)=2\pi iRe^{2\pi it}dt$, $|d\xi(t)|=2\pi Rdt$, $|\xi|=R$ and by (2.15) we get for $A,\ B\in\mathcal{B}(H)$ that*

$$\begin{aligned} &|\langle C^*[f(A)-f(0)I]Bx,y\rangle| \\ &\leq M(f,A;R)\left\langle \left||A|^\alpha B\right|^2x,x\right\rangle^{1/2}\left\langle \left||A^*|^{1-\alpha}C\right|^2y,y\right\rangle^{1/2} \end{aligned} \tag{2.19}$$

for $x,\ y\in H$, where $\alpha\in[0,1]$ and

$$M(f,A;R) := \frac{1}{R-\|A\|}\int_0^1\left|f\left(Re^{2\pi it}\right)\right|dt.$$

In particular,

$$|\langle C^{*}[f(A)-f(0)I]Bx,y\rangle| \le M(f,A;R)\left\langle \left||A|^{1/2}B\right|^{2}x,x\right\rangle^{1/2}\left\langle \left||A^{*}|^{1/2}C\right|^{2}y,y\right\rangle^{1/2} \tag{2.20}$$

for $x,\ y\in H$.

Moreover, if $\|f\|_{R,\infty}:=\sup_{t\in[0,1]}\left|f\left(Re^{2\pi it}\right)\right|<\infty$, *then we have the simpler inequalities*

$$|\langle C^{*}[f(A)-f(0)I]Bx,y\rangle| \le \frac{\|f\|_{R,\infty}}{R-\|A\|}\left\langle \left||A|^{\alpha}B\right|^{2}x,x\right\rangle^{1/2}\left\langle \left||A^{*}|^{1-\alpha}C\right|^{2}y,y\right\rangle^{1/2} \tag{2.21}$$

for $x,\ y\in H$, *where* $\alpha\in[0,1]$ *and*

$$|\langle C^{*}[f(A)-f(0)I]Bx,y\rangle| \le \frac{\|f\|_{R,\infty}}{R-\|A\|}\left\langle \left||A|^{1/2}B\right|^{2}x,x\right\rangle^{1/2}\left\langle \left||A^{*}|^{1/2}C\right|^{2}y,y\right\rangle^{1/2}. \tag{2.22}$$

3. Norm and Numerical Radius Inequalities

The following vector inequality for positive operators $A\ge 0$, obtained by C. A. McCarthy in [21] is well known,

$$\langle Ax,x\rangle^{p}\le\langle A^{p}x,x\rangle,\ p\ge 1$$

for $x\in H$, $\|x\|=1$.

Buzano's inequality [6],

$$\frac{1}{2}\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right]\ge|\langle x,e\rangle\langle e,y\rangle| \tag{3.1}$$

that holds for any $x,\ y,\ e\in H$ with $\|e\|=1$ will also be used in the sequel.

We also have the following norm and numerical radius inequalities:

Theorem 4. *Let* $f:G\subset\mathbb{C}\to\mathbb{C}$ *be an analytic function on the domain* G *with* $0\in G$ *and* $A\in\mathcal{B}(H)$ *with* $\operatorname{Sp}(A)\subset G$ *and* γ *a closed rectifiable path in* $G\setminus\{0\}$ *and such that* $\operatorname{Sp}(A)\subset\operatorname{ins}(\gamma)$. *If* $B,\ C\in\mathcal{B}(H)$, *then we have the norm inequality*

$$\|C^{*}[f(A)-f(0)I]B\|\le M(f,A;\gamma)\,\||A|^{\alpha}B\|\left\||A^{*}|^{1-\alpha}C\right\|. \tag{3.2}$$

We also have the numerical radius inequalities

$$\omega(C^{*}[f(A)-f(0)I]B)\le\frac{1}{2}M(f,A;\gamma)\left\|\left||A|^{\alpha}B\right|^{2}+\left||A^{*}|^{1-\alpha}C\right|^{2}\right\| \tag{3.3}$$

and

$$\begin{aligned}\omega^{2}(C^{*}[f(A)-f(0)I]B) &\le\frac{1}{2}M^{2}(f,A;\gamma)\\ &\times\left[\||A|^{\alpha}B\|^{2}\left\||A^{*}|^{1-\alpha}C\right\|^{2}+\omega\left(\left||A^{*}|^{1-\alpha}C\right|^{2}\left||A|^{\alpha}B\right|^{2}\right)\right].\end{aligned} \tag{3.4}$$

Proof. We have from (2.2), by taking the supremum over $\|x\| = \|y\| = 1$, that

$$\begin{aligned}
&\|C^* [f(A) - f(0) I] B\|^2 \\
&= \sup_{\|x\|=\|y\|=1} |\langle C^* [f(A) - f(0) I] Bx, y\rangle|^2 \\
&\leq M^2(f, A; \gamma) \sup_{\|x\|=1} \left\langle ||A|^{\alpha} B|^2 x, x\right\rangle \sup_{\|y\|=1} \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \\
&= M^2(f, A; \gamma) \left\| ||A|^{\alpha} B|^2 \right\| \left\| \left||A^*|^{1-\alpha} C\right|^2 \right\| \\
&= M^2(f, A; \gamma) \||A|^{\alpha} B\|^2 \left\||A^*|^{1-\alpha} C\right\|^2,
\end{aligned}$$

which gives (3.2).

From (2.2) we get, by taking $y = x$, the square root and using the *A-G-mean inequality*, that

$$\begin{aligned}
&|\langle C^* [f(A) - f(0) I] Bx, x\rangle| \\
&\leq M(f, A; \gamma) \left\langle ||A|^{\alpha} B|^2 x, x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} C\right|^2 x, x\right\rangle^{1/2} \\
&\leq \frac{1}{2} M(f, A; \gamma) \left(\left\langle ||A|^{\alpha} B|^2 x, x\right\rangle + \left\langle \left||A^*|^{1-\alpha} C\right|^2 x, x\right\rangle\right) \\
&= \frac{1}{2} M(f, A; \gamma) \left\langle \left(||A|^{\alpha} B|^2 + \left||A^*|^{1-\alpha} C\right|^2\right) x, x\right\rangle
\end{aligned} \tag{3.5}$$

for all $x \in H$.

By taking the supremum over $\|x\| = 1$ in (3.5) we get that

$$\begin{aligned}
&\omega(C^* [f(A) - f(0) I] B) \\
&= \sup_{\|x\|=1} |\langle C^* [f(A) - f(0) I] Bx, x\rangle| \\
&\leq \frac{1}{2} M(f, A; \gamma) \sup_{\|x\|=1} \left\langle \left(||A|^{\alpha} B|^2 + \left||A^*|^{1-\alpha} C\right|^2\right) x, x\right\rangle \\
&= \frac{1}{2} M(f, A; \gamma) \left\| ||A|^{\alpha} B|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right\|,
\end{aligned}$$

which proves (3.3).

From (2.2) for $y=x$ and Buzano's inequality we derive that

$$\begin{aligned}
&|\langle C^{*}[f(A)-f(0)I]Bx,x\rangle|^{2} \\
&\leq M^{2}(f,A;\gamma)\left\langle ||A|^{\alpha}B|^{2}x,x\right\rangle\left\langle x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle \\
&\leq \frac{1}{2}M^{2}(f,A;\gamma) \\
&\times\left[\left\|||A|^{\alpha}B|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle ||A|^{\alpha}B|^{2}x,\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\rangle\right|\right] \\
&=\frac{1}{2}M^{2}(f,A;\gamma) \\
&\times\left[\left\|||A|^{\alpha}B|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}||A|^{\alpha}B|^{2}x,x\right\rangle\right|\right]
\end{aligned} \tag{3.6}$$

for all $x\in H$.

By taking the supremum over $\|x\|=1$ in (3.6) we get that

$$\begin{aligned}
&\omega^{2}(C^{*}[f(A)-f(0)I]B) \\
&=\sup_{\|x\|=1}|\langle C^{*}[f(A)-f(0)I]Bx,x\rangle|^{2} \\
&\leq \frac{1}{2}M^{2}(f,A;\gamma) \\
&\times\sup_{\|x\|=1}\left[\left\|||A|^{\alpha}B|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}||A|^{\alpha}B|^{2}x,x\right\rangle\right|\right] \\
&\leq \frac{1}{2}M^{2}(f,A;\gamma) \\
&\times\left[\sup_{\|x\|=1}\left\{\left\|||A|^{\alpha}B|^{2}x\right\|\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\|\right\}\right. \\
&\left.+\sup_{\|x\|=1}\left|\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}||A|^{\alpha}B|^{2}x,x\right\rangle\right|\right] \\
&\leq \frac{1}{2}M^{2}(f,A;\gamma) \\
&\times\left[\sup_{\|x\|=1}\left\|||A|^{\alpha}B|^{2}x\right\|\sup_{\|x\|=1}\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}x\right\|\right. \\
&\left.+\sup_{\|x\|=1}\left|\left\langle\left||A^{*}|^{1-\alpha}C\right|^{2}||A|^{\alpha}B|^{2}x,x\right\rangle\right|\right] \\
&=\frac{1}{2}M^{2}(f,A;\gamma) \\
&\times\left[\left\|||A|^{\alpha}B|^{2}\right\|\left\|\left||A^{*}|^{1-\alpha}C\right|^{2}\right\|+\omega\left(\left||A^{*}|^{1-\alpha}C\right|^{2}||A|^{\alpha}B|^{2}\right)\right] \\
&=\frac{1}{2}M^{2}(f,A;\gamma)\left[\||A|^{\alpha}B\|^{2}\left\||A^{*}|^{1-\alpha}C\right\|^{2}+\omega\left(\left||A^{*}|^{1-\alpha}C\right|^{2}||A|^{\alpha}B|^{2}\right)\right],
\end{aligned}$$

which proves (6.10). □

Remark 3. *If we take $\alpha = 1/2$ in Theorem 4, then we get the norm inequality*

$$\|C^* [f(A) - f(0) I] B\| \le M(f, A; \gamma) \left\| |A|^{1/2} B \right\| \left\| |A^*|^{1/2} C \right\| \tag{3.7}$$

and the numerical radius inequalities

$$\omega(C^* [f(A) - f(0) I] B) \le \frac{1}{2} M(f, A; \gamma) \left\| \left| |A|^{1/2} B \right|^2 + \left| |A^*|^{1/2} C \right|^2 \right\| \tag{3.8}$$

and

$$\begin{aligned} &\omega^2 (C^* [f(A) - f(0) I] B) \\ &\le \frac{1}{2} M^2 (f, A; \gamma) \\ &\times \left[\left\| |A|^{1/2} B \right\|^2 \left\| |A^*|^{1/2} C \right\|^2 + \omega\left(\left| |A^*|^{1/2} C \right|^2 \left| |A|^{1/2} B \right|^2 \right) \right]. \end{aligned} \tag{3.9}$$

