

NUMERICAL RADIUS AND p -SCHATTEN NORM INEQUALITIES FOR ANALYTIC FUNCTIONS OF OPERATORS IN HILBERT SPACES

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ABSTRACT. Let H be a complex Hilbert space, $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$ an analytic function on the domain G and $A \in \mathcal{B}(H)$ with $\text{Sp}(A) \subset G$ and γ a closed rectifiable path in G and such that $\text{Sp}(A) \subset \text{ins}(\gamma)$. If we denote

$$B(f, \gamma; A) := \frac{1}{2\pi} \int_{\gamma} |f(\xi)| (|\xi| - \|A\|)^{-1} |d\xi|,$$

then for $B, C \in \mathcal{B}(H)$ we have

$$|\langle C^* A f(A) B x, y \rangle| \leq B(f, \gamma; A) \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1-\alpha} C \|^2 y, y \right\rangle^{1/2}$$

for $\alpha \in [0, 1]$ and $x, y \in H$. Some natural applications for *numerical radius* and *p -Schatten norm* are also provided.

1. INTRODUCTION

In 1988, F. Kittaneh obtained the following generalization of Schwarz inequality [17]:

Theorem 1. *Assume that h and g are non-negative functions on $[0, \infty)$ which are continuous and satisfying the relation $h(t)g(t) = t$ for all $t \in [0, \infty)$. For any $T \in \mathcal{B}(H)$*

$$(1.1) \quad |\langle Tx, y \rangle| \leq \|h(|T|)x\| \|g(|T^*|)y\|$$

for all $x, y \in H$.

If we take $h(t) = t^\alpha$, $g(t) = t^{1-\alpha}$ with $\alpha \in [0, 1]$, then we obtain *Kato's inequality*

$$(1.2) \quad |\langle Tx, y \rangle| \leq \| |T|^\alpha x \| \| |T^*|^{1-\alpha} y \| \text{ for all } x, y \in H.$$

The *numerical radius* $w(T)$ of an operator T on H is given by

$$(1.3) \quad \omega(T) = \sup \{ |\langle Tx, x \rangle|, \|x\| = 1 \}.$$

Obviously, by (1.3), for any $x \in H$ one has

$$(1.4) \quad |\langle Tx, x \rangle| \leq w(T) \|x\|^2.$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \rightarrow H$, i.e.,

- (i) $\omega(T) \geq 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
- (ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;

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(iii) $\omega(T + V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$(1.5) \quad \omega(T) \leq \|T\| \leq 2\omega(T)$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [18], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (1.5):

$$(1.6) \quad \omega(T) \leq \frac{1}{2} \left(\|T\| + \|T^2\|^{1/2} \right).$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [19] improved the inequality (1.5) as follows:

$$(1.7) \quad \frac{1}{4} \|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2} \|T^*T + TT^*\|$$

for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [13]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$(1.8) \quad \omega^r(T) \leq \frac{1}{2} \left\| |T|^{2\alpha r} + |T^*|^{2(1-\alpha)r} \right\|$$

and

$$(1.9) \quad \omega^{2r}(T) \leq \left\| \alpha |T|^{2r} + (1-\alpha) |T^*|^{2r} \right\|,$$

where $\alpha \in (0, 1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (1.8) that

$$(1.10) \quad \omega(T) \leq \frac{1}{2} \left(\| |T| + |T^*| \| \right)$$

and from (1.9) that

$$(1.11) \quad \omega^2(T) \leq \frac{1}{2} \left\| |T|^2 + |T^*|^2 \right\|.$$

For more related results, see the recent books on inequalities for numerical radii [10] and [5].

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H . If $\{e_i\}_{i \in I}$ an orthonormal basis of H , we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$(1.12) \quad \|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty.$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$(1.13) \quad \text{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle,$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H . Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.13) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) *If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and*

$$(1.14) \quad \operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)};$$

(ii) *If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_1(H)$ and*

$$(1.15) \quad \operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|;$$

(iii) *$\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;*

(iv) *If $A, B \in \mathcal{B}_2(H)$ then $AB, BA \in \mathcal{B}_1(H)$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;*

(v) *$\mathcal{B}_{fin}(H)$, the space of operators of finite rank, is a dense subspace of $\mathcal{B}_1(H)$.*

For a large number of results concerning trace inequalities, see the recent survey paper [11].

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p -Schatten norm* is finite [25, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} = \left(\sum_{i \in I} \langle |A|^p e_i, e_i \rangle \right)^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$(1.16) \quad \mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H)$$

and

$$(1.17) \quad \|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|.$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$ -ideal $\mathcal{B}_p(H)$ and $(\mathcal{B}_p(H), \|\cdot\|_p)$ is a Banach space.

Also, see for instance [25, p. 60-64],

$$(1.18) \quad \|A\|_p = \|A^*\|_p, \quad A \in \mathcal{B}_p(H)$$

$$(1.19) \quad \|AB\|_p \leq \|A\|_p \|B\|_p, \quad A, B \in \mathcal{B}_p(H)$$

and

$$(1.20) \quad \|AB\|_p \leq \|A\|_p \|B\|, \quad \|BA\|_p \leq \|B\| \|A\|_p, \quad A \in \mathcal{B}_p(H), \quad B \in \mathcal{B}(H).$$

This implies that

$$(1.21) \quad \|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \quad A \in \mathcal{B}_p(H), \quad B, C \in \mathcal{B}(H).$$

In terms of *p -Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(1.22) \quad (|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \quad A \in \mathcal{B}_p(H), \quad B \in \mathcal{B}_q(H).$$

For the theory of trace functionals and their applications the reader is referred to [24] and [25].

For $\mathcal{E} := \{e_i\}_{i \in I}$ an orthonormal basis of H we define for $A \in \mathcal{B}_p(H)$, $p \geq 1$

$$\|A\|_{\mathcal{E}, p} := \left(\sum_{i \in I} |\langle Ae_i, e_i \rangle|^p \right)^{1/p}.$$

We observe that $\|\cdot\|_{\mathcal{E},p}$ is a norm on $\mathcal{B}_p(H)$ and

$$\|A\|_{\mathcal{E},p} \leq \|A\|_p \text{ for } A \in \mathcal{B}_p(H).$$

Further, we can take the supremum over all orthonormal basis in H we can also define, for $A \in \mathcal{B}_p(H)$, that

$$\omega_p(A) := \sup_{\mathcal{E}} \|A\|_{\mathcal{E},p} \leq \|A\|_p,$$

which is a *norm* on $\mathcal{B}_p(H)$.

It is also known that, if $\mathcal{E} = \{e_i\}_{i \in I}$ and $\mathcal{F} = \{f_i\}_{i \in I}$ are orthonormal basis, then [22]

$$(1.23) \quad \sup_{\mathcal{E}, \mathcal{F}} \sum_{i \in I} |\langle Te_i, f_i \rangle|^s = \|T\|_s^s \text{ for } s \geq 1.$$

Let \mathcal{B} be a unital Banach algebra, $A \in \mathcal{B}$ and G be a domain of \mathbb{C} with $\text{Sp}(A) \subset G$. If $f : G \rightarrow \mathbb{C}$ is analytic on G , we define an element $f(A)$ in \mathcal{B} by

$$(1.24) \quad f(A) := \frac{1}{2\pi i} \int_{\delta} f(\xi) (\xi - A)^{-1} d\xi,$$

where $\delta \subset G$ is taken to be closed rectifiable curve in G and such that $\text{Sp}(A) \subset \text{ins}(\delta)$, the inside of δ .

It is well known (see for instance [7, pp. 201-204]) that $f(A)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$(1.25) \quad \text{Sp}(f(A)) = f(\text{Sp}(A))$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [9] and [23].

2. VECTOR INEQUALITIES

In 1988, F. Kittaneh [17, Corollary 7] obtained the following Schwarz type inequality for natural powers of operators:

Lemma 1. *Let $A \in \mathcal{B}(H)$ and $\alpha \in [0, 1]$. Then for natural number $n \geq 1$ we have*

$$(2.1) \quad |\langle A^n x, y \rangle|^2 \leq \|A\|^{2n-2} \langle |A|^{2\alpha} x, x \rangle \langle |A^*|^{2(1-\alpha)} y, y \rangle$$

for all $x, y \in H$.

We can state the following result as well:

Corollary 1. *Let $A, B, C \in \mathcal{B}(H)$ and $\alpha \in [0, 1]$. Then for $n \geq 1$ we have*

$$(2.2) \quad |\langle C^* A^n Bx, y \rangle|^2 \leq \|A\|^{2n-2} \langle |A|^\alpha B^2 x, x \rangle \langle |A^*|^{1-\alpha} C \Big|^2 y, y \rangle$$

for all $x, y \in H$.

Proof. If we replace x by Bx and y by Cy in (2.1), then we get

$$(2.3) \quad |\langle C^* A^n Bx, y \rangle|^2 \leq \|A\|^{2n-2} \langle B^* |A|^{2\alpha} Bx, x \rangle \langle C^* |A^*|^{2(1-\alpha)} Cy, y \rangle$$

for all $x, y \in H$.

Observe that $B^* |A|^{2\alpha} B = \|A\|^\alpha B^2$ and $C^* |A^*|^{2(1-\alpha)} C = \left| |A^*|^{1-\alpha} C \right|^2$, then by (2.3) we get (2.2). \square

We also have:

Lemma 2. *Assume that $A, B, C \in \mathcal{B}(H)$ with $\|A\| < 1$, then*

$$(2.4) \quad \left| \left\langle C^* A (I - A)^{-1} Bx, y \right\rangle \right| \leq (1 - \|A\|)^{-1} \\ \times \left\langle \|A\|^\alpha |B|^2 x, x \right\rangle^{1/2} \left\langle \|A^*\|^{1-\alpha} |C|^2 y, y \right\rangle^{1/2}$$

for $\alpha \in [0, 1]$ and $x, y \in H$. In particular,

$$(2.5) \quad \left| \left\langle C^* A (I - A)^{-1} Bx, y \right\rangle \right| \leq (1 - \|A\|)^{-1} \\ \times \left\langle \|A\|^{1/2} |B|^2 x, x \right\rangle^{1/2} \left\langle \|A^*\|^{1/2} |C|^2 y, y \right\rangle^{1/2}$$

for $x, y \in H$.

Proof. If we put $n = k + 1$, $k \in \mathbb{N}$ in (2.2) and take the square root, then we get

$$\left| \left\langle C^* A A^k Bx, y \right\rangle \right| \leq \|A\|^k \left\langle \|A\|^\alpha |B|^2 x, x \right\rangle^{1/2} \left\langle \|A^*\|^{1-\alpha} |C|^2 y, y \right\rangle^{1/2}$$

for all $x, y \in H$.