The second main result is as follows:

Theorem 5. *Assume that the conditions of Theorem 4 are satisfied. If $\alpha \in [0, 1]$, $r > 0$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 1$, then*

$$\begin{aligned} &\omega^{2r} (C^* [f(A) - f(0) I] B) \\ &\le M^{2r} (f, A; \gamma) \left\| \frac{1}{p} \left| |A|^{\alpha} B \right|^{2rp} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2rq} \right\|. \end{aligned} \tag{3.10}$$

If $r \ge 1$, then

$$\begin{aligned} &\omega^{2r} (C^* [f(A) - f(0) I] B) \\ &\le \frac{1}{2} M^{2r} (f, A; \gamma) \left[\left\| |A|^{\alpha} B \right\|^{2r} \left\| |A^*|^{1-\alpha} C \right\|^{2r} \right. \\ &\left. + \omega^r \left(\left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^{\alpha} B \right|^2 \right) \right]. \end{aligned} \tag{3.11}$$

If $r \ge 1$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 2$, then also

$$\begin{aligned} &\omega^{2r} (C^* [f(A) - f(0) I] B) \\ &\le \frac{1}{2} M^{2r} (f, A; \gamma) \left(\left\| \frac{1}{p} \left| |A|^{\alpha} B \right|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2qr} \right\| \right. \\ &\left. + \omega^r \left(\left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^{\alpha} B \right|^2 \right) \right). \end{aligned} \tag{3.12}$$

Proof. If we take the power $r>0$ in (2.2) written for $y=x$ then we get, by Young and McCarthy inequalities that

$$\begin{aligned}
&\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Bx,x\right\rangle\right|^{2r}\\
&\leq M^{2r}\left(f,A;\gamma\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{r}\\
&\leq M^{2r}\left(f,A;\gamma\right)\left[\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{rp}+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{rq}\right]\\
&\leq M^{2r}\left(f,A;\gamma\right)\left[\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2rp}x,x\right\rangle+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}x,x\right\rangle\right]\\
&=M^{2r}\left(f,A;\gamma\right)\left[\left\langle\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}x,x\right\rangle\right]
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, then we get that

$$\begin{aligned}
&\omega^{2r}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right)\\
&=\sup_{\|x\|=1}\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Bx,x\right\rangle\right|^{2r}\\
&\leq M^{2r}\left(f,A;\gamma\right)\sup_{\|x\|=1}\left[\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}\right)x,x\right\rangle\right]\\
&=M^{2r}\left(f,A;\gamma\right)\left\|\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2rq}\right\|,
\end{aligned}$$

which proves (3.10).

If we take the power $r\geq 1$ in (3.6) and by using the convexity of the power function, we get

$$\begin{aligned}
&\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Bx,x\right\rangle\right|^{2r}\\
&=M^{2r}\left(f,A;\gamma\right)\\
&\times\left[\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|}{2}\right]^{r}\\
&\leq M^{2r}\left(f,A;\gamma\right)\\
&\times\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r}+\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}}{2}
\end{aligned}\tag{3.13}$$

for $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, then we get that

$$\begin{aligned}
&\omega^{2r}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right)\\
&\leq M^{2r}\left(f,A;\gamma\right)\\
&\times\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|^{r}+\omega^{r}\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)}{2}\\
&=M^{2r}\left(f,A;\gamma\right)\\
&\times\frac{\left\|\left|A\right|^{\alpha}B\right\|^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2r}+\omega^{r}\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)}{2},
\end{aligned}$$

which proves (3.11).

Also, observe that

$$\begin{aligned}
&\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r}\\
&\leq\frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{pr}+\frac{1}{q}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{qr}\\
&=\frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{2\frac{pr}{2}}+\frac{1}{q}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{2\frac{qr}{2}}\\
&=\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{4}x,x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}x,x\right\rangle^{\frac{qr}{2}}\\
&\leq\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}x,x\right\rangle\\
&=\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle,
\end{aligned}$$

for $x\in H$ with $\|x\|=1$. Then

$$\begin{aligned}
&\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r}+\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}}{2}\\
&\leq\frac{1}{2}\left[\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle\right.\\
&\left.+\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}\right]
\end{aligned}$$

and by (3.13)

$$\begin{aligned}
&\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Bx,x\right\rangle\right|^{2r}\\
&\leq\frac{1}{2}B^{2r}\left(f,A;\gamma\right)\left[\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle\right.\\
&\left.+\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}\right]
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$, we derive (3.12). □

Remark 4. *If we take $r=1$ and $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ in (3.10), then we obtain*

$$\begin{aligned} &\omega^2\left(C^*\left[f\left(A\right)-f\left(0\right)I\right]B\right) \\ &\leq M^2\left(f,A;\gamma\right)\left\|\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2p}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{2q}\right\|, \end{aligned} \tag{3.14}$$

which for $p=q=2$ gives

$$\omega^2\left(C^*\left[f\left(A\right)-f\left(0\right)I\right]B\right)\leq\frac{1}{2}M^2\left(f,A;\gamma\right)\left\|\left|\left|A\right|^{\alpha}B\right|^{4}+\left|\left|A^*\right|^{1-\alpha}C\right|^{4}\right\|. \tag{3.15}$$

If we take $r=1$ and $p=q=2$ in (3.12), then we get

$$\begin{aligned} &\omega^2\left(C^*\left[f\left(A\right)-f\left(0\right)I\right]B\right) \\ &\leq\frac{1}{2}M^2\left(f,A;\gamma\right)\left(\frac{1}{2}\left\|\left|\left|A\right|^{\alpha}B\right|^{4}+\left|\left|A^*\right|^{1-\alpha}C\right|^{4}\right\|\right. \\ &\left.+\omega\left(\left|\left|A^*\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\right). \end{aligned} \tag{3.16}$$

If we take $r=2$ and $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ in (3.12), then we get

$$\begin{aligned} &\omega^4\left(C^*\left[f\left(A\right)-f\left(0\right)I\right]B\right) \\ &\leq\frac{1}{2}M^4\left(f,A;\gamma\right)\left(\left\|\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{4p}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{4q}\right\|\right. \\ &\left.+\omega^2\left(\left|\left|A^*\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\right). \end{aligned} \tag{3.17}$$

We also have:

Theorem 6. *With the assumptions of Theorem 4, we have for $r\geq 1,\ \lambda\in[0,1]$ that*

$$\begin{aligned} &\omega^2\left(C^*\left[f\left(A\right)-f\left(0\right)I\right]B\right) \\ &\leq M^2\left(f,A;\gamma\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r} \\ &\times\left\|\left|A\right|^{\alpha}B\right\|^{2\lambda}\left\|\left|A^*\right|^{1-\alpha}C\right\|^{2(1-\lambda)} \end{aligned} \tag{3.18}$$

for all $\alpha\in[0,1]$.

Also, we have

$$\begin{aligned} &\omega^2\left(C^*\left[f\left(A\right)-f\left(0\right)I\right]B\right) \\ &\leq M^2\left(f,A;\gamma\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r} \\ &\times\left\|\lambda\left|\left|A\right|^{\alpha}B\right|^{2r}+\left(1-\lambda\right)\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r} \end{aligned} \tag{3.19}$$

for all $\alpha\in[0,1]$ and $r\geq 1$.

Proof. From the first part of (3.6) we have

$$
\begin{aligned}
&\left|\left\langle C^{*}\left[f(A)-f(0)I\right]Bx,x\right\rangle\right|^{2}\\
&\leq M^{2}(f,A;\gamma)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle\\
&=M^{2}(f,A;\gamma)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{1-\lambda}\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{\lambda}\\
&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{\alpha}\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{1-\lambda}\\
&\leq M^{2}(f,A;\gamma)\left[(1-\lambda)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle+\lambda\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle\right]\\
&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{\lambda}\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{1-\lambda}
\end{aligned}
$$

for all $x\in H$, $\|x\|=1$.

If we take the power $r\geq 1$, then we get by the convexity of power r that

$$
\begin{aligned}
&\left|\left\langle C^{*}\left[f(A)-f(0)I\right]Bx,x\right\rangle\right|^{2r}\\
&\leq M^{2r}(f,A;\gamma)\left[(1-\lambda)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle+\lambda\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle\right]^{r}\\
&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r\lambda}\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{r(1-\lambda)}\\
&\leq M^{2r}(f,A;\gamma)\left[(1-\lambda)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r}+\lambda\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{r}\right]\\
&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r\lambda}\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{r(1-\lambda)}
\end{aligned}
\tag{3.20}
$$

for all $x\in H$, $\|x\|=1$.

If we use McCarthy inequality for power $r\geq 1$, then we get

$$
\begin{aligned}
&(1-\lambda)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r}+\lambda\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{r}\\
&\leq(1-\lambda)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2r}x,x\right\rangle+\lambda\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}x\right\rangle\\
&=\left\langle\left[(1-\lambda)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle
\end{aligned}
$$

and by (3.20)

$$
\begin{aligned}
&\left|\left\langle C^{*}\left[f(A)-f(0)I\right]Bx,x\right\rangle\right|^{2r}\\
&\leq M^{2r}(f,A;\gamma)\left[\left\langle\left[(1-\lambda)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\\
&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r\lambda}\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{r(1-\lambda)}
\end{aligned}
\tag{3.21}
$$

for all $x\in H$, $\|x\|=1$.

If we take the supremum over $\|x\|=1$, then we get

$$\begin{aligned}
&\omega^{2r}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right)\\
&=\sup_{\|x\|=1}\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Bx,x\right\rangle\right|^{2r}\\
&\leq M^{2r}\left(f,A;\gamma\right)\sup_{\|x\|=1}\left[\left\langle\left[\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\\
&\times\sup_{\|x\|=1}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r\lambda}\sup_{\|x\|=1}\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{r(1-\lambda)}\\
&=M^{2r}\left(f,A;\gamma\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\\
&\times\left\|\left|A\right|^{\alpha}B\right\|^{2r\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2r(1-\lambda)},
\end{aligned}$$

which gives (3.18).

We also have

$$\begin{aligned}
&\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Bx,x\right\rangle\right|^{2r}\\
&\leq M^{2r}\left(f,A;\gamma\right)\left[\left\langle\left[\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\\
&\times\left[\left\langle\left[\lambda\left|\left|A\right|^{\alpha}B\right|^{2r}+\left(1-\lambda\right)\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]
\end{aligned}$$

for all $x\in H$, $\|x\|=1$, which proves (3.19). □

Remark 5. *If we take $r=1$ in Theorem 6, then we get*

$$\begin{aligned}
&\omega^{2}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right)\\
&\leq M^{2}\left(f,A;\gamma\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\\
&\times\left\|\left|A\right|^{\alpha}B\right\|^{2\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2(1-\lambda)}
\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}
&\omega^{2}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right)\\
&\leq M^{2}\left(f,A;\gamma\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|\\
&\times\left\|\lambda\left|\left|A\right|^{\alpha}B\right|^{2}+\left(1-\lambda\right)\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|
\end{aligned}\tag{3.23}$$

for all α, $\lambda\in[0,1]$.

If we take $\lambda=1/2$ in (3.22), then we obtain

$$\begin{aligned}
&\omega^{2}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right)\\
&\leq\frac{1}{2}M^{2}\left(f,A;\gamma\right)\left\|\left|\left|A\right|^{\alpha}B\right|^{2}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\left\|\left|A\right|^{\alpha}B\right\|\left\|\left|A^{*}\right|^{1-\alpha}C\right\|.
\end{aligned}\tag{3.24}$$

If we take $r=2$ in Theorem 6, then we get

$$\begin{aligned}(3.25)\qquad &\omega^2\left(C^*\left[f(A)-f(0)I\right]B\right)\\ &\le M^2(f,A;\gamma)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^4+\lambda\left||A^*|^{1-\alpha}C\right|^4\right\|^{1/2}\\ &\times\left\||A|^{\alpha}B\right\|^{2\lambda}\left\||A^*|^{1-\alpha}C\right\|^{2(1-\lambda)}\end{aligned}$$

and

$$\begin{aligned}(3.26)\qquad &\omega^2\left(C^*\left[f(A)-f(0)I\right]B\right)\\ &\le M^2(f,A;\gamma)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^4+\lambda\left||A^*|^{1-\alpha}C\right|^4\right\|^{1/2}\\ &\times\left\|\lambda\left||A|^{\alpha}B\right|^4+(1-\lambda)\left||A^*|^{1-\alpha}C\right|^4\right\|^{1/2}\end{aligned}$$

for all $\alpha,\ \lambda\in[0,1]$.

If we take $\lambda=1/2$ in (3.25), then we obtain

$$\begin{aligned}(3.27)\quad &\omega^2\left(C^*\left[f(A)-f(0)I\right]B\right)\\ &\le\frac{\sqrt{2}}{2}M^2(f,A;\gamma)\left\|\left||A|^{\alpha}B\right|^4+\left||A^*|^{1-\alpha}C\right|^4\right\|^{1/2}\left\||A|^{\alpha}B\right\|\left\||A^*|^{1-\alpha}C\right\|.\end{aligned}$$

4. Inequalities for Trace of Operators

We have the following result for trace of operators:

Theorem 7. *Let $r\ge 1/2$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\ qr\ge 1$. Let $f:G\subset\mathbb{C}\to\mathbb{C}$ be an analytic function on the domain G with $0\in G$ and $A\in\mathcal{B}(H)$ with $\operatorname{Sp}(A)\subset G$ and γ a closed rectifiable path in $G\setminus\{0\}$ and such that $\operatorname{Sp}(A)\subset\operatorname{ins}(\gamma)$. If $B,\ C\in\mathcal{B}(H)$ with $|A|^{\alpha}B\in\mathcal{B}_{2pr}(H)$ and $|A^*|^{1-\alpha}C\in\mathcal{B}_{2qr}(H)$ for $\alpha\in[0,1]$, then $C^*\left[f(A)-f(0)I\right]B\in\mathcal{B}_{2r}(H)$ and*

$$(4.1)\qquad \left\|C^*\left[f(A)-f(0)I\right]B\right\|_{2r}\le M(f,A;\gamma)\left\||A|^{\alpha}B\right\|_{2pr}\left\||A^*|^{1-\alpha}C\right\|_{2qr}.$$

In particular,

$$(4.2)\qquad \left\|C^*\left[f(A)-f(0)I\right]B\right\|_{2r}\le M(f,A;\gamma)\left\||A|^{1/2}B\right\|_{2pr}\left\||A^*|^{1/2}C\right\|_{2qr}$$

for $|A|^{1/2}B\in\mathcal{B}_{2pr}(H)$ and $|A^*|^{1/2}C\in\mathcal{B}_{2qr}(H)$.

Proof. If we take in (2.2) the power $r>0$ and $x=e_i$, $y=f_i$ where $\mathcal{E}=\{e_i\}_{i\in I}$ and $\mathcal{F}=\{f_i\}_{i\in I}$ are orthonormal basis and sum, then we get

$$\begin{aligned}(4.3)\qquad &\sum_{i\in I}\left|\left\langle C^*\left[f(A)-f(0)I\right]Be_i,f_i\right\rangle\right|^{2r}\\ &\le M^{2r}(f,A;\gamma)\sum_{i\in I}\left\langle\left||A|^{\alpha}B\right|^2e_i,e_i\right\rangle^r\left\langle\left||A^*|^{1-\alpha}C\right|^2f_i,f_i\right\rangle^r.\end{aligned}$$

If we use the Hölder's inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we get

$$\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{r} \le\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{pr}\right)^{1/p}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\right)^{1/q}. \tag{4.4}$$

By the McCarthy inequality for pr, $qr\geq 1$, we have

$$\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{pr}\le\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2pr}e_{i},e_{i}\right\rangle$$

and

$$\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\le\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}f_{i},f_{i}\right\rangle,$$

therefore

$$\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{pr}\right)^{1/p}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\right)^{1/q}$$
$$\le\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2pr}e_{i},e_{i}\right\rangle\right)^{1/p}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\right)^{1/q}$$
$$=\left(\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2pr}\right)^{1/p}\left(\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2qr}\right)^{1/q}=\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}.$$

By (4.3) and (4.4) we derive

$$\sum_{i\in I}\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Be_{i},f_{i}\right\rangle\right|^{2r} \le M^{2r}\left(f,A;\gamma\right)\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}. \tag{4.5}$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (2.9), then by (1.23) we get

$$\left\|C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right\|_{2r}^{2r}\le M^{2r}\left(f,A;\gamma\right)\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}$$

and the inequality (4.1) is obtained. □

Remark 6. *If we take $r=1/2$ and $p=q=2$, then by (4.1) we get*

$$\left\|C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right\|_{1}\le M\left(f,A;\gamma\right)\left\|\left|A\right|^{\alpha}B\right\|_{2}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2} \tag{4.6}$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_{2}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2}\left(H\right)$ for $\alpha\in[0,1]$.

Also, if $r=1$ and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (4.1) we get

$$\left\|C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right\|_{2}\le M\left(f,A;\gamma\right)\left\|\left|A\right|^{\alpha}B\right\|_{2p}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2q} \tag{4.7}$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_{2p}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2q}\left(H\right)$ for $\alpha\in[0,1]$.