Further, if we sum over k from 0 to m , then we obtain

$$(2.6) \quad \left| \left\langle C^* A \sum_{k=0}^m A^k Bx, y \right\rangle \right| \\ = \left| \sum_{k=0}^m \left\langle C^* A A^k Bx, y \right\rangle \right| \leq \sum_{k=0}^m \left| \left\langle C^* A A^k Bx, y \right\rangle \right| \\ \leq \sum_{k=0}^m \|A\|^k \left\langle \|A\|^\alpha |B|^2 x, x \right\rangle^{1/2} \left\langle \|A^*\|^{1-\alpha} |C|^2 y, y \right\rangle^{1/2}$$

for all $x, y \in H$.

Since $\|A\| < 1$, then series $\sum_{k=0}^{\infty} A^k$ and $\sum_{k=0}^{\infty} \|A\|^k$ are convergent and

$$\sum_{k=0}^{\infty} A^k = (I - A)^{-1} \quad \text{and} \quad \sum_{k=0}^{\infty} \|A\|^k = (1 - \|A\|)^{-1}.$$

By taking now the limit over $m \rightarrow \infty$ in (2.6) we deduce the desired result (2.4). \square

Our first main result is as follows:

Theorem 3. *Let $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the domain G and $A \in \mathcal{B}(H)$ with $\text{Sp}(A) \subset G$ and γ a closed rectifiable path in G and such that $\text{Sp}(A) \subset \text{ins}(\gamma)$. If we denote*

$$B(f, \gamma; A) := \frac{1}{2\pi} \int_{\gamma} |f(\xi)| (|\xi| - \|A\|)^{-1} |d\xi|,$$

then for $B, C \in \mathcal{B}(H)$ we have

$$(2.7) \quad \left| \left\langle C^* A f(A) Bx, y \right\rangle \right| \leq B(f, \gamma; A) \left\langle \|A\|^\alpha |B|^2 x, x \right\rangle^{1/2} \left\langle \|A^*\|^{1-\alpha} |C|^2 y, y \right\rangle^{1/2}$$

for $\alpha \in [0, 1]$ and $x, y \in H$. In particular,

$$(2.8) \quad \begin{aligned} & |\langle C^* A f(A) Bx, y \rangle| \\ & \leq B(f, \gamma; A) \left\langle \| |A|^{1/2} B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1/2} C \|^2 y, y \right\rangle^{1/2} \end{aligned}$$

for $x, y \in H$.

Proof. We have, by the representation

$$f(A) := \frac{1}{2\pi i} \int_{\gamma} f(\xi) (\xi I - A)^{-1} d\xi,$$

that

$$\begin{aligned} \langle C^* A f(A) Bx, y \rangle &= \left\langle C^* A \frac{1}{2\pi i} \int_{\gamma} f(\xi) (\xi I - A)^{-1} d\xi Bx, y \right\rangle \\ &= \frac{1}{2\pi i} \int_{\gamma} f(\xi) \langle C^* A (\xi I - A)^{-1} Bx, y \rangle d\xi \end{aligned}$$

for $x, y \in H$.

By taking the modulus and using the complex integral properties, we get

$$(2.9) \quad \begin{aligned} |\langle C^* A f(A) Bx, y \rangle| &\leq \frac{1}{2\pi} \int_{\gamma} |f(\xi)| \left| \langle C^* A (\xi I - A)^{-1} Bx, y \rangle \right| |d\xi| \\ &= \frac{1}{2\pi} \int_{\gamma} |f(\xi)| |\xi|^{-1} \left| \left\langle C^* A \left(I - \frac{A}{\xi} \right)^{-1} Bx, y \right\rangle \right| |d\xi| \end{aligned}$$

for $x, y \in H$.

Since $\left\| \frac{A}{\xi} \right\| < 1$ for $\xi \in \gamma$, then by Lemma 2 for $\frac{A}{\xi}$ we have

$$(2.10) \quad \begin{aligned} |\xi|^{-1} \left| \left\langle C^* A \left(I - \frac{A}{\xi} \right)^{-1} Bx, y \right\rangle \right| &= \left| \left\langle C^* \frac{A}{\xi} \left(I - \frac{A}{\xi} \right)^{-1} Bx, y \right\rangle \right| \\ &\leq \left(1 - \left\| \frac{A}{\xi} \right\| \right)^{-1} \left\langle \left\| \frac{A}{\xi} \right\|^\alpha B \right\rangle^{1/2} \left\langle \left\| \frac{A^*}{\bar{\xi}} \right\|^{1-\alpha} C \right\rangle^{1/2} \\ &= \left(\frac{|\xi| - \|A\|}{|\xi|} \right)^{-1} \left\langle \left\| \frac{A}{\xi} \right\|^\alpha B \right\rangle^{1/2} \left\langle \left\| \frac{A^*}{\bar{\xi}} \right\|^{1-\alpha} C \right\rangle^{1/2} \\ &= \frac{|\xi|}{|\xi|^\alpha |\bar{\xi}|^{1-\alpha}} (|\xi| - \|A\|)^{-1} \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1-\alpha} C \|^2 y, y \right\rangle^{1/2} \\ &= (|\xi| - \|A\|)^{-1} \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1-\alpha} C \|^2 y, y \right\rangle^{1/2} \end{aligned}$$

for $x, y \in H$.

By utilizing (2.10) we derive

$$\begin{aligned}
 (2.11) \quad & \frac{1}{2\pi} \int_{\gamma} |f(\xi)| |\xi|^{-1} \left| \left\langle C^* A \left(I - \frac{A}{\xi} \right)^{-1} Bx, y \right\rangle \right| |d\xi| \\
 & \leq \left(\frac{1}{2\pi} \int_{\gamma} |f(\xi)| (|\xi| - \|A\|)^{-1} |d\xi| \right) \\
 & \quad \times \left\langle \|A\|^\alpha B|^2 x, x \right\rangle^{1/2} \left\langle \|A^*\|^{1-\alpha} C|^2 y, y \right\rangle^{1/2}
 \end{aligned}$$

for $x, y \in H$.

By making use of (2.9) and (2.11) we obtain (2.7). \square

Remark 1. For $B = C = I$ in (2.7) we get the one operator inequalities

$$(2.12) \quad |\langle Af(A)x, y \rangle| \leq B(f, \gamma; A) \left\langle |A|^{2\alpha} x, x \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle^{1/2}$$

for $\alpha \in [0, 1]$ and $x, y \in H$. In particular,

$$(2.13) \quad |\langle Af(A)x, y \rangle| \leq B(f, \gamma; A) \langle |A|x, x \rangle^{1/2} \langle |A^*|y, y \rangle^{1/2}$$

for all $x, y \in H$.

If A is invertible and take $C = I$, $B = A^{-1}$ in (2.7), then we get

$$(2.14) \quad |\langle f(A)x, y \rangle| \leq B(f, \gamma; A) \left\langle |A|^{-2(1-\alpha)} x, x \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle^{1/2}$$

for $\alpha \in [0, 1]$ and $x, y \in H$. In particular,

$$(2.15) \quad |\langle f(A)x, y \rangle| \leq B(f, \gamma; A) \left\langle |A|^{-1} x, x \right\rangle^{1/2} \langle |A^*|y, y \rangle^{1/2}$$

for $x, y \in H$.

If $A > 0$ and we take $B = A^{-\beta}$, $C = A^{-1+\beta}$, $\beta \in [0, 1]$, then we derive

$$(2.16) \quad |\langle f(A)x, y \rangle| \leq B(f, \gamma; A) \left\langle A^{2(\alpha-\beta)} x, x \right\rangle^{1/2} \left\langle A^{2(\beta-\alpha)} y, y \right\rangle^{1/2}$$

for $\alpha \in [0, 1]$ and $x, y \in H$. In particular, for $\alpha = \beta$ we obtain

$$(2.17) \quad |\langle f(A)x, y \rangle| \leq B(f, \gamma; A) \|x\| \|y\|$$

for $x, y \in H$.

Corollary 2. With the assumptions of Theorem 3 and if

$$\|f\|_{\gamma, \infty} := \sup_{\xi \in \gamma} |f(\xi)| < \infty,$$

then, by denoting

$$B_\infty(f, \gamma; A) := \frac{1}{2\pi} \|f\|_{\gamma, \infty} \int_{\gamma} (|\xi| - \|A\|)^{-1} |d\xi|,$$

we have

$$\begin{aligned}
 (2.18) \quad & |\langle C^* Af(A)Bx, y \rangle| \\
 & \leq B_\infty(f, \gamma; A) \left\langle \|A\|^\alpha B|^2 x, x \right\rangle^{1/2} \left\langle \|A^*\|^{1-\alpha} C|^2 y, y \right\rangle^{1/2}
 \end{aligned}$$

for $\alpha \in [0, 1]$ and $x, y \in H$. In particular,

$$(2.19) \quad |\langle C^* A f(A) Bx, y \rangle| \\ \leq B_\infty(f, \gamma; A) \left\langle \| |A|^{1/2} B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1/2} C \|^2 y, y \right\rangle^{1/2}$$

for $x, y \in H$.

Remark 2. If we assume that $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$ is an analytic function on the domain G and $A \in \mathcal{B}(H)$ with $\text{Sp}(A) \subset D(0, R) \subset D$ where $D(0, R)$ is an open disk centered in 0 and of radius R , then by taking γ parametrized by $\xi(t) = Re^{2\pi it}$ where $t \in [0, 1]$, then $d\xi(t) = 2\pi i Re^{2\pi it} dt$, $|d\xi(t)| = 2\pi R dt$, $|\xi| = R$ and by (2.18) we get for $A, B \in \mathcal{B}(H)$ that

$$(2.20) \quad |\langle C^* A f(A) Bx, y \rangle| \leq \frac{R}{R - \|A\|} \int_0^1 |f(Re^{2\pi it})| dt \\ \times \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1-\alpha} C \|^2 y, y \right\rangle^{1/2}$$

where $\alpha \in [0, 1]$ and $x, y \in H$. In particular,

$$(2.21) \quad |\langle C^* A f(A) Bx, y \rangle| \leq \frac{R}{R - \|A\|} \int_0^1 |f(Re^{2\pi it})| dt \\ \times \left\langle \| |A|^{1/2} B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1/2} C \|^2 y, y \right\rangle^{1/2}$$

for $x, y \in H$.