We also have:

Theorem 8. *Let $r \geq 1/2$, p, $q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Let $f : G \subset \mathbb{C} \to \mathbb{C}$ be an analytic function on the domain G with $0 \in G$ and $A \in \mathcal{B}(H)$ with $\operatorname{Sp}(A) \subset G$ and γ a closed rectifiable path in $G \setminus \{0\}$ and such that $\operatorname{Sp}(A) \subset \operatorname{ins}(\gamma)$. If $|A|^{\alpha} B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2q}(H)$ for $\alpha \in [0,1]$, then $C^*[f(A)-f(0)I]B \in \mathcal{B}_{2r}(H)$ and*

$$\|C^*[f(A)-f(0)I]B\|_{2r} \leq M(f,A;\gamma)\,\||A|^{\alpha} B\|_{2p} \left\||A^*|^{1-\alpha} C\right\|_{2q}. \tag{4.8}$$

In particular,

$$\|C^*[f(A)-f(0)I]B\|_{2r} \leq M(f,A;\gamma) \left\||A|^{1/2} B\right\|_{2p} \left\||A^*|^{1/2} C\right\|_{2q} \tag{4.9}$$

for $|A|^{1/2} B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2q}(H)$.

Proof. Assume that $\mathcal{E}=\{e_i\}_{i\in I}$ and $\mathcal{F}=\{f_i\}_{i\in I}$ are orthonormal basis in H. Observe that we have $\frac{1}{\frac{p}{r}}+\frac{1}{\frac{q}{r}}=1$ and by Hölder's inequality for $\frac{p}{r}$ and $\frac{q}{r}$ we have

$$\begin{aligned}
&\sum_{i\in I}\left\langle \left||A|^{\alpha} B\right|^2 e_i, e_i\right\rangle^r \left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i\right\rangle^r \\
&= \sum_{i\in I}\left[\left\langle \left||A|^{\alpha} B\right|^2 e_i, e_i\right\rangle^p\right]^{\frac{r}{p}} \left[\left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i\right\rangle^q\right]^{\frac{r}{q}} \\
&\leq \left(\sum_{i\in I}\left\langle \left||A|^{\alpha} B\right|^2 e_i, e_i\right\rangle^p\right)^{r/p} \left(\sum_{i\in I}\left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i\right\rangle^q\right)^{r/q}.
\end{aligned} \tag{4.10}$$

By McCarthy inequality for p, $q > 1$ we get

$$\sum_{i\in I}\left\langle \left||A|^{\alpha} B\right|^2 e_i, e_i\right\rangle^p \leq \sum_{i\in I}\left\langle \left||A|^{\alpha} B\right|^{2p} e_i, e_i\right\rangle$$

and

$$\sum_{i\in I}\left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i\right\rangle^q \leq \sum_{i\in I}\left\langle \left||A^*|^{1-\alpha} C\right|^{2q} f_i, f_i\right\rangle$$

and by (4.10)

$$\begin{aligned}
&\sum_{i\in I}\left\langle \left||A|^{\alpha} B\right|^2 e_i, e_i\right\rangle^r \left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i\right\rangle^r \\
&\leq \left(\sum_{i\in I}\left\langle \left||A|^{\alpha} B\right|^{2p} e_i, e_i\right\rangle\right)^{r/p} \left(\sum_{i\in I}\left\langle \left||A^*|^{1-\alpha} C\right|^{2q} f_i, f_i\right\rangle\right)^{r/q} \\
&= \||A|^{\alpha} B\|_{2p}^{2r} \left\||A^*|^{1-\alpha} C\right\|_{2q}^{2r}.
\end{aligned} \tag{4.11}$$

By (4.3) and (4.11) we get

$$\begin{aligned}
&\sum_{i\in I}\left|\left\langle C^*[f(A)-f(0)I]Be_i, f_i\right\rangle\right|^{2r} \\
&\leq M^{2r}(f,A;\gamma)\,\||A|^{\alpha} B\|_{2p}^{2r} \left\||A^*|^{1-\alpha} C\right\|_{2q}^{2r}.
\end{aligned} \tag{4.12}$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (4.12) we get

$$\|C^*[f(A)-f(0)I]B\|_{2r}^{2r} \leq M^{2r}(f,A;\gamma)\,\||A|^{\alpha} B\|_{2p}^{2r} \left\||A^*|^{1-\alpha} C\right\|_{2q}^{2r}$$

and the inequality (4.8) is thus proved. □

Remark 7. *If we take $p=q=2r=s\geq 1$, then by (4.8) we get*

$$\left\| C^{*}\left[f\left(A\right) -f\left(0\right) I\right] B\right\| _{s}\leq M\left(f,A;\gamma \right) \left\| \left\vert A\right\vert ^{\alpha }B\right\| _{2s}\left\| \left\vert A^{*}\right\vert ^{1-\alpha }C\right\| _{2s} \tag{4.13}$$

provided that $\left\vert A\right\vert ^{\alpha }B\in \mathcal{B}_{2s}\left(H\right)$ and $\left\vert A^{*}\right\vert ^{1-\alpha }C\in \mathcal{B}_{2s}\left(H\right)$ for $\alpha \in \left[0,1\right]$.

For $\alpha =1/2$ we have

$$\left\| C^{*}\left[f\left(A\right) -f\left(0\right) I\right] B\right\| _{s}\leq M\left(f,A;\gamma \right) \left\| \left\vert A\right\vert ^{1/2}B\right\| _{2s}\left\| \left\vert A^{*}\right\vert ^{1/2}C\right\| _{2s} \tag{4.14}$$

provided that $\left\vert A\right\vert ^{1/2}B\in \mathcal{B}_{2s}\left(H\right)$ and $\left\vert A^{*}\right\vert ^{1/2}C\in \mathcal{B}_{2s}\left(H\right)$.

If $r=2$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, then

$$\left\| C^{*}\left[f\left(A\right) -f\left(0\right) I\right] B\right\| _{4}\leq M\left(f,A;\gamma \right) \left\| \left\vert A\right\vert ^{\alpha }B\right\| _{2p}\left\| \left\vert A^{*}\right\vert ^{1-\alpha }C\right\| _{2q} \tag{4.15}$$

provided that $\left\vert A\right\vert ^{\alpha }B\in \mathcal{B}_{2p}\left(H\right)$ and $\left\vert A^{*}\right\vert ^{1-\alpha }C\in \mathcal{B}_{2q}\left(H\right)$ for $\alpha \in \left[0,1\right]$.

In particular,

$$\left\| C^{*}\left[f\left(A\right) -f\left(0\right) I\right] B\right\| _{4}\leq M\left(f,A;\gamma \right) \left\| \left\vert A\right\vert ^{1/2}B\right\| _{2p}\left\| \left\vert A^{*}\right\vert ^{1/2}C\right\| _{2q} \tag{4.16}$$

for $\left\vert A\right\vert ^{1/2}B\in \mathcal{B}_{2p}\left(H\right)$ and $\left\vert A^{*}\right\vert ^{1/2}C\in \mathcal{B}_{2q}\left(H\right)$.

Theorem 9. *Let $f:G\subset \mathbb{C}\rightarrow \mathbb{C}$ be an analytic function on the domain G with $0\in G$ and $A\in \mathcal{B}\left(H\right)$ with $\operatorname{Sp}\left(A\right) \subset G$ and γ a closed rectifiable path in $G\setminus \left\{ 0\right\}$ and such that $\operatorname{Sp}\left(A\right) \subset \operatorname{ins}\left(\gamma \right)$. If $r\geq 1/2,$ $p,$ $q>1$ with $\frac{1}{p}+\frac{1}{q}=1,$ $pr,$ $qr\geq 1$ and $\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2pr},$ $\left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2qr}\in \mathcal{B}_{1}\left(H\right)$, then $C^{*}\left[f\left(A\right) -f\left(0\right) I\right] B\in \mathcal{B}_{2r}\left(H\right)$ and*

$$\begin{aligned} &\omega _{2r}^{2r}\left(C^{*}\left[f\left(A\right) -f\left(0\right) I\right] B\right) \\ &\leq M^{2r}\left(f,A;\gamma \right) \operatorname{tr}\left(\frac{1}{p}\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2pr}+\frac{1}{q}\left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2qr}\right) . \end{aligned} \tag{4.17}$$

If $r\geq 1$ and $\left\vert A\right\vert ^{\alpha }B,\left\vert A^{*}\right\vert ^{1-\alpha }C\in \mathcal{B}_{4r}\left(H\right)$, then $C^{*}\left[f\left(A\right) -f\left(0\right) I\right] B\in \mathcal{B}_{2r}\left(H\right)$ and

$$\begin{aligned} &\omega _{2r}^{2r}\left(C^{*}\left[f\left(A\right) -f\left(0\right) I\right] B\right) \\ &\leq \frac{1}{2}M^{2r}\left(f,A;\gamma \right) \\ &\times \left(\left\| \left\vert A\right\vert ^{\alpha }B\right\| _{4r}^{2r}\left\| \left\vert A^{*}\right\vert ^{1-\alpha }C\right\| _{4r}^{2r}+\omega _{r}^{r}\left(\left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2}\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2}\right) \right) \\ &\leq \frac{1}{2}M^{2r}\left(f,A;\gamma \right) \\ &\times \left(\left\| \left\vert A\right\vert ^{\alpha }B\right\| _{4r}^{2r}\left\| \left\vert A^{*}\right\vert ^{1-\alpha }C\right\| _{4r}^{2r}+\left\| \left\vert \left\vert A^{*}\right\vert ^{1-\alpha }C\right\vert ^{2}\left\vert \left\vert A\right\vert ^{\alpha }B\right\vert ^{2}\right\| _{r}^{r}\right) . \end{aligned} \tag{4.18}$$

If $r \geq 1$, p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, $pr, qr \geq 2$, *then*

$$\begin{aligned} &\omega_{2r}^{2r}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right) \\ &\leq \frac{1}{2}M^{2r}\left(f,A;\gamma\right)\left[\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)\right. \\ &\left.+\omega_{r}^{r}\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\right] \\ &\leq \frac{1}{2}M^{2r}\left(f,A;\gamma\right)\left[\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)\right. \\ &\left.+\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right\|_{r}^{r}\right]. \end{aligned} \tag{4.19}$$

Proof. From (2.2) for $y = x$ we have that

$$\begin{aligned} &\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Bx,x\right\rangle\right|^{2} \\ &\leq M^{2}\left(f,A;\gamma\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle \end{aligned} \tag{4.20}$$

for $x \in H$ with $\|x\| = 1$.

If we take the power $r > 0$, we get, by Young and McCarthy inequalities, that

$$\begin{aligned} &\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Bx,x\right\rangle\right|^{2r} \\ &\leq M^{2r}\left(f,A;\gamma\right)\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{r} \\ &\leq M^{2r}\left(f,A;\gamma\right)\left[\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{pr}+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{qr}\right] \\ &\leq M^{2r}\left(f,A;\gamma\right)\left[\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}x,x\right\rangle\right] \\ &= M^{2r}\left(f,A;\gamma\right)\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

If $\mathcal{E} = \{e_i\}_{i \in I}$ is an orthonormal basis, then by taking $x = e_i$ and summing over $i \in I$ we get

$$\begin{aligned} &\left\|C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right\|_{\mathcal{E},2r}^{2r} \\ &= \sum_{i\in I}\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Be_{i},e_{i}\right\rangle\right|^{2r} \\ &\leq M^{2r}\left(f,A;\gamma\right)\sum_{i\in I}\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)e_{i},e_{i}\right\rangle \\ &= M^{2r}\left(f,A;\gamma\right)\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right), \end{aligned}$$

which, by taking the supremum over $\mathcal{E}$, proves (4.17).

By Buzano's inequality we have

$$\begin{aligned}
&\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle \left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle \\
&\leq \frac{1}{2}\left[\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\| + \left|\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle\right|\right] \\
&= \frac{1}{2}\left[\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\| + \left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right]
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

If we take the power $r\geq 1$ and use the convexity of power function, then we get

$$\begin{aligned}
&\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r} \left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{r} \\
&\leq \left[\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\| + \left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|}{2}\right]^{r} \\
&\leq \frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r} + \left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}}{2} \\
&= \frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{2\frac{r}{2}}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{2\frac{r}{2}} + \left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}}{2} \\
&= \frac{\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}x,x\right\rangle^{\frac{r}{2}}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}x,x\right\rangle^{\frac{r}{2}} + \left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}}{2}
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

Therefore

$$\begin{aligned}
(4.21)\qquad &\left\|C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right\|_{\mathcal{E},2r}^{2r} \\
&= \sum_{i\in I}\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Be_{i},e_{i}\right\rangle\right|^{2r} \\
&\leq M^{2r}\left(f,A;\gamma\right)\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r}\left\langle e_{i},\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i}\right\rangle^{r} \\
&\leq \frac{1}{2}M^{2r}\left(f,A;\gamma\right)\left[\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}e_{i},e_{i}\right\rangle^{\frac{r}{2}}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}e_{i},e_{i}\right\rangle^{\frac{r}{2}}\right. \\
&\left. + \sum_{i\in I}\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle\right|^{r}\right].
\end{aligned}$$

Using Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}e_i,e_i\right\rangle^{\frac{r}{2}}\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^{4}e_i,e_i\right\rangle^{\frac{r}{2}}\\
&\le\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}e_i,e_i\right\rangle^{r}\right)^{1/2}\left(\sum_{i\in I}\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^{4}e_i,e_i\right\rangle^{r}\right)^{1/2}\\
&\le\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4r}e_i,e_i\right\rangle\right)^{1/2}\left(\sum_{i\in I}\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^{4r}e_i,e_i\right\rangle\right)^{1/2}\\
&=\left\|\left|A\right|^{\alpha}B\right\|_{4r}^{2r}\left\|\left|A^*\right|^{1-\alpha}C\right\|_{4r}^{2r},
\end{aligned}$$

where for the last inequality we used McCarthy's result for $r\geq 1$. This proves (4.18).

Further, if we use Young's inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$ab\le\frac{1}{p}a^p+\frac{1}{q}b^q,\ a,b\ge 0,$$

then we get

$$\begin{aligned}
\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^{2}x\right\|^{r}&\le\frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{pr}+\frac{1}{q}\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^{2}x\right\|^{qr}\\
&=\frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{2\frac{pr}{2}}+\frac{1}{q}\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^{2}x\right\|^{2\frac{qr}{2}}\\
&=\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{4}x,x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle\left|\left|A^*\right|^{1-\alpha}C\right|^{4}x,x\right\rangle^{\frac{qr}{2}}\\
&\le\frac{1}{p}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle\left|\left|A^*\right|^{1-\alpha}C\right|^{2qr}x,x\right\rangle\\
&=\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle
\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

Therefore

$$\begin{aligned}
&\left\|C^*\left[f\left(A\right)-f\left(0\right)I\right]B\right\|_{\mathcal{E},2r}^{2r}\\
&=\sum_{i\in I}\left|\left\langle C^*\left[f\left(A\right)-f\left(0\right)I\right]Be_i,e_i\right\rangle\right|^{2r}\\
&\le M^{2r}\left(f,A;\gamma\right)\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r}\left\langle e_i,\left|\left|A^*\right|^{1-\alpha}C\right|^{2}e_i\right\rangle^{r}
\end{aligned}$$

$$\leq \frac{1}{2}M^{2r}\left(f,A;\gamma\right)\left[\sum_{i\in I}\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)e_{i},e_{i}\right\rangle\right.$$
$$\left.+\sum_{i\in I}\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle\right|^{r}\right]$$
$$=\frac{1}{2}M^{2r}\left(f,A;\gamma\right)\left[\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)\right.$$
$$\left.+\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right\|_{\mathcal{E},r}^{r}\right],$$

which proves, by taking the supremum over $\mathcal{E}$, the desired inequality (4.19). □

Remark 8. *Let* $\alpha\in[0,1]$. *If* $r=1/2$, p, $q=2$ *and* $\left|\left|A\right|^{\alpha}B\right|^{2}$, $\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\in\mathcal{B}_{1}(H)$, *then* $C^{*}\left[f(A)-f(0)I\right]B\in\mathcal{B}_{1}(H)$ *and by (4.17) we get*

$$\omega_{1}\left(C^{*}\left[f(A)-f(0)I\right]B\right)\leq\frac{1}{2}M\left(f,A;\gamma\right)\operatorname{tr}\left(\left|\left|A\right|^{\alpha}B\right|^{2}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right).\tag{4.22}$$

If $r=1$ *and* p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then by (4.17) we obtain*

$$\omega_{2}^{2}\left(C^{*}\left[f(A)-f(0)I\right]B\right)\leq M^{2}\left(f,A;\gamma\right)\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2p}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2q}\right),\tag{4.23}$$

provided that $\left|\left|A\right|^{\alpha}B\right|^{2p}$, $\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2q}\in\mathcal{B}_{1}(H)$.

If we take $r=1$ *in (4.18), then we get*

$$\begin{aligned}\omega_{2}^{2}\left(C^{*}\left[f(A)-f(0)I\right]B\right)&\leq\frac{1}{2}M^{2}\left(f,A;\gamma\right)\\&\times\left(\left\|\left|A\right|^{\alpha}B\right\|_{4}^{2}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{4}^{2}+\omega_{1}\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\right)\\&\leq\frac{1}{2}M^{2}\left(f,A;\gamma\right)\\&\times\left(\left\|\left|A\right|^{\alpha}B\right\|_{4}^{2}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{4}^{2}+\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right\|_{1}\right),\end{aligned}\tag{4.24}$$

provided that $\left|A\right|^{\alpha}B$, $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{4}(H)$.

If $r=1$ and $p=q=2$ in (4.19), then we get for $\left|\left|A\right|^{\alpha}B\right|^{2p}$, $\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2q}\in\mathcal{B}_{1}\left(H\right)$ that

$$\begin{aligned}
(4.25)\qquad & \omega_{2}^{2}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right)\\
&\leq\frac{1}{4}M^{2}\left(f,A;\gamma\right)\left[\operatorname{tr}\left(\left|\left|A\right|^{\alpha}B\right|^{2p}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2q}\right)\right.\\
&\left.+\frac{1}{2}M^{2}\left(f,A;\gamma\right)\omega_{1}\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\right.\\
&\leq\frac{1}{4}M^{2}\left(f,A;\gamma\right)\operatorname{tr}\left(\left|\left|A\right|^{\alpha}B\right|^{2p}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2q}\right)\\
&+\frac{1}{2}M^{2}\left(f,A;\gamma\right)\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right\|_{1}.
\end{aligned}$$

We also have:

Theorem 10. *With the assumptions of Theorem 9, we have for $r\geq1$, $\lambda\in[0,1]$ that*

$$\begin{aligned}
(4.26)\qquad & \omega_{2r}^{2r}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right)\\
&\leq M^{2r}\left(f,A;\gamma\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\\
&\times\left\|\left|A\right|^{\alpha}B\right\|_{2r}^{2r\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)},
\end{aligned}$$

provided that $\left|A\right|^{\alpha}B$, $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2r}\left(H\right)$.