Moreover, if $\|f\|_{R, \infty} := \sup_{t \in [0, 1]} |f(Re^{2\pi it})| < \infty$, then we have the simpler inequalities

$$(2.22) \quad |\langle C^* A f(A) Bx, y \rangle| \leq \frac{R \|f\|_{R, \infty}}{R - \|A\|} \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1-\alpha} C \|^2 y, y \right\rangle^{1/2}$$

for $x, y \in H$. In particular,

$$(2.23) \quad |\langle C^* A f(A) Bx, y \rangle| \leq \frac{R \|f\|_{R, \infty}}{R - \|A\|} \\ \times \left\langle \| |A|^{1/2} B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1/2} C \|^2 y, y \right\rangle^{1/2}$$

for $x, y \in H$.

3. NORM AND NUMERICAL RADIUS INEQUALITIES

The following vector inequality for positive operators $A \geq 0$, obtained by C. A. McCarthy in [21] is well known,

$$\langle Ax, x \rangle^p \leq \langle A^p x, x \rangle, \quad p \geq 1$$

for $x \in H$, $\|x\| = 1$.

Buzano's inequality [6],

$$(3.1) \quad \frac{1}{2} [\|x\| \|y\| + |\langle x, y \rangle|] \geq |\langle x, e \rangle \langle e, y \rangle|$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$ will also be used in the sequel.

We also have the following norm and numerical radius inequalities:

Theorem 4. *Let $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the domain G and $A \in \mathcal{B}(H)$ with $\text{Sp}(A) \subset G$ and γ a closed rectifiable path in G and such that $\text{Sp}(A) \subset \text{ins}(\gamma)$. If $B, C \in \mathcal{B}(H)$, then we have the norm inequality*

$$(3.2) \quad \|C^* A f(A) B\| \leq B(f, \gamma; A) \| |A|^\alpha B \| \| |A^*|^{1-\alpha} C \|\|.$$

We also have the numerical radius inequalities

$$(3.3) \quad \omega(C^* A f(A) B) \leq \frac{1}{2} B(f, \gamma; A) \left\| \| |A|^\alpha B \|^2 + \| |A^*|^{1-\alpha} C \|^2 \right\|$$

and

$$(3.4) \quad \begin{aligned} \omega^2(C^* A f(A) B) &\leq \frac{1}{2} B^2(f, \gamma; A) \left[\| |A|^\alpha B \|^2 \| |A^*|^{1-\alpha} C \|^2 + \omega \left(\| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \right) \right]. \end{aligned}$$

Proof. We have from (2.7), by taking the supremum over $\|x\| = \|y\| = 1$, that

$$\begin{aligned} \|C^* A f(A) B\|^2 &= \sup_{\|x\|=\|y\|=1} |\langle C^* A f(A) Bx, y \rangle|^2 \\ &\leq B^2(f, \gamma; A) \sup_{\|x\|=1} \langle \| |A|^\alpha B \|^2 x, x \rangle \sup_{\|y\|=1} \langle \| |A^*|^{1-\alpha} C \|^2 y, y \rangle \\ &= B^2(f, \gamma; A) \left\| \| |A|^\alpha B \|^2 \right\| \left\| \| |A^*|^{1-\alpha} C \|^2 \right\| \\ &= B^2(f, \gamma; A) \| |A|^\alpha B \|^2 \| |A^*|^{1-\alpha} C \|^2, \end{aligned}$$

which gives (3.2).

From (2.7) we get, by taking $y = x$, the square root and using the *A-G-mean inequality*, that

$$(3.5) \quad \begin{aligned} |\langle C^* A f(A) Bx, x \rangle| &\leq B(f, \gamma; A) \langle \| |A|^\alpha B \|^2 x, x \rangle^{1/2} \langle \| |A^*|^{1-\alpha} C \|^2 x, x \rangle^{1/2} \\ &\leq \frac{1}{2} B(f, \gamma; A) \left(\langle \| |A|^\alpha B \|^2 x, x \rangle + \langle \| |A^*|^{1-\alpha} C \|^2 x, x \rangle \right) \\ &= \frac{1}{2} B(f, \gamma; A) \left\langle \left(\| |A|^\alpha B \|^2 + \| |A^*|^{1-\alpha} C \|^2 \right) x, x \right\rangle \end{aligned}$$

for all $x \in H$.

By taking the supremum over $\|x\| = 1$ in (3.5) we get that

$$\begin{aligned} \omega(C^* A f(A) B) &= \sup_{\|x\|=1} |\langle C^* A f(A) Bx, x \rangle| \\ &\leq \frac{1}{2} B(f, \gamma; A) \sup_{\|x\|=1} \left\langle \left(\| |A|^\alpha B \|^2 + \| |A^*|^{1-\alpha} C \|^2 \right) x, x \right\rangle \\ &= \frac{1}{2} B(f, \gamma; A) \left\| \| |A|^\alpha B \|^2 + \| |A^*|^{1-\alpha} C \|^2 \right\|, \end{aligned}$$

which proves (5.2).

From (2.4) for $y = x$ and Buzano's inequality we derive that

$$\begin{aligned}
(3.6) \quad & |\langle C^* A f(A) B x, x \rangle|^2 \\
& \leq B^2(f, \gamma; A) \left\langle \| |A|^\alpha B|^2 x, x \right\rangle \left\langle x, \| |A^*|^{1-\alpha} C|^2 x \right\rangle \\
& \leq \frac{1}{2} B^2(f, \gamma; A) \\
& \times \left[\left\| \| |A|^\alpha B|^2 x \right\| \left\| \| |A^*|^{1-\alpha} C|^2 x \right\| + \left| \left\langle \| |A|^\alpha B|^2 x, \| |A^*|^{1-\alpha} C|^2 x \right\rangle \right| \right] \\
& = \frac{1}{2} B^2(f, \gamma; A) \\
& \times \left[\left\| \| |A|^\alpha B|^2 x \right\| \left\| \| |A^*|^{1-\alpha} C|^2 x \right\| + \left| \left\langle \| |A^*|^{1-\alpha} C|^2 \| |A|^\alpha B|^2 x, x \right\rangle \right| \right]
\end{aligned}$$

for all $x \in H$.

By taking the supremum over $\|x\| = 1$ in (3.6) we get that

$$\begin{aligned}
& \omega^2(C^* A f(A) B) \\
& = \sup_{\|x\|=1} |\langle C^* A f(A) B x, x \rangle|^2 \\
& \leq \frac{1}{2} B^2(f, \gamma; A) \\
& \times \sup_{\|x\|=1} \left[\left\| \| |A|^\alpha B|^2 x \right\| \left\| \| |A^*|^{1-\alpha} C|^2 x \right\| + \left| \left\langle \| |A^*|^{1-\alpha} C|^2 \| |A|^\alpha B|^2 x, x \right\rangle \right| \right] \\
& \leq \frac{1}{2} B^2(f, \gamma; A) \\
& \times \left[\sup_{\|x\|=1} \left\{ \left\| \| |A|^\alpha B|^2 x \right\| \left\| \| |A^*|^{1-\alpha} C|^2 x \right\| \right\} \right. \\
& \left. + \sup_{\|x\|=1} \left| \left\langle \| |A^*|^{1-\alpha} C|^2 \| |A|^\alpha B|^2 x, x \right\rangle \right| \right] \\
& \leq \frac{1}{2} B^2(f, \gamma; A) \\
& \times \left[\sup_{\|x\|=1} \left\| \| |A|^\alpha B|^2 x \right\| \sup_{\|x\|=1} \left\| \| |A^*|^{1-\alpha} C|^2 x \right\| \right. \\
& \left. + \sup_{\|x\|=1} \left| \left\langle \| |A^*|^{1-\alpha} C|^2 \| |A|^\alpha B|^2 x, x \right\rangle \right| \right] \\
& = \frac{1}{2} B^2(f, \gamma; A) \left[\left\| \| |A|^\alpha B|^2 \right\| \left\| \| |A^*|^{1-\alpha} C|^2 \right\| + \omega \left(\| |A^*|^{1-\alpha} C|^2 \| |A|^\alpha B|^2 \right) \right] \\
& = \frac{1}{2} B^2(f, \gamma; A) \left[\left\| \| |A|^\alpha B|^2 \right\|^2 \left\| \| |A^*|^{1-\alpha} C|^2 \right\|^2 + \omega \left(\| |A^*|^{1-\alpha} C|^2 \| |A|^\alpha B|^2 \right) \right],
\end{aligned}$$

which proves (5.13). \square

Remark 3. If we take $\alpha = 1/2$ in Theorem 4, then we get the norm inequality

$$(3.7) \quad \|C^* Af(A) B\| \leq B(f, \gamma; A) \left\| |A|^{1/2} B \right\| \left\| |A^*|^{1/2} C \right\|$$

and the numerical radius inequalities

$$(3.8) \quad \omega(C^* Af(A) B) \leq \frac{1}{2} B(f, \gamma; A) \left\| |A|^{1/2} B \right\|^2 + \left\| |A^*|^{1/2} C \right\|^2$$

and

$$(3.9) \quad \omega^2(C^* Af(A) B) \leq \frac{1}{2} B^2(f, \gamma; A) \left[\left\| |A|^{1/2} B \right\|^2 \left\| |A^*|^{1/2} C \right\|^2 + \omega \left(\left\| |A^*|^{1/2} C \right\|^2 \left\| |A|^{1/2} B \right\|^2 \right) \right].$$

The second main result is as follows:

Theorem 5. Assume that the conditions of Theorem 4 are satisfied. If $\alpha \in [0, 1]$, $r > 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 1$, then

$$(3.10) \quad \omega^{2r}(C^* Af(A) B) \leq B^{2r}(f, \gamma; A) \left\| \frac{1}{p} |A|^\alpha B \right\|^{2rp} + \left\| \frac{1}{q} |A^*|^{1-\alpha} C \right\|^{2rq}.$$

If $r \geq 1$, then

$$(3.11) \quad \omega^{2r}(C^* Af(A) B) \leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[\left\| |A|^\alpha B \right\|^{2r} \left\| |A^*|^{1-\alpha} C \right\|^{2r} + \omega^r \left(\left\| |A^*|^{1-\alpha} C \right\|^2 \left\| |A|^\alpha B \right\|^2 \right) \right].$$

If $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$, then also

$$(3.12) \quad \omega^{2r}(C^* Af(A) B) \leq \frac{1}{2} B^{2r}(f, \gamma; A) \left(\left\| \frac{1}{p} |A|^\alpha B \right\|^{2pr} + \left\| \frac{1}{q} |A^*|^{1-\alpha} C \right\|^{2qr} + \omega^r \left(\left\| |A^*|^{1-\alpha} C \right\|^2 \left\| |A|^\alpha B \right\|^2 \right) \right).$$