In particular,

$$\begin{aligned}
(4.27)\qquad & \omega_{2r}^{2r}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right)\\
&\leq\frac{1}{2}M^{2r}\left(f,A;\gamma\right)\left\|\left|\left|A\right|^{\alpha}B\right|^{2r}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\\
&\times\left\|\left|A\right|^{\alpha}B\right\|_{2r}^{r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2r}^{r}.
\end{aligned}$$

Proof. If $\mathcal{E}=\left\{e_{i}\right\}_{i\in I}$ is an orthonormal basis, then by taking $x=e_{i}$ in (3.21) and summing over $i\in I$ we get

$$\begin{aligned}
(4.28)\qquad & \sum_{i\in I}\left|\left\langle C^{*}\left[f\left(A\right)-f\left(0\right)I\right]Be_{i},e_{i}\right\rangle\right|^{2r}\\
&\leq M^{2r}\left(f,A;\gamma\right)\sum_{i\in I}\left[\left\langle\left[\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]e_{i},e_{i}\right\rangle\right]\\
&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r\lambda}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i},e_{i}\right\rangle^{r(1-\lambda)}\\
&\leq M^{2r}\left(f,A;\gamma\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\\
&\times\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r\lambda}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i},e_{i}\right\rangle^{r(1-\lambda)}.
\end{aligned}$$

If we use Hölder's inequality for $p=\frac{1}{\lambda}$, $q=\frac{1}{1-\lambda}$, then we have

$$\begin{aligned}
&\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r\lambda}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i},e_{i}\right\rangle^{r(1-\lambda)}\\
&\leq\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{r}\right)^{\lambda}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_{i},e_{i}\right\rangle^{r}\right)^{1-\lambda}\\
&\leq\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2r}e_{i},e_{i}\right\rangle\right)^{\lambda}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}e_{i},e_{i}\right\rangle\right)^{1-\lambda}\\
&=\left\|\left|A\right|^{\alpha}B\right\|_{2r}^{2r\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)},
\end{aligned}$$

which proves (4.26). □

Remark 9. *If we take $r=1$ in Theorem 10, then we get for $\alpha\in[0,1]$ that*

$$\begin{aligned}
&\omega_{2}^{2}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right)\\
&\leq M^{2}\left(f,A;\gamma\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|\\
&\times\left\|\left|A\right|^{\alpha}B\right\|_{2}^{2\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2}^{2(1-\lambda)},
\end{aligned}\tag{4.29}$$

provided that $\left|A\right|^{\alpha}B,\ \left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2}\left(H\right).$

In particular,

$$\begin{aligned}
&\omega_{2}^{2}\left(C^{*}\left[f\left(A\right)-f\left(0\right)I\right]B\right)\\
&\leq\frac{1}{2}M^{2}\left(f,A;\gamma\right)\left\|\left|\left|A\right|^{\alpha}B\right|^{2}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|\left\|\left|A\right|^{\alpha}B\right\|_{2}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2}.
\end{aligned}\tag{4.30}$$

5. Applications for Complex Perspectives

Let f be a continuous function defined on the interval I of real numbers, Q a self-adjoint operator on the Hilbert space H and P a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(P^{-1/2}QP^{-1/2}\right)\subset\mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_{f}\left(Q,P\right)$ by setting

$$\mathcal{P}_{f}\left(Q,P\right):=P^{1/2}f\left(P^{-1/2}QP^{-1/2}\right)P^{1/2}.$$

If P and B are commutative, then

$$\mathcal{P}_{\Phi}\left(Q,P\right)=Pf\left(QP^{-1}\right)$$

provided $\operatorname{Sp}\left(QP^{-1}\right)\subset\mathring{I}$.

Let f be an analytic function on the domain G, $P>0$ and $T\in B\left(H\right)$ with $\operatorname{Sp}\left(P^{-1/2}TP^{-1/2}\right)\subset G$ and γ a closed path in G with $\operatorname{Sp}\left(P^{-1/2}TP^{-1/2}\right)\subset\operatorname{ins}\left(\gamma\right)$. We can define the *complex operator valued perspective* by,

$$\mathcal{P}_{f}\left(T,P\right):=P^{1/2}f\left(P^{-1/2}TP^{-1/2}\right)P^{1/2}.$$

By using the analytic functional calculus we have the representation

$$\mathcal{P}_{f}\left(T,P\right):=\frac{1}{2\pi i}\int_{\gamma}f\left(\xi\right)P^{1/2}\left(\xi I-P^{-1/2}TP^{-1/2}\right)^{-1}P^{1/2}d\xi.$$

We have the following bounds:

Proposition 1. *Let f be an analytic function on the domain G, $0 \in G$ and $f(0) = 0$. If $P > 0$ and $T \in B(H)$ with $\operatorname{Sp}\left(P^{-1/2}TP^{-1/2}\right) \subset G$ and γ a closed path in $G \setminus \{0\}$ with $\operatorname{Sp}\left(P^{-1/2}TP^{-1/2}\right) \subset \operatorname{ins}(\gamma)$, then*

$$\begin{aligned}
|\langle \mathcal{P}_f(T,P)x,y\rangle| &\le M\left(f, P^{-1/2}TP^{-1/2};\gamma\right) \\
&\quad \times \left\langle \left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^2 x,x\right\rangle^{1/2} \\
&\quad \times \left\langle \left|\left|P^{-1/2}T^*P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^2 y,y\right\rangle^{1/2} \\
&\le M\left(f, P^{-1/2}TP^{-1/2};\gamma\right) \\
&\quad \times \left\|P^{-1/2}TP^{-1/2}\right\| \langle Px,x\rangle^{1/2} \langle Py,y\rangle^{1/2}
\end{aligned} \tag{5.1}$$

for all $\alpha \in [0,1]$ and x, $y \in H$. Here

$$M\left(f, P^{-1/2}TP^{-1/2};\gamma\right) = \frac{1}{2\pi}\int_\gamma \frac{|f(\xi)|\,|d\xi|}{|\xi|\left(|\xi| - \left\|P^{-1/2}TP^{-1/2}\right\|\right)}.$$

Proof. Observe that for x, $y \in H$,

$$\begin{aligned}
\left\langle \left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^2 x,x\right\rangle &= \left\|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}x\right\|^2 \\
&\le \left\|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha}\right\|^2 \left\|P^{1/2}x\right\|^2 \\
&= \left\|P^{-1/2}TP^{-1/2}\right\|^{2\alpha} \langle Px,x\rangle
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \left|\left|P^{-1/2}T^*P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^2 y,y\right\rangle &= \left\|\left|P^{-1/2}T^*P^{-1/2}\right|^{1-\alpha} P^{1/2}y\right\|^2 \\
&\le \left\|\left|P^{-1/2}T^*P^{-1/2}\right|^{1-\alpha}\right\|^2 \left\|P^{1/2}y\right\|^2 \\
&= \left\|P^{-1/2}TP^{-1/2}\right\|^{2(1-\alpha)} \langle Py,y\rangle.
\end{aligned}$$

From (2.11) we have for $B = C = P^{1/2}$ and $A = P^{-1/2}TP^{-1/2}$ that

$$\begin{aligned}
&\left|\left\langle P^{1/2} f\left(P^{-1/2}TP^{-1/2}\right) P^{1/2}x,y\right\rangle\right| \\
&\le M\left(f, P^{-1/2}TP^{-1/2};\gamma\right) \\
&\times \left\langle \left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^2 x,x\right\rangle^{1/2} \left\langle \left|\left|P^{-1/2}T^*P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^2 y,y\right\rangle^{1/2} \\
&\le M\left(f, P^{-1/2}TP^{-1/2};\gamma\right) \left\|P^{-1/2}TP^{-1/2}\right\| \langle Px,x\rangle^{1/2} \langle Py,y\rangle^{1/2}
\end{aligned}$$

for x, $y \in H$. This proves the desired result (5.1). □

Corollary 2. *Let f be an analytic function on the domain G, $0 \in G$ and $f(0) = 0$. If $P > 0$ and $T \in B(H)$ with* $\operatorname{Sp}\left(P^{-1/2}TP^{-1/2}\right) \subset D(0,R) \subset G$, *then*

$$|\langle \mathcal{P}_f(T,P)x,y\rangle| \le M\left(f, P^{-1/2}TP^{-1/2}; R\right) \times \left\langle \left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^2 x,x\right\rangle^{1/2} \times \left\langle \left|\left|P^{-1/2}T^*P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^2 y,y\right\rangle^{1/2} \le M\left(f, P^{-1/2}TP^{-1/2}; R\right) \times \left\|P^{-1/2}TP^{-1/2}\right\| \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} \tag{5.2}$$

for all $\alpha \in [0,1]$ and x, $y \in H$.

Moreover, if $\|f\|_{R,\infty} := \sup_{t\in[0,1]}\left|f\left(Re^{2\pi it}\right)\right| < \infty$, *then we have the simpler inequalities*

$$|\langle \mathcal{P}_f(T,P)x,y\rangle| \le \frac{\|f\|_{R,\infty}}{R - \left\|P^{-1/2}TP^{-1/2}\right\|} \times \left\langle \left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^2 x,x\right\rangle^{1/2} \times \left\langle \left|\left|P^{-1/2}T^*P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^2 y,y\right\rangle^{1/2} \le \frac{\|f\|_{R,\infty}\left\|P^{-1/2}TP^{-1/2}\right\|}{R - \left\|P^{-1/2}TP^{-1/2}\right\|}\langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} \tag{5.3}$$

for all $\alpha \in [0,1]$ and x, $y \in H$.

Observe that

$$\left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^2 = P\left|P^{-1/2}TP^{-1/2}\right|^{2\alpha} P = P\left(P^{-1/2}T^*P^{-1}TP^{-1/2}\right)^{\alpha} P$$

Since $0 < P^{-1} \le \left\|P^{-1}\right\| I$, then $0 < T^*P^{-1}T \le \left\|P^{-1}\right\| |T|^2$ and

$$P^{-1/2}T^*P^{-1}TP^{-1/2} \le \left\|P^{-1}\right\| P^{-1/2}|T|^2 P^{-1/2} \le \left\|P^{-1}\right\|^2 |T|^2.$$

If we use Heinz inequality for $\alpha \in [0,1]$, then we get

$$\left(P^{-1/2}T^*P^{-1}TP^{-1/2}\right)^{\alpha} \le \left\|P^{-1}\right\|^{2\alpha} |T|^{2\alpha},$$

which implies that

$$\left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^2 = P\left(P^{-1/2}T^*P^{-1}TP^{-1/2}\right)^{\alpha} P \le \left\|P^{-1}\right\|^{2\alpha} P|T|^{2\alpha} P.$$

Similarly,

$$\left|\left|P^{-1/2}T^*P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^2 \le \left\|P^{-1}\right\|^{2(1-\alpha)} P|T^*|^{2(1-\alpha)} P.$$

Therefore

$$\begin{aligned} 0 &< \left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^{2} + \left|\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^{2} \\ &\leq \left\|P^{-1}\right\|^{2\alpha} P\left|T\right|^{2\alpha} P + \left\|P^{-1}\right\|^{2(1-\alpha)} P\left|T^{*}\right|^{2(1-\alpha)} P \\ &= P\left(\left\|P^{-1}\right\|^{2\alpha}\left|T\right|^{2\alpha} + \left\|P^{-1}\right\|^{2(1-\alpha)}\left|T^{*}\right|^{2(1-\alpha)}\right)P, \end{aligned}$$

which implies that

$$\begin{aligned} &\left\|\left|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right|^{2} + \left|\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha} P^{1/2}\right|^{2}\right\| \\ &\leq \left\|P\left(\left\|P^{-1}\right\|^{2\alpha}\left|T\right|^{2\alpha} + \left\|P^{-1}\right\|^{2(1-\alpha)}\left|T^{*}\right|^{2(1-\alpha)}\right)P\right\| \\ &\leq \left\|\left\|P^{-1}\right\|^{2\alpha}\left|T\right|^{2\alpha} + \left\|P^{-1}\right\|^{2(1-\alpha)}\left|T^{*}\right|^{2(1-\alpha)}\right\|\left\|P\right\|^{2}. \end{aligned}$$

From (3.3) we get

$$\begin{aligned} \omega\left(\mathcal{P}_{f}\left(T,P\right)\right) &\leq \frac{1}{2}M\left(f,P^{-1/2}TP^{-1/2};\gamma\right) \\ &\quad\times\left\|\left\|P^{-1}\right\|^{2\alpha}\left|T\right|^{2\alpha} + \left\|P^{-1}\right\|^{2(1-\alpha)}\left|T^{*}\right|^{2(1-\alpha)}\right\|\left\|P\right\|^{2} \end{aligned} \tag{5.4}$$

for $\alpha \in [0,1]$.

For $\alpha = 1/2$ we derive

$$\omega\left(\mathcal{P}_{f}\left(T,P\right)\right) \leq \frac{1}{2}M\left(f,P^{-1/2}TP^{-1/2};\gamma\right)\left\|\left|T\right| + \left|T^{*}\right|\right\|\left\|P^{-1}\right\|\left\|P\right\|^{2}. \tag{5.5}$$

If $P > 0$ and $T \in B(H)$ with $\operatorname{Sp}\left(P^{-1/2}TP^{-1/2}\right) \subset D(0,R)$ and $\|f\|_{R,\infty} := \sup_{t\in[0,1]}\left|f\left(Re^{2\pi it}\right)\right| < \infty$, then we have the simpler inequalities

$$\begin{aligned} \omega\left(\mathcal{P}_{f}\left(T,P\right)\right) &\leq \frac{\|f\|_{R,\infty}}{R - \left\|P^{-1/2}TP^{-1/2}\right\|} \\ &\quad\times\left\|\left\|P^{-1}\right\|^{2\alpha}\left|T\right|^{2\alpha} + \left\|P^{-1}\right\|^{2(1-\alpha)}\left|T^{*}\right|^{2(1-\alpha)}\right\|\left\|P\right\|^{2} \end{aligned} \tag{5.6}$$

for $\alpha \in [0,1]$.

For $\alpha = 1/2$ we derive

$$\omega\left(\mathcal{P}_{f}\left(T,P\right)\right) \leq \frac{\|f\|_{R,\infty}}{R - \left\|P^{-1/2}TP^{-1/2}\right\|}\left\|\left|T\right| + \left|T^{*}\right|\right\|\left\|P^{-1}\right\|\left\|P\right\|^{2}. \tag{5.7}$$

If we use now the inequality (4.1) for $C = B = P^{1/2}$ and $A = P^{-1/2}TP^{-1/2}$ we get that

$$\begin{aligned} \left\|\mathcal{P}_{f}\left(T,P\right)\right\|_{2r} &\leq M\left(f,P^{-1/2}TP^{-1/2};\gamma\right) \\ &\quad\times\left\|\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2}\right\|_{2pr}\left\|\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha} P^{1/2}\right\|_{2qr} \end{aligned} \tag{5.8}$$

provided that $\alpha \in [0,1]$, $r \geq 1/2$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$ while $\left|P^{-1/2}TP^{-1/2}\right|^{\alpha} P^{1/2} \in \mathcal{B}_{2pr}(H)$ and $\left|P^{-1/2}T^{*}P^{-1/2}\right|^{1-\alpha} P^{1/2} \in \mathcal{B}_{2qr}(H)$.

Now, we observe that

$$\left\| \left| P^{-1/2}TP^{-1/2} \right|^{\alpha} P^{1/2} \right\|_{2pr} \leq \left\| \left| P^{-1/2}TP^{-1/2} \right|^{\alpha} \right\| \left\| P^{1/2} \right\|_{2pr} = \left\| P^{-1/2}TP^{-1/2} \right\|^{\alpha} \|P\|_{pr}$$

and

$$\left\| \left| P^{-1/2}T^{*}P^{-1/2} \right|^{1-\alpha} P^{1/2} \right\|_{2qr} \leq \left\| \left| P^{-1/2}T^{*}P^{-1/2} \right|^{1-\alpha} \right\| \left\| P^{1/2} \right\|_{2qr} = \left\| P^{-1/2}T^{*}P^{-1/2} \right\|^{1-\alpha} \|P\|_{qr}.$$

These imply that

$$\begin{aligned} &\left\| \left| P^{-1/2}TP^{-1/2} \right|^{\alpha} P^{1/2} \right\|_{2pr} \left\| \left| P^{-1/2}T^{*}P^{-1/2} \right|^{1-\alpha} P^{1/2} \right\|_{2qr} \\ &\leq \left\| P^{-1/2}TP^{-1/2} \right\|^{\alpha} \|P\|_{pr} \left\| P^{-1/2}T^{*}P^{-1/2} \right\|^{1-\alpha} \|P\|_{qr} \\ &= \left\| P^{-1/2}TP^{-1/2} \right\|^{\alpha} \left\| P^{-1/2}TP^{-1/2} \right\|^{1-\alpha} \|P\|_{pr} \|P\|_{qr} \\ &= \left\| P^{-1/2}TP^{-1/2} \right\| \|P\|_{pr} \|P\|_{qr} \end{aligned}$$

and by (5.8) we obtain the simpler inequality

$$\|\mathcal{P}_f(T,P)\|_{2r} \leq M\left(f, P^{-1/2}TP^{-1/2}; \gamma\right) \left\| P^{-1/2}TP^{-1/2} \right\| \|P\|_{pr} \|P\|_{qr}. \tag{5.9}$$

If $P > 0$ and $T \in B(H)$ with $\operatorname{Sp}\left(P^{-1/2}TP^{-1/2}\right) \subset D(0,R)$ and $\|f\|_{R,\infty} := \sup_{t\in[0,1]} \left|f\left(Re^{2\pi it}\right)\right| < \infty$, then we have the simpler inequality

$$\|\mathcal{P}_f(T,P)\|_{2r} \leq \frac{\|f\|_{R,\infty}}{R - \left\| P^{-1/2}TP^{-1/2} \right\|} \left\| P^{-1/2}TP^{-1/2} \right\| \|P\|_{pr} \|P\|_{qr} \tag{5.10}$$

for $r \geq 1/2$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \geq 1$ and $P \in \mathcal{B}_{pr}(H) \cap \mathcal{B}_{qr}(H)$.