Proof. If we take the power $r > 0$ in (2.4) written for $y = x$ then we get, by Young and McCarthy inequalities that

$$\begin{aligned} & |\langle C^* Af(A) Bx, x \rangle|^{2r} \\ & \leq B^{2r}(f, \gamma; A) \left\langle |A|^\alpha B \right\rangle^r \left\langle |A^*|^{1-\alpha} C \right\rangle^r \\ & \leq B^{2r}(f, \gamma; A) \left[\frac{1}{p} \left\langle |A|^\alpha B \right\rangle^{rp} + \frac{1}{q} \left\langle |A^*|^{1-\alpha} C \right\rangle^{rq} \right] \\ & \leq B^{2r}(f, \gamma; A) \left[\frac{1}{p} \left\langle |A|^\alpha B \right\rangle^{2rp} + \frac{1}{q} \left\langle |A^*|^{1-\alpha} C \right\rangle^{2rq} \right] \\ & = B^{2r}(f, \gamma; A) \left[\left\langle \frac{1}{p} |A|^\alpha B \right\rangle^{2rp} + \left\langle \frac{1}{q} |A^*|^{1-\alpha} C \right\rangle^{2rq} \right] \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$\begin{aligned}
& \omega^{2r} (C^* A f(A) B) \\
&= \sup_{\|x\|=1} |\langle C^* A f(A) B x, x \rangle|^{2r} \\
&\leq B^{2r} (f, \gamma; A) \sup_{\|x\|=1} \left[\left\langle \left(\frac{1}{p} \|A\|^\alpha B^{2rp} + \frac{1}{q} \|A^*\|^{1-\alpha} C^{2rq} \right) x, x \right\rangle \right] \\
&= B^{2r} (f, \gamma; A) \left\| \frac{1}{p} \|A\|^\alpha B^{2rp} + \frac{1}{q} \|A^*\|^{1-\alpha} C^{2rq} \right\|,
\end{aligned}$$

which proves (3.10).

If we take the power $r \geq 1$ in (3.6) and by using the convexity of the power function, we get

$$\begin{aligned}
(3.13) \quad & |\langle C^* A f(A) B x, x \rangle|^{2r} \\
&= B^{2r} (f, \gamma; A) \\
&\times \left[\frac{\left\| \|A\|^\alpha B^2 x \right\| \left\| \|A^*\|^{1-\alpha} C^2 x \right\| + \left| \left\langle \|A^*\|^{1-\alpha} C^2 \|A\|^\alpha B^2 x, x \right\rangle \right|}{2} \right]^r \\
&\leq B^{2r} (f, \gamma; A) \\
&\times \frac{\left\| \|A\|^\alpha B^2 x \right\|^r \left\| \|A^*\|^{1-\alpha} C^2 x \right\|^r + \left| \left\langle \|A^*\|^{1-\alpha} C^2 \|A\|^\alpha B^2 x, x \right\rangle \right|^r}{2}
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$\begin{aligned}
& \omega^{2r} (C^* A f(A) B) \\
&\leq B^{2r} (f, \gamma; A) \\
&\times \frac{\left\| \|A\|^\alpha B^2 \right\|^r \left\| \|A^*\|^{1-\alpha} C^2 \right\|^r + \omega^r \left(\left\| \|A^*\|^{1-\alpha} C^2 \|A\|^\alpha B^2 \right\| \right)}{2} \\
&= B^{2r} (f, \gamma; A) \\
&\times \frac{\left\| \|A\|^\alpha B \right\|^{2r} \left\| \|A^*\|^{1-\alpha} C \right\|^{2r} + \omega^r \left(\left\| \|A^*\|^{1-\alpha} C^2 \|A\|^\alpha B^2 \right\| \right)}{2},
\end{aligned}$$

which proves (3.11).

Also, observe that

$$\begin{aligned}
 & \left\| \| |A|^\alpha B|^2 x \right\|^r \left\| \| |A^*|^{1-\alpha} C|^2 x \right\|^r \\
 & \leq \frac{1}{p} \left\| \| |A|^\alpha B|^2 x \right\|^{pr} + \frac{1}{q} \left\| \| |A^*|^{1-\alpha} C|^2 x \right\|^{qr} \\
 & = \frac{1}{p} \left\| \| |A|^\alpha B|^2 x \right\|^{2\frac{pr}{2}} + \frac{1}{q} \left\| \| |A^*|^{1-\alpha} C|^2 x \right\|^{2\frac{qr}{2}} \\
 & = \frac{1}{p} \left\langle \| |A|^\alpha B|^4 x, x \right\rangle^{\frac{pr}{2}} + \frac{1}{q} \left\langle \| |A^*|^{1-\alpha} C|^4 x, x \right\rangle^{\frac{qr}{2}} \\
 & \leq \frac{1}{p} \left\langle \| |A|^\alpha B|^{2pr} x, x \right\rangle + \frac{1}{q} \left\langle \| |A^*|^{1-\alpha} C|^{2qr} x, x \right\rangle \\
 & = \left\langle \left(\frac{1}{p} \| |A|^\alpha B|^{2pr} + \frac{1}{q} \| |A^*|^{1-\alpha} C|^{2qr} \right) x, x \right\rangle,
 \end{aligned}$$

for $x \in H$ with $\|x\| = 1$. Then

$$\begin{aligned}
 & \frac{\left\| \| |A|^\alpha B|^2 x \right\|^r \left\| \| |A^*|^{1-\alpha} C|^2 x \right\|^r + \left| \left\langle \| |A^*|^{1-\alpha} C|^2 \| |A|^\alpha B|^2 x, x \right\rangle \right|^r}{2} \\
 & \leq \frac{1}{2} \left[\left\langle \left(\frac{1}{p} \| |A|^\alpha B|^{2pr} + \frac{1}{q} \| |A^*|^{1-\alpha} C|^{2qr} \right) x, x \right\rangle \right. \\
 & \quad \left. + \left| \left\langle \| |A^*|^{1-\alpha} C|^2 \| |A|^\alpha B|^2 x, x \right\rangle \right|^r \right]
 \end{aligned}$$

and by (3.13)

$$\begin{aligned}
 & |C^* A f(A) B x, x|^{2r} \\
 & \leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[\left\langle \left(\frac{1}{p} \| |A|^\alpha B|^{2pr} + \frac{1}{q} \| |A^*|^{1-\alpha} C|^{2qr} \right) x, x \right\rangle \right. \\
 & \quad \left. + \left| \left\langle \| |A^*|^{1-\alpha} C|^2 \| |A|^\alpha B|^2 x, x \right\rangle \right|^r \right]
 \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, we derive (3.12). \square

Remark 4. If we take $r = 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ in (3.10), then we obtain

$$(3.14) \quad \omega^2(C^* A f(A) B) \leq B^2(f, \gamma; A) \left\| \frac{1}{p} \| |A|^\alpha B|^{2p} + \frac{1}{q} \| |A^*|^{1-\alpha} C|^{2q} \right\|,$$

which for $p = q = 2$ gives

$$(3.15) \quad \omega^2(C^* A f(A) B) \leq \frac{1}{2} B^2(f, \gamma; A) \left\| \| |A|^\alpha B|^4 + \| |A^*|^{1-\alpha} C|^4 \right\|.$$

If we take $r = 1$ and $p = q = 2$ in (3.12), then we get

$$\begin{aligned}
 (3.16) \quad \omega^2(C^* A f(A) B) & \leq \frac{1}{2} B^2(f, \gamma; A) \left(\frac{1}{2} \left\| \| |A|^\alpha B|^4 + \| |A^*|^{1-\alpha} C|^4 \right\| \right. \\
 & \quad \left. + \omega \left(\| |A^*|^{1-\alpha} C|^2 \| |A|^\alpha B|^2 \right) \right).
 \end{aligned}$$

If we take $r = 2$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ in (3.12), then we get

$$(3.17) \quad \omega^4(C^*Af(A)B) \leq \frac{1}{2}B^4(f, \gamma; A) \left(\left\| \frac{1}{p} \|A|^\alpha B|^{4p} + \frac{1}{q} |A^*|^{1-\alpha} C|^{4q} \right\| + \omega^2 \left(|A^*|^{1-\alpha} C|^2 \|A|^\alpha B|^2 \right) \right).$$

We also have:

Theorem 6. *With the assumptions of Theorem 4, we have for $r \geq 1$, $\lambda \in [0, 1]$ that*

$$(3.18) \quad \omega^2(C^*Af(A)B) \leq B^2(f, \gamma; A) \left\| (1-\lambda) \|A|^\alpha B|^{2r} + \lambda |A^*|^{1-\alpha} C|^{2r} \right\|^{1/r} \\ \times \| \|A|^\alpha B\|^{2\lambda} \| |A^*|^{1-\alpha} C \|^{2(1-\lambda)}$$

for all $\alpha \in [0, 1]$.

Also, we have

$$(3.19) \quad \omega^2(C^*Af(A)B) \leq B^2(f, \gamma; A) \left\| (1-\lambda) \|A|^\alpha B|^{2r} + \lambda |A^*|^{1-\alpha} C|^{2r} \right\|^{1/r} \\ \times \left\| \lambda \|A|^\alpha B|^{2r} + (1-\lambda) |A^*|^{1-\alpha} C|^{2r} \right\|^{1/r}$$

for all $\alpha \in [0, 1]$ and $r \geq 1$.

Proof. From the first part of (3.6) we have

$$\begin{aligned} & |\langle C^*Af(A)Bx, x \rangle|^2 \\ & \leq B^2(f, \gamma; A) \langle \|A|^\alpha B|^2 x, x \rangle \langle x, |A^*|^{1-\alpha} C|^2 x \rangle \\ & = B^2(f, \gamma; A) \langle \|A|^\alpha B|^2 x, x \rangle^{1-\lambda} \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^\lambda \\ & \times \langle \|A|^\alpha B|^2 x, x \rangle^\alpha \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^{1-\lambda} \\ & \leq B^2(f, \gamma; A) \left[(1-\lambda) \langle \|A|^\alpha B|^2 x, x \rangle + \lambda \langle x, |A^*|^{1-\alpha} C|^2 x \rangle \right] \\ & \times \langle \|A|^\alpha B|^2 x, x \rangle^\lambda \langle x, |A^*|^{1-\alpha} C|^2 x \rangle^{1-\lambda} \end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

If we take the power $r \geq 1$, then we get by the convexity of power r that

$$\begin{aligned}
 (3.20) \quad & |\langle C^* A f(A) B x, x \rangle|^{2r} \\
 & \leq B^{2r}(f, \gamma; A) \left[(1-\lambda) \langle \|A\|^\alpha B^2 x, x \rangle + \lambda \langle x, |A^*|^{1-\alpha} C^2 x \rangle \right]^r \\
 & \quad \times \langle \|A\|^\alpha B^2 x, x \rangle^{r\lambda} \langle x, |A^*|^{1-\alpha} C^2 x \rangle^{r(1-\lambda)} \\
 & \leq B^{2r}(f, \gamma; A) \left[(1-\lambda) \langle \|A\|^\alpha B^2 x, x \rangle^r + \lambda \langle x, |A^*|^{1-\alpha} C^2 x \rangle^r \right] \\
 & \quad \times \langle \|A\|^\alpha B^2 x, x \rangle^{r\lambda} \langle x, |A^*|^{1-\alpha} C^2 x \rangle^{r(1-\lambda)}
 \end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

If we use McCarthy inequality for power $r \geq 1$, then we get

$$\begin{aligned}
 & (1-\lambda) \langle \|A\|^\alpha B^2 x, x \rangle^r + \lambda \langle x, |A^*|^{1-\alpha} C^2 x \rangle^r \\
 & \leq (1-\lambda) \langle \|A\|^\alpha B^{2r} x, x \rangle + \lambda \langle x, |A^*|^{1-\alpha} C^{2r} x \rangle \\
 & = \left\langle \left[(1-\lambda) \|A\|^\alpha B^{2r} + \lambda |A^*|^{1-\alpha} C^{2r} \right] x, x \right\rangle
 \end{aligned}$$

and by (3.20)

$$\begin{aligned}
 (3.21) \quad & |\langle C^* A f(A) B x, x \rangle|^{2r} \\
 & \leq B^{2r}(f, \gamma; A) \left[\left\langle \left[(1-\lambda) \|A\|^\alpha B^{2r} + \lambda |A^*|^{1-\alpha} C^{2r} \right] x, x \right\rangle \right] \\
 & \quad \times \langle \|A\|^\alpha B^2 x, x \rangle^{r\lambda} \langle x, |A^*|^{1-\alpha} C^2 x \rangle^{r(1-\lambda)}
 \end{aligned}$$

for all $x \in H$, $\|x\| = 1$.

If we take the supremum over $\|x\| = 1$, then we get

$$\begin{aligned}
 & \omega^{2r}(C^* A f(A) B) \\
 & = \sup_{\|x\|=1} |\langle C^* A f(A) B x, x \rangle|^{2r} \\
 & \leq B^{2r}(f, \gamma; A) \sup_{\|x\|=1} \left[\left\langle \left[(1-\lambda) \|A\|^\alpha B^{2r} + \lambda |A^*|^{1-\alpha} C^{2r} \right] x, x \right\rangle \right] \\
 & \quad \times \sup_{\|x\|=1} \langle \|A\|^\alpha B^2 x, x \rangle^{r\lambda} \sup_{\|x\|=1} \langle x, |A^*|^{1-\alpha} C^2 x \rangle^{r(1-\lambda)} \\
 & = B^{2r}(f, \gamma; A) \left\| (1-\lambda) \|A\|^\alpha B^{2r} + \lambda |A^*|^{1-\alpha} C^{2r} \right\| \\
 & \quad \times \| \|A\|^\alpha B \|^{2r\lambda} \| |A^*|^{1-\alpha} C \|^{2r(1-\lambda)},
 \end{aligned}$$

which gives (3.18).

We also have

$$\begin{aligned} & |\langle C^* Af(A) Bx, x \rangle|^{2r} \\ & \leq B^{2r}(f, \gamma; A) \left[\left\langle \left[(1-\lambda) \|A\|^\alpha B^{2r} + \lambda \|A^*\|^{1-\alpha} C^{2r} \right] x, x \right\rangle \right] \\ & \times \left[\left\langle \left[\lambda \|A\|^\alpha B^{2r} + (1-\lambda) \|A^*\|^{1-\alpha} C^{2r} \right] x, x \right\rangle \right] \end{aligned}$$

for all $x \in H$, $\|x\| = 1$, which proves (3.19). \square

Remark 5. If we take $r = 1$ in Theorem 6, then we get

$$(3.22) \quad \omega^2(C^* Af(A) B) \leq B^2(f, \gamma; A) \left\| (1-\lambda) \|A\|^\alpha B^2 + \lambda \|A^*\|^{1-\alpha} C^2 \right\| \\ \times \left\| \|A\|^\alpha B \right\|^{2\lambda} \left\| \|A^*\|^{1-\alpha} C \right\|^{2(1-\lambda)}$$

and

$$(3.23) \quad \omega^2(C^* Af(A) B) \leq B^2(f, \gamma; A) \left\| (1-\lambda) \|A\|^\alpha B^2 + \lambda \|A^*\|^{1-\alpha} C^2 \right\| \\ \times \left\| \lambda \|A\|^\alpha B^2 + (1-\lambda) \|A^*\|^{1-\alpha} C^2 \right\|$$

for all $\alpha, \lambda \in [0, 1]$.

If we take $\lambda = 1/2$ in (3.22), then we obtain

$$(3.24) \quad \omega^2(C^* Af(A) B) \\ \leq \frac{1}{2} B^2(f, \gamma; A) \left\| \|A\|^\alpha B^2 + \|A^*\|^{1-\alpha} C^2 \right\| \left\| \|A\|^\alpha B \right\| \left\| \|A^*\|^{1-\alpha} C \right\|.$$

If we take $r = 2$ in Theorem 6, then we get

$$(3.25) \quad \omega^2(C^* Af(A) B) \leq B^2(f, \gamma; A) \left\| (1-\lambda) \|A\|^\alpha B^4 + \lambda \|A^*\|^{1-\alpha} C^4 \right\|^{1/2} \\ \times \left\| \|A\|^\alpha B \right\|^{2\lambda} \left\| \|A^*\|^{1-\alpha} C \right\|^{2(1-\lambda)}$$

and

$$(3.26) \quad \omega^2(C^* Af(A) B) \leq B^2(f, \gamma; A) \left\| (1-\lambda) \|A\|^\alpha B^4 + \lambda \|A^*\|^{1-\alpha} C^4 \right\|^{1/2} \\ \times \left\| \lambda \|A\|^\alpha B^4 + (1-\lambda) \|A^*\|^{1-\alpha} C^4 \right\|^{1/2}$$

for all $\alpha, \lambda \in [0, 1]$.

If we take $\lambda = 1/2$ in (3.25), then we obtain

$$(3.27) \quad \omega^2(C^* Af(A) B) \\ \leq \frac{\sqrt{2}}{2} B^2(f, \gamma; A) \left\| \|A\|^\alpha B^4 + \|A^*\|^{1-\alpha} C^4 \right\|^{1/2} \left\| \|A\|^\alpha B \right\| \left\| \|A^*\|^{1-\alpha} C \right\|.$$

4. INEQUALITIES FOR TRACE OF OPERATORS

We have the following result for trace of operators:

Theorem 7. *Let $r \geq 1/2$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 1$. Let $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the domain G and $A \in \mathcal{B}(H)$ with $\text{Sp}(A) \subset G$ and γ a closed rectifiable path in G and such that $\text{Sp}(A) \subset \text{ins}(\gamma)$. If $B, C \in \mathcal{B}(H)$ with $|A|^\alpha B \in \mathcal{B}_{2pr}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2qr}(H)$ for $\alpha \in [0, 1]$, then $C^* A f(A) B \in \mathcal{B}_{2r}(H)$ and*

$$(4.1) \quad \|C^* A f(A) B\|_{2r} \leq B(f, \gamma; A) \| |A|^\alpha B \|_{2pr} \left\| |A^*|^{1-\alpha} C \right\|_{2qr}.$$

In particular,

$$(4.2) \quad \|C^* A f(A) B\|_{2r} \leq B(f, \gamma; A) \left\| |A|^{1/2} B \right\|_{2pr} \left\| |A^*|^{1/2} C \right\|_{2qr}$$

for $|A|^{1/2} B \in \mathcal{B}_{2pr}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2qr}(H)$.

Proof. If we take in (2.7) the power $r > 0$ and $x = e_i$, $y = f_i$ where $\mathcal{E} = \{e_i\}_{i \in I}$ and $\mathcal{F} = \{f_i\}_{i \in I}$ are orthonormal basis and sum, then we get

$$(4.3) \quad \sum_{i \in I} |\langle C^* A f(A) B e_i, f_i \rangle|^{2r} \\ \leq B^{2r}(f, \gamma; A) \sum_{i \in I} \left\langle |A|^\alpha B|^2 e_i, e_i \right\rangle^r \left\langle |A^*|^{1-\alpha} C|^2 f_i, f_i \right\rangle^r.$$

If we use the Hölder's inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we get

$$(4.4) \quad \sum_{i \in I} \left\langle |A|^\alpha B|^2 e_i, e_i \right\rangle^r \left\langle |A^*|^{1-\alpha} C|^2 f_i, f_i \right\rangle^r \\ \leq \left(\sum_{i \in I} \left\langle |A|^\alpha B|^2 e_i, e_i \right\rangle^{pr} \right)^{1/p} \left(\sum_{i \in I} \left\langle |A^*|^{1-\alpha} C|^2 f_i, f_i \right\rangle^{qr} \right)^{1/q}.$$

By the McCarthy inequality for $pr, qr \geq 1$, we have

$$\sum_{i \in I} \left\langle |A|^\alpha B|^2 e_i, e_i \right\rangle^{pr} \leq \sum_{i \in I} \left\langle |A|^\alpha B|^{2pr} e_i, e_i \right\rangle$$

and

$$\sum_{i \in I} \left\langle |A^*|^{1-\alpha} C|^2 f_i, f_i \right\rangle^{qr} \leq \sum_{i \in I} \left\langle |A^*|^{1-\alpha} C|^{2qr} f_i, f_i \right\rangle,$$

therefore

$$\left(\sum_{i \in I} \left\langle |A|^\alpha B|^2 e_i, e_i \right\rangle^{pr} \right)^{1/p} \left(\sum_{i \in I} \left\langle |A^*|^{1-\alpha} C|^2 f_i, f_i \right\rangle^{qr} \right)^{1/q} \\ \leq \left(\sum_{i \in I} \left\langle |A|^\alpha B|^{2pr} e_i, e_i \right\rangle \right)^{1/p} \left(\sum_{i \in I} \left\langle |A^*|^{1-\alpha} C|^{2qr} f_i, f_i \right\rangle \right)^{1/q} \\ = \left(\| |A|^\alpha B \|_{2pr}^{2pr} \right)^{1/p} \left(\| |A^*|^{1-\alpha} C \|_{2qr}^{2qr} \right)^{1/q} = \| |A|^\alpha B \|_{2pr}^{2r} \| |A^*|^{1-\alpha} C \|_{2qr}^{2r}.$$

By (4.3) and (4.4) we derive

$$(4.5) \quad \sum_{i \in I} |\langle C^* A f(A) B e_i, f_i \rangle|^{2r} \leq B^{2r}(f, \gamma; A) \| |A|^\alpha B \|_{2pr}^{2r} \| |A^*|^{1-\alpha} C \|_{2qr}^{2r}.$$

Now, if we take the supremum over \mathcal{E} and \mathcal{F} in (2.6), then by (1.23) we get

$$\| C^* A f(A) B \|_{2r}^{2r} \leq B^{2r}(f, \gamma; A) \| |A|^\alpha B \|_{2pr}^{2r} \| |A^*|^{1-\alpha} C \|_{2qr}^{2r}$$

and the inequality (4.1) is obtained. \square

Remark 6. If we take $r = 1/2$ and $p = q = 2$, then by (4.1) we get

$$(4.6) \quad \| C^* A f(A) B \|_1 \leq B(f, \gamma; A) \| |A|^\alpha B \|_2 \| |A^*|^{1-\alpha} C \|_2$$

provided that $|A|^\alpha B \in \mathcal{B}_2(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_2(H)$ for $\alpha \in [0, 1]$.

Also, if $r = 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (4.1) we get

$$(4.7) \quad \| C^* A f(A) B \|_2 \leq B(f, \gamma; A) \| |A|^\alpha B \|_{2p} \| |A^*|^{1-\alpha} C \|_{2q}$$

provided that $|A|^\alpha B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2q}(H)$ for $\alpha \in [0, 1]$.

We also have:

Theorem 8. Let $r \geq 1/2$, $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the domain G and $A \in \mathcal{B}(H)$ with $\text{Sp}(A) \subset G$ and γ a closed rectifiable path in G and such that $\text{Sp}(A) \subset \text{ins}(\gamma)$. If $|A|^\alpha B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2q}(H)$ for $\alpha \in [0, 1]$, then $C^* A f(A) B \in \mathcal{B}_{2r}(H)$ and

$$(4.8) \quad \| C^* A f(A) B \|_{2r} \leq B(f, \gamma; A) \| |A|^\alpha B \|_{2p} \| |A^*|^{1-\alpha} C \|_{2q}.$$

In particular,

$$(4.9) \quad \| C^* A f(A) B \|_{2r} \leq B(f, \gamma; A) \| |A|^{1/2} B \|_{2p} \| |A^*|^{1/2} C \|_{2q}$$

for $|A|^{1/2} B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2q}(H)$.

Proof. Assume that $\mathcal{E} = \{e_i\}_{i \in I}$ and $\mathcal{F} = \{f_i\}_{i \in I}$ are orthonormal basis in H . Observe that we have $\frac{1}{p} + \frac{1}{q} = 1$ and by Hölder's inequality for $\frac{p}{r}$ and $\frac{q}{r}$ we have

$$(4.10) \quad \begin{aligned} & \sum_{i \in I} \left\langle |A|^\alpha B^2 e_i, e_i \right\rangle^r \left\langle |A^*|^{1-\alpha} C^2 f_i, f_i \right\rangle^r \\ &= \sum_{i \in I} \left[\left\langle |A|^\alpha B^2 e_i, e_i \right\rangle^p \right]^{\frac{r}{p}} \left[\left\langle |A^*|^{1-\alpha} C^2 f_i, f_i \right\rangle^q \right]^{\frac{r}{q}} \\ &\leq \left(\sum_{i \in I} \left\langle |A|^\alpha B^2 e_i, e_i \right\rangle^p \right)^{r/p} \left(\sum_{i \in I} \left\langle |A^*|^{1-\alpha} C^2 f_i, f_i \right\rangle^q \right)^{r/q}. \end{aligned}$$

By McCarthy inequality for $p, q > 1$ we get

$$\sum_{i \in I} \left\langle |A|^\alpha B^2 e_i, e_i \right\rangle^p \leq \sum_{i \in I} \left\langle |A|^\alpha B^{2p} e_i, e_i \right\rangle$$

and

$$\sum_{i \in I} \left\langle \left| |A^*|^{1-\alpha} C \right|^2 f_i, f_i \right\rangle^q \leq \sum_{i \in I} \left\langle \left| |A^*|^{1-\alpha} C \right|^{2q} f_i, f_i \right\rangle$$

and by (4.10)

$$(4.11) \quad \begin{aligned} & \sum_{i \in I} \left\langle \left| |A|^\alpha B \right|^2 e_i, e_i \right\rangle^r \left\langle \left| |A^*|^{1-\alpha} C \right|^2 f_i, f_i \right\rangle^r \\ & \leq \left(\sum_{i \in I} \left\langle \left| |A|^\alpha B \right|^{2p} e_i, e_i \right\rangle \right)^{r/p} \left(\sum_{i \in I} \left\langle \left| |A^*|^{1-\alpha} C \right|^{2q} f_i, f_i \right\rangle \right)^{r/q} \\ & = \left\| |A|^\alpha B \right\|_{2p}^{2r} \left\| |A^*|^{1-\alpha} C \right\|_{2q}^{2r}. \end{aligned}$$

By (4.3) and (4.11) we get

$$(4.12) \quad \sum_{i \in I} |\langle C^* A f(A) B e_i, f_i \rangle|^{2r} \leq B^{2r}(f, \gamma; A) \left\| |A|^\alpha B \right\|_{2p}^{2r} \left\| |A^*|^{1-\alpha} C \right\|_{2q}^{2r}.$$

Now, if we take the supremum over \mathcal{E} and \mathcal{F} in (4.12) we get

$$\|C^* A f(A) B\|_{2r}^{2r} \leq B^{2r}(f, \gamma; A) \left\| |A|^\alpha B \right\|_{2p}^{2r} \left\| |A^*|^{1-\alpha} C \right\|_{2q}^{2r}$$

and the inequality (4.8) is thus proved. \square

Remark 7. If we take $p = q = 2r = s \geq 1$, then by (4.8) we get

$$(4.13) \quad \|C^* A f(A) B\|_s \leq B(f, \gamma; A) \left\| |A|^\alpha B \right\|_{2s} \left\| |A^*|^{1-\alpha} C \right\|_{2s}$$

provided that $|A|^\alpha B \in \mathcal{B}_{2s}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2s}(H)$ for $\alpha \in [0, 1]$.

For $\alpha = 1/2$ we have

$$(4.14) \quad \|C^* A f(A) B\|_s \leq B(f, \gamma; A) \left\| |A|^{1/2} B \right\|_{2s} \left\| |A^*|^{1/2} C \right\|_{2s}$$

provided that $|A|^{1/2} B \in \mathcal{B}_{2s}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2s}(H)$.

If $r = 2$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, then

$$(4.15) \quad \|C^* A f(A) B\|_4 \leq B(f, \gamma; A) \left\| |A|^\alpha B \right\|_{2p} \left\| |A^*|^{1-\alpha} C \right\|_{2q}$$

provided that $|A|^\alpha B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2q}(H)$ for $\alpha \in [0, 1]$.

In particular,

$$(4.16) \quad \|C^* A f(A) B\|_4 \leq B(f, \gamma; A) \left\| |A|^{1/2} B \right\|_{2p} \left\| |A^*|^{1/2} C \right\|_{2q}$$

for $|A|^{1/2} B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2q}(H)$.

Theorem 9. Let $f : G \subset \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function on the domain G and $A \in \mathcal{B}(H)$ with $\text{Sp}(A) \subset G$ and γ a closed rectifiable path in G and such that $\text{Sp}(A) \subset \text{ins}(\gamma)$.

If $r \geq 1/2$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $pr, qr \geq 1$ and $\left\| |A|^\alpha B \right\|_{2pr}^{2pr}, \left\| |A^*|^{1-\alpha} C \right\|_{2qr}^{2qr} \in \mathcal{B}_1(H)$, then $C^* A f(A) B \in \mathcal{B}_{2r}(H)$ and

$$(4.17) \quad \omega_{2r}^{2r}(C^* A f(A) B) \leq B^{2r}(f, \gamma; A) \text{tr} \left(\frac{1}{p} \left\| |A|^\alpha B \right\|_{2pr}^{2pr} + \frac{1}{q} \left\| |A^*|^{1-\alpha} C \right\|_{2qr}^{2qr} \right).$$

If $r \geq 1$ and $|A|^\alpha B, |A^*|^{1-\alpha} C \in \mathcal{B}_{4r}(H)$, then $C^* Af(A)B \in \mathcal{B}_{2r}(H)$ and

(4.18)

$$\begin{aligned} & \omega_{2r}^{2r}(C^* Af(A)B) \\ & \leq \frac{1}{2} B^{2r}(f, \gamma; A) \left(\| |A|^\alpha B \|_{4r}^{2r} \| |A^*|^{1-\alpha} C \|_{4r}^{2r} + \omega_r^r \left(\| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \right) \right) \\ & \leq \frac{1}{2} B^{2r}(f, \gamma; A) \left(\| |A|^\alpha B \|_{4r}^{2r} \| |A^*|^{1-\alpha} C \|_{4r}^{2r} + \left\| \| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \right\|_r^r \right). \end{aligned}$$

If $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $pr, qr \geq 2$, then

$$\begin{aligned} (4.19) \quad \omega_{2r}^{2r}(C^* Af(A)B) & \leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[\text{tr} \left(\frac{1}{p} \| |A|^\alpha B \|^{2pr} + \frac{1}{q} \| |A^*|^{1-\alpha} C \|^{2qr} \right) \right. \\ & \quad \left. + \omega_r^r \left(\| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \right) \right] \\ & \leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[\text{tr} \left(\frac{1}{p} \| |A|^\alpha B \|^{2pr} + \frac{1}{q} \| |A^*|^{1-\alpha} C \|^{2qr} \right) \right. \\ & \quad \left. + \left\| \| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \right\|_r^r \right]. \end{aligned}$$

Proof. From (2.7) for $y = x$ we have that

$$(4.20) \quad |\langle C^* Af(A)Bx, x \rangle|^2 \leq B^2(f, \gamma; A) \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle \left\langle \| |A^*|^{1-\alpha} C \|^2 x, x \right\rangle$$

for $x \in H$ with $\|x\| = 1$.

If we take the power $r > 0$, we get, by Young and McCarthy inequalities, that

$$\begin{aligned} & |\langle C^* Af(A)Bx, x \rangle|^{2r} \\ & \leq B^{2r}(f, \gamma; A) \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^r \left\langle \| |A^*|^{1-\alpha} C \|^2 x, x \right\rangle^r \\ & \leq B^{2r}(f, \gamma; A) \left[\frac{1}{p} \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^{pr} + \frac{1}{q} \left\langle \| |A^*|^{1-\alpha} C \|^2 x, x \right\rangle^{qr} \right] \\ & \leq B^{2r}(f, \gamma; A) \left[\frac{1}{p} \left\langle \| |A|^\alpha B \|^{2pr} x, x \right\rangle + \frac{1}{q} \left\langle \| |A^*|^{1-\alpha} C \|^{2qr} x, x \right\rangle \right] \\ & = B^{2r}(f, \gamma; A) \left\langle \left(\frac{1}{p} \| |A|^\alpha B \|^{2pr} + \frac{1}{q} \| |A^*|^{1-\alpha} C \|^{2qr} \right) x, x \right\rangle \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

If $\mathcal{E} = \{e_i\}_{i \in I}$ is an orthonormal basis, then by taking $x = e_i$ and summing over $i \in I$ we get

$$\begin{aligned}
 & \|C^* A f(A) B\|_{\mathcal{E}, 2r}^{2r} \\
 &= \sum_{i \in I} |\langle C^* A f(A) B e_i, e_i \rangle|^{2r} \\
 &\leq B^{2r}(f, \gamma; A) \sum_{i \in I} \left\langle \left(\frac{1}{p} \|A|^\alpha B|^{2pr} + \frac{1}{q} \|A^*|^{1-\alpha} C|^{2qr} \right) e_i, e_i \right\rangle \\
 &= B^{2r}(f, \gamma; A) \operatorname{tr} \left(\frac{1}{p} \|A|^\alpha B|^{2pr} + \frac{1}{q} \|A^*|^{1-\alpha} C|^{2qr} \right),
 \end{aligned}$$

which, by taking the supremum over \mathcal{E} , proves (4.17).

By Buzano's inequality we have

$$\begin{aligned}
 & \left\langle \|A|^\alpha B|^2 x, x \right\rangle \left\langle x, \|A^*|^{1-\alpha} C|^2 x \right\rangle \\
 &\leq \frac{1}{2} \left[\left\| \|A|^\alpha B|^2 x \right\| \left\| \|A^*|^{1-\alpha} C|^2 x \right\| + \left| \left\langle \|A|^\alpha B|^2 x, \|A^*|^{1-\alpha} C|^2 x \right\rangle \right| \right] \\
 &= \frac{1}{2} \left[\left\| \|A|^\alpha B|^2 x \right\| \left\| \|A^*|^{1-\alpha} C|^2 x \right\| + \left| \left\langle \|A^*|^{1-\alpha} C|^2 \|A|^\alpha B|^2 x, x \right\rangle \right| \right]
 \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

If we take the power $r \geq 1$ and use the convexity of power function, then we get

$$\begin{aligned}
 & \left\langle \|A|^\alpha B|^2 x, x \right\rangle^r \left\langle x, \|A^*|^{1-\alpha} C|^2 x \right\rangle^r \\
 &\leq \left[\frac{\left\| \|A|^\alpha B|^2 x \right\| \left\| \|A^*|^{1-\alpha} C|^2 x \right\| + \left| \left\langle \|A^*|^{1-\alpha} C|^2 \|A|^\alpha B|^2 x, x \right\rangle \right|}{2} \right]^r \\
 &\leq \frac{\left\| \|A|^\alpha B|^2 x \right\|^r \left\| \|A^*|^{1-\alpha} C|^2 x \right\|^r + \left| \left\langle \|A^*|^{1-\alpha} C|^2 \|A|^\alpha B|^2 x, x \right\rangle \right|^r}{2} \\
 &= \frac{\left\| \|A|^\alpha B|^2 x \right\|^{2\frac{r}{2}} \left\| \|A^*|^{1-\alpha} C|^2 x \right\|^{2\frac{r}{2}} + \left| \left\langle \|A^*|^{1-\alpha} C|^2 \|A|^\alpha B|^2 x, x \right\rangle \right|^r}{2} \\
 &= \frac{\left\langle \|A|^\alpha B|^4 x, x \right\rangle^{\frac{r}{2}} \left\langle \|A^*|^{1-\alpha} C|^4 x, x \right\rangle^{\frac{r}{2}} + \left| \left\langle \|A^*|^{1-\alpha} C|^2 \|A|^\alpha B|^2 x, x \right\rangle \right|^r}{2}
 \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

Therefore

$$\begin{aligned}
(4.21) \quad & \|C^* A f(A) B\|_{\mathcal{E}, 2r}^{2r} \\
&= \sum_{i \in I} |\langle C^* A f(A) B e_i, e_i \rangle|^{2r} \\
&\leq B^{2r}(f, \gamma; A) \sum_{i \in I} \left\langle \| |A|^\alpha B \|^2 e_i, e_i \right\rangle^r \left\langle e_i, \| |A^*|^{1-\alpha} C \|^2 e_i \right\rangle^r \\
&\leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[\sum_{i \in I} \left\langle \| |A|^\alpha B \|^4 e_i, e_i \right\rangle^{\frac{r}{2}} \left\langle \| |A^*|^{1-\alpha} C \|^4 e_i, e_i \right\rangle^{\frac{r}{2}} \right. \\
&\quad \left. + \sum_{i \in I} \left| \left\langle \| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 e_i, e_i \right\rangle \right|^r \right].
\end{aligned}$$

Using Cauchy-Schwarz inequality we have

$$\begin{aligned}
& \sum_{i \in I} \left\langle \| |A|^\alpha B \|^4 e_i, e_i \right\rangle^{\frac{r}{2}} \left\langle \| |A^*|^{1-\alpha} C \|^4 e_i, e_i \right\rangle^{\frac{r}{2}} \\
&\leq \left(\sum_{i \in I} \left\langle \| |A|^\alpha B \|^4 e_i, e_i \right\rangle^r \right)^{1/2} \left(\sum_{i \in I} \left\langle \| |A^*|^{1-\alpha} C \|^4 e_i, e_i \right\rangle^r \right)^{1/2} \\
&\leq \left(\sum_{i \in I} \left\langle \| |A|^\alpha B \|^4 e_i, e_i \right\rangle^{2r} \right)^{1/2} \left(\sum_{i \in I} \left\langle \| |A^*|^{1-\alpha} C \|^4 e_i, e_i \right\rangle^{2r} \right)^{1/2} \\
&= \| |A|^\alpha B \|_{4r}^{2r} \| |A^*|^{1-\alpha} C \|_{4r}^{2r},
\end{aligned}$$

where for the last inequality we used McCarthy's result for $r \geq 1$. This proves (4.18).

Further, if we use Young's inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q, \quad a, b \geq 0,$$

then we get

$$\begin{aligned}
\| \| |A|^\alpha B \|^2 x \| \| |A^*|^{1-\alpha} C \|^2 x \|^r &\leq \frac{1}{p} \| \| |A|^\alpha B \|^2 x \|^{pr} + \frac{1}{q} \| \| |A^*|^{1-\alpha} C \|^2 x \|^{qr} \\
&= \frac{1}{p} \| \| |A|^\alpha B \|^2 x \|^{2\frac{pr}{2}} + \frac{1}{q} \| \| |A^*|^{1-\alpha} C \|^2 x \|^{2\frac{qr}{2}} \\
&= \frac{1}{p} \left\langle \| |A|^\alpha B \|^4 x, x \right\rangle^{\frac{pr}{2}} + \frac{1}{q} \left\langle \| |A^*|^{1-\alpha} C \|^4 x, x \right\rangle^{\frac{qr}{2}} \\
&\leq \frac{1}{p} \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^{pr} + \frac{1}{q} \left\langle \| |A^*|^{1-\alpha} C \|^2 x, x \right\rangle^{qr} \\
&= \left\langle \left(\frac{1}{p} \| |A|^\alpha B \|^2 + \frac{1}{q} \| |A^*|^{1-\alpha} C \|^2 \right) x, x \right\rangle
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

Therefore

$$\begin{aligned}
 \|C^*Af(A)B\|_{\mathcal{E},2r}^{2r} &= \sum_{i \in I} |\langle C^*Af(A)Be_i, e_i \rangle|^{2r} \\
 &\leq B^{2r}(f, \gamma; A) \sum_{i \in I} \left\langle \| |A|^\alpha B \|^2 e_i, e_i \right\rangle^r \left\langle e_i, \left\| |A^*|^{1-\alpha} C \right\|^2 e_i \right\rangle^r \\
 &\leq \frac{1}{2} B^{2r}(f, \gamma; A) \left[\sum_{i \in I} \left\langle \left(\frac{1}{p} \| |A|^\alpha B \|^2 + \frac{1}{q} \left\| |A^*|^{1-\alpha} C \right\|^2 \right) e_i, e_i \right\rangle \right. \\
 &\quad \left. + \sum_{i \in I} \left| \left\langle \left\| |A^*|^{1-\alpha} C \right\|^2 \| |A|^\alpha B \|^2 e_i, e_i \right\rangle \right|^r \right] \\
 &= \frac{1}{2} B^{2r}(f, \gamma; A) \left[\text{tr} \left(\frac{1}{p} \| |A|^\alpha B \|^2 + \frac{1}{q} \left\| |A^*|^{1-\alpha} C \right\|^2 \right) \right. \\
 &\quad \left. + \left\| \left\| |A^*|^{1-\alpha} C \right\|^2 \| |A|^\alpha B \|^2 \right\|_{\mathcal{E},r}^r \right],
 \end{aligned}$$

which proves, by taking the supremum over \mathcal{E} , the desired inequality (4.19). \square

Remark 8. Let $\alpha \in [0, 1]$. If $r = 1/2$, $p, q = 2$ and $\| |A|^\alpha B \|^2, \left\| |A^*|^{1-\alpha} C \right\|^2 \in \mathcal{B}_1(H)$, then $C^*Af(A)B \in \mathcal{B}_1(H)$ and by (4.17) we get

$$(4.22) \quad \omega_1(C^*Af(A)B) \leq \frac{1}{2} B(f, \gamma; A) \text{tr} \left(\| |A|^\alpha B \|^2 + \left\| |A^*|^{1-\alpha} C \right\|^2 \right).$$

If $r = 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (4.17) we obtain

$$(4.23) \quad \omega_2^2(C^*Af(A)B) \leq B^2(f, \gamma; A) \text{tr} \left(\frac{1}{p} \| |A|^\alpha B \|^2 + \frac{1}{q} \left\| |A^*|^{1-\alpha} C \right\|^2 \right),$$

provided that $\| |A|^\alpha B \|^2, \left\| |A^*|^{1-\alpha} C \right\|^2 \in \mathcal{B}_1(H)$.

If we take $r = 1$ in (4.18), then we get

$$\begin{aligned}
 (4.24) \quad \omega_2^2(C^*Af(A)B) &\leq \frac{1}{2} B^2(f, \gamma; A) \left(\left\| \| |A|^\alpha B \|^2 \right\|_4^2 \left\| \left\| |A^*|^{1-\alpha} C \right\|^2 \right\|_4^2 + \omega_1 \left(\left\| |A^*|^{1-\alpha} C \right\|^2 \| |A|^\alpha B \|^2 \right) \right) \\
 &\leq \frac{1}{2} B^2(f, \gamma; A) \left(\left\| \| |A|^\alpha B \|^2 \right\|_4^2 \left\| \left\| |A^*|^{1-\alpha} C \right\|^2 \right\|_4^2 + \left\| \left\| |A^*|^{1-\alpha} C \right\|^2 \| |A|^\alpha B \|^2 \right\|_1 \right),
 \end{aligned}$$

provided that $|A|^\alpha B, |A^*|^{1-\alpha} C \in \mathcal{B}_4(H)$.

If $r = 1$ and $p = q = 2$ in (4.19), then we get for $\| |A|^\alpha B \|^{2p}$, $\| |A^*|^{1-\alpha} C \|^{2q} \in \mathcal{B}_1(H)$ that

$$\begin{aligned}
(4.25) \quad \omega_2^2(C^* A f(A) B) &\leq \frac{1}{4} B^2(f, \gamma; A) \left[\text{tr} \left(\| |A|^\alpha B \|^{2p} + \| |A^*|^{1-\alpha} C \|^{2q} \right) \right. \\
&\quad \left. + \frac{1}{2} B^2(f, \gamma; A) \omega_1 \left(\| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \right) \right] \\
&\leq \frac{1}{4} B^2(f, \gamma; A) \text{tr} \left(\| |A|^\alpha B \|^{2p} + \| |A^*|^{1-\alpha} C \|^{2q} \right) \\
&\quad + \frac{1}{2} B^2(f, \gamma; A) \left\| \| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \right\|_1.
\end{aligned}$$

We also have:

Theorem 10. *With the assumptions of Theorem 9, we have for $r \geq 1$, $\lambda \in [0, 1]$ that*

$$\begin{aligned}
(4.26) \quad \omega_{2r}^{2r}(C^* A f(A) B) &\leq B^{2r}(f, \gamma; A) \left\| (1-\lambda) \| |A|^\alpha B \|^{2r} + \lambda \| |A^*|^{1-\alpha} C \|^{2r} \right\| \\
&\quad \times \left\| \| |A|^\alpha B \|_{2r}^{2r\lambda} \| |A^*|^{1-\alpha} C \|_{2r}^{2r(1-\lambda)} \right\|,
\end{aligned}$$

provided that $|A|^\alpha B, |A^*|^{1-\alpha} C \in \mathcal{B}_{2r}(H)$.

In particular,

$$\begin{aligned}
(4.27) \quad \omega_{2r}^{2r}(C^* A f(A) B) &\leq \frac{1}{2} B^{2r}(f, \gamma; A) \left\| \| |A|^\alpha B \|^{2r} + \| |A^*|^{1-\alpha} C \|^{2r} \right\| \\
&\quad \times \left\| \| |A|^\alpha B \|_{2r}^r \| |A^*|^{1-\alpha} C \|_{2r}^r \right\|.
\end{aligned}$$

Proof. If $\mathcal{E} = \{e_i\}_{i \in I}$ is an orthonormal basis, then by taking $x = e_i$ in (3.21) and summing over $i \in I$ we get

$$\begin{aligned}
(4.28) \quad &\sum_{i \in I} |(C^* A f(A) B e_i, e_i)|^{2r} \\
&\leq B^{2r}(f, \gamma; A) \sum_{i \in I} \left[\left\langle \left[(1-\lambda) \| |A|^\alpha B \|^{2r} + \lambda \| |A^*|^{1-\alpha} C \|^{2r} \right] e_i, e_i \right\rangle \right] \\
&\quad \times \left\langle \| |A|^\alpha B \|^2 e_i, e_i \right\rangle^{r\lambda} \left\langle \| |A^*|^{1-\alpha} C \|^2 e_i, e_i \right\rangle^{r(1-\lambda)} \\
&\leq B^{2r}(f, \gamma; A) \left\| (1-\lambda) \| |A|^\alpha B \|^{2r} + \lambda \| |A^*|^{1-\alpha} C \|^{2r} \right\| \\
&\quad \times \sum_{i \in I} \left\langle \| |A|^\alpha B \|^2 e_i, e_i \right\rangle^{r\lambda} \left\langle \| |A^*|^{1-\alpha} C \|^2 e_i, e_i \right\rangle^{r(1-\lambda)}.
\end{aligned}$$

If we use Hölder's inequality for $p = \frac{1}{\lambda}$, $q = \frac{1}{1-\lambda}$, then we have

$$\begin{aligned}
 & \sum_{i \in I} \left\langle \| |A|^\alpha B \|^2 e_i, e_i \right\rangle^{r\lambda} \left\langle \| |A^*|^{1-\alpha} C \|^2 e_i, e_i \right\rangle^{r(1-\lambda)} \\
 & \leq \left(\sum_{i \in I} \left\langle \| |A|^\alpha B \|^2 e_i, e_i \right\rangle^r \right)^\lambda \left(\sum_{i \in I} \left\langle \| |A^*|^{1-\alpha} C \|^2 e_i, e_i \right\rangle^r \right)^{1-\lambda} \\
 & \leq \left(\sum_{i \in I} \left\langle \| |A|^\alpha B \|^2 e_i, e_i \right\rangle^{2r} \right)^\lambda \left(\sum_{i \in I} \left\langle \| |A^*|^{1-\alpha} C \|^2 e_i, e_i \right\rangle^{2r} \right)^{1-\lambda} \\
 & = \| |A|^\alpha B \|_{2r}^{2r\lambda} \| |A^*|^{1-\alpha} C \|_{2r}^{2r(1-\lambda)},
 \end{aligned}$$

which proves (4.26). \square

Remark 9. If we take $r = 1$ in Theorem 10, then we get for $\alpha \in [0, 1]$ that

$$\begin{aligned}
 (4.29) \quad \omega_2^2(C^* A f(A) B) & \leq B^2(f, \gamma; A) \left\| (1-\lambda) \| |A|^\alpha B \|^2 + \lambda \| |A^*|^{1-\alpha} C \|^2 \right\| \\
 & \quad \times \| |A|^\alpha B \|_2^{2\lambda} \| |A^*|^{1-\alpha} C \|_2^{2(1-\lambda)},
 \end{aligned}$$

provided that $|A|^\alpha B, |A^*|^{1-\alpha} C \in \mathcal{B}_2(H)$.

In particular,

$$\begin{aligned}
 (4.30) \quad \omega_2^2(C^* A f(A) B) & \leq \frac{1}{2} B^2(f, \gamma; A) \left\| \| |A|^\alpha B \|^2 + \| |A^*|^{1-\alpha} C \|^2 \right\| \\
 & \quad \times \| |A|^\alpha B \|_2 \| |A^*|^{1-\alpha} C \|_2.
 \end{aligned}$$

5. SOME EXAMPLES

Consider the exponential function $f(A) = \exp A$, $A \in \mathcal{B}(H)$. Assume that $A \in \mathcal{B}(H)$ and $\|A\| < R$ for some $R > 0$. Observe that for $t \in [0, 1]$,

$$\left| \exp(R e^{2\pi i t}) \right| = \left| \exp[R(\cos(2\pi t) + i \sin(2\pi t))] \right| = \exp[R \cos(2\pi t)]$$

and then by (2.20) we get for $B, C \in \mathcal{B}(H)$ that

$$\begin{aligned}
 (5.1) \quad |\langle C^* A \exp(A) B x, y \rangle| & \leq \frac{R}{R - \|A\|} \int_0^1 \exp[R \cos(2\pi t)] dt \\
 & \quad \times \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1-\alpha} C \|^2 y, y \right\rangle^{1/2}
 \end{aligned}$$

for $x, y \in H$

The modified Bessel function of the first kind $I_\nu(z)$ for real number ν can be defined by the power series as [2, p. 376]

$$I_\nu(z) = \left(\frac{1}{2} z \right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4} z^2 \right)^k}{k! \Gamma(\nu + k + 1)},$$

where Γ is the gamma function. For $n = 0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4} z^2 \right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(\nu \theta) d\theta - \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-z \cosh t - \nu t} dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(n\theta) d\theta.$$

For $n = 0$ we have

$$I_0(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} d\theta.$$

If we change the variable $\theta = 2\pi t$, then $dt = \frac{1}{2\pi} d\theta$ and

$$\begin{aligned} \int_0^1 \exp[R \cos(2\pi t)] dt &= \frac{1}{2\pi} \int_0^{2\pi} \exp[R \cos \theta] d\theta \\ &= \frac{1}{2} \left(\frac{1}{\pi} \int_0^\pi \exp[R \cos \theta] d\theta + \frac{1}{\pi} \int_\pi^{2\pi} \exp[R \cos \theta] d\theta \right) \\ &= \frac{1}{2} (I_0(R) + I_0(-R)) = I_0(R). \end{aligned}$$

From (5.1) we then get

$$(5.2) \quad |\langle C^* A \exp(A) Bx, y \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \times \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1-\alpha} C \|^2 y, y \right\rangle^{1/2}$$

for $\alpha \in [0, 1]$, $x, y \in H$, $A, B, C \in \mathcal{B}(H)$ with $\|A\| < R$.

By taking $B = C = I$ in (5.2) we get for $\|A\| < R$ that

$$(5.3) \quad |\langle A \exp(A) x, y \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \left\langle |A|^{2\alpha} x, x \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle^{1/2}$$

for $x, y \in H$. In particular,

$$(5.4) \quad |\langle A \exp(A) x, x \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \langle |A| x, x \rangle^{1/2} \langle |A^*| y, y \rangle^{1/2}$$

for $x, y \in H$.

If A is invertible and take $C = I$, $B = A^{-1}$ in (5.2), then we get for $\|A\| < R$ that

$$(5.5) \quad |\langle \exp(A) x, y \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \left\langle |A|^{-2(1-\alpha)} x, x \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle^{1/2}$$

for $x, y \in H$. In particular,

$$(5.6) \quad |\langle \exp(A) x, x \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \left\langle |A|^{-1} x, x \right\rangle^{1/2} \langle |A^*| y, y \rangle^{1/2}$$

for $x, y \in H$.

If $0 < A < R$ and we take $B = A^{-\beta}$, $C = A^{-1+\beta}$, $\alpha, \beta \in [0, 1]$, then by (5.2) we derive

$$(5.7) \quad |\langle \exp(A) x, y \rangle| \leq \frac{RI_0(R)}{R - \|A\|} \left\langle A^{2(\alpha-\beta)} x, x \right\rangle^{1/2} \left\langle A^{2(\beta-\alpha)} y, y \right\rangle^{1/2}$$

for $x, y \in H$.

By Theorem 4 we get the norm inequality

$$(5.8) \quad \|C^* A \exp