6. Two Examples

Consider the exponential function $f(A) = \exp A$, $A \in \mathcal{B}(H)$. Assume that $A \in \mathcal{B}(H)$ and $\|A\| < R$ for some $R > 0$. Observe that for $t \in [0,1]$,

$$\left|\exp\left(Re^{2\pi it}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and then by (2.19) we get for B, $C \in \mathcal{B}(H)$ that

$$\begin{aligned} \left|\left\langle C^{*}\left[\exp(A) - I\right]Bx, y\right\rangle\right| &\leq \frac{1}{R - \|A\|} \int_0^1 \exp\left[R\cos(2\pi t)\right] dt \\ &\times \left\langle \left||A|^{\alpha} B\right|^2 x, x \right\rangle^{1/2} \left\langle \left||A^{*}|^{1-\alpha} C\right|^2 y, y \right\rangle^{1/2} \end{aligned} \tag{6.1}$$

for x, $y \in H$

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [2, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^{\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu + k + 1)},$$

where Γ is the *gamma function*. For $n=0$ we have $I_0(z)$ given by

$$I_0(z)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta-\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

If we change the variable $\theta=2\pi t$, then $dt=\frac{1}{2\pi}d\theta$ and

$$\begin{aligned}\int_0^1\exp\left[R\cos(2\pi t)\right]dt&=\frac{1}{2\pi}\int_0^{2\pi}\exp\left[R\cos\theta\right]d\theta\\&=\frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi\exp\left[R\cos\theta\right]d\theta+\frac{1}{\pi}\int_\pi^{2\pi}\exp\left[R\cos\theta\right]d\theta\right)\\&=\frac{1}{2}\left(I_0(R)+I_0(-R)\right)=I_0(R).\end{aligned}$$

From (6.1) we then get

$$\begin{aligned}(6.2)\qquad\left|\left\langle C^*\left[\exp(A)-I\right]Bx,y\right\rangle\right|&\le\frac{I_0(R)}{R-\|A\|}\\&\quad\times\left\langle\left||A|^\alpha B\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}C\right|^2y,y\right\rangle^{1/2}\end{aligned}$$

for $\alpha\in[0,1]$, x, $y\in H$, A, B, $C\in\mathcal{B}(H)$ with $\|A\|<R$.

By taking $B=C=I$ in (6.2) we get for $\|A\|<R$ that

$$(6.3)\qquad\left|\left\langle\left[\exp(A)-I\right]x,y\right\rangle\right|\le\frac{I_0(R)}{R-\|A\|}\left\langle|A|^{2\alpha}x,x\right\rangle^{1/2}\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle^{1/2}$$

for x, $y\in H$. In particular,

$$(6.4)\qquad\left|\left\langle\left[\exp(A)-I\right]x,y\right\rangle\right|\le\frac{I_0(R)}{R-\|A\|}\left\langle|A|x,x\right\rangle^{1/2}\left\langle|A^*|y,y\right\rangle^{1/2}$$

for x, $y\in H$.

By Theorem 4 we get the norm inequality

$$(6.5)\qquad\left\|C^*\left[\exp(A)-I\right]BB\right\|\le\frac{I_0(R)}{R-\|A\|}\left\||A|^\alpha B\right\|\left\||A^*|^{1-\alpha}C\right\|.$$

We also have the numerical radius inequalities

$$(6.6)\qquad\omega\left(C^*\left[\exp(A)-I\right]B\right)\le\frac{1}{2}\frac{I_0(R)}{R-\|A\|}\left\|\left||A|^\alpha B\right|^2+\left||A^*|^{1-\alpha}C\right|^2\right\|$$

and

$$\omega^2\left(C^*\left[\exp(A)-I\right]B\right)\le\frac{1}{2}\left(\frac{I_0(R)}{R-\|A\|}\right)^2 \times\left[\left\||A|^\alpha B\right\|^2\left\||A^*|^{1-\alpha}C\right\|^2+\omega\left(\left||A^*|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right)\right]. \tag{6.7}$$

Let $r\ge 1/2$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,\ qr\ge 1$. If $B,\ C\in\mathcal{B}(H)$ with $|A|^\alpha B\in\mathcal{B}_{2pr}(H)$ and $|A^*|^{1-\alpha}C\in\mathcal{B}_{2qr}(H)$ for $\alpha\in[0,1]$, then $C^*\left[\exp(A)-I\right]B\in\mathcal{B}_{2r}(H)$ and by (4.1)

$$\left\|C^*\left[\exp(A)-I\right]B\right\|_{2r}\le\frac{I_0(R)}{R-\|A\|}\left\||A|^\alpha B\right\|_{2pr}\left\||A^*|^{1-\alpha}C\right\|_{2qr}. \tag{6.8}$$

If $r\ge 1/2$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $pr,\ qr\ge 1$ and $\left||A|^\alpha B\right|^{2pr},\ \left||A^*|^{1-\alpha}C\right|^{2qr}\in\mathcal{B}_1(H)$, then $C^*\left[\exp(A)-I\right]B\in\mathcal{B}_{2r}(H)$ and by (4.17)

$$\begin{aligned}&\omega_{2r}^{2r}\left(C^*\left[\exp(A)-I\right]B\right)\\&\le\left(\frac{I_0(R)}{R-\|A\|}\right)^{2r}\operatorname{tr}\left(\frac{1}{p}\left||A|^\alpha B\right|^{2pr}+\frac{1}{q}\left||A^*|^{1-\alpha}C\right|^{2qr}\right).\end{aligned} \tag{6.9}$$

Consider the power series

$$f(z):=\ln(1-z)^{-1}=\sum_{n=1}^\infty\frac{1}{n}z^n.$$

We observe that for $|z|<1$

$$\left|\ln(1-z)^{-1}\right|\le\sum_{n=1}^\infty\frac{1}{n}|z|^n=\ln(1-|z|)^{-1}.$$

Now if we assume that $A,\ B,\ C\in\mathcal{B}(H)$ and $\|A\|<R<1$, then by (2.17) we get

$$\left|\left\langle C^*\ln(1-A)^{-1}Bx,y\right\rangle\right|\le\frac{\ln(1-R)^{-1}}{R-\|A\|} \times\left\langle\left||A|^\alpha B\right|^2x,x\right\rangle^{1/2}\left\langle\left||A^*|^{1-\alpha}C\right|^2y,y\right\rangle^{1/2} \tag{6.10}$$

for $\alpha\in[0,1]$, $x,\ y\in H$.

By taking $B=C=I$ in (6.10) we get for $\|A\|<R<1$ that

$$\left|\left\langle\ln(1-A)^{-1}x,y\right\rangle\right|\le\frac{\ln(1-R)^{-1}}{R-\|A\|}\left\langle|A|^{2\alpha}x,x\right\rangle^{1/2}\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle^{1/2} \tag{6.11}$$

for $x,\ y\in H$. In particular,

$$\left|\left\langle\ln(1-A)^{-1}x,y\right\rangle\right|\le\frac{\ln(1-R)^{-1}}{R-\|A\|}\langle|A|x,x\rangle^{1/2}\langle|A^*|y,y\rangle^{1/2} \tag{6.12}$$

for $x,\ y\in H$.

One can state some norm, numerical radius and p-Schatten norm inequalities for $\ln(1-A)^{-1}$, however the details are omitted.

References

[1] A. Aluthge, Some generalized theorems on p-hyponormal operators, *Integral Equations Operator Theory,* **24** (1996), 497-501.

[2] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standars, Applied Mathematics Series, **55**, 1972.

[3] A. Abu-Omar and F. Kittaneh, A numerical radius inequality involving the generalized Aluthge transform, *Studia Math.* **216** (1) (2013) 69-75.

[4] P. Bhunia, S. Bag, and K. Paul, Numerical radius inequalities and its applications in estimation of zeros of polynomials, *Linear Algebra and its Applications*, vol. **573** (2019) pp. 166–177.

[5] P. Bhunia, S. S. Dragomir, M. S. Moslehian and K. Paul, *Lectures on Numerical Radius Inequalities*, Springer Cham, 2022. https://doi.org/10.1007/978-3-031-13670-2.

[6] M. L. Buzano, Generalizzazione della diseguaglianza di Cauchy-Schwarz. (Italian), *Rend. Sem. Mat. Univ. e Politech. Torino*, **31** (1971/73), 405–409 (1974).

[7] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.

[8] M. Cho and K. Tanahashi, Spectral relations for Aluthge transform, *Scientiae Mathematicae Japonicae*, **55** (1) (2002), 77-83.

[9] R. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, 1972.

[10] S. S. Dragomir, *Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces*, SpringerBriefs in Mathematics, 2013. https://doi.org/10.1007/978-3-319-01448-7.

[11] S. S. Dragomir, Trace inequalities for operators in Hilbert spaces: a survey of recent results, *Aust. J. Math. Anal. Appl.* Vol. **19** (2022), No. 1, Art. 1, 202 pp.

[12] S. S. Dragomir, Generalizations of Furuta's inequality, Volume **61**, 2013 - Issue 5, 617-626.

[13] M. El-Haddad and F. Kittaneh, Numerical radius inequalities for Hilbert space operators. II, *Studia Math.* **182** (2007), No. 2, 133-140

[14] T. Furuta, An extension of the Heinz-Kato theorem. *Proc. Amer. Math. Soc.* **120** (1994), no. 3, 785–787.

[15] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.

[16] T. Kato, Notes on some inequalities for linear operators, *Math. Ann.* **125**(1952), 208-212.

[17] F. Kittaneh, Notes on some inequalities for Hilbert space operators, *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 2, 283–293.

[18] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, *Studia Math.* **158** (2003), No. 1, 11-17.

[19] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, *Studia Math.*, **168** (2005), No. 1, 73-80.

[20] T. Kato, Notes on some inequalities for linear operators, *Math. Ann.*, **125** (1952), 208-212.

[21] C. A. McCarthy, C_p, *Israel J. Math.* **5** (1967), 249–271.

[22] J. R. Ringrose, *Compact Non-self-adjoint Operators,* Van Nostrand Reinhold, New York, 1971.

[23] W. Rudin, *Functional Analysis*, McGraw Hill, 1973.

[24] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.

[25] V. A. Zagrebnov, *Gibbs Semigroups,* Operator Theory: Advances and Applications, Volume 273, Birkhäuser, 2019.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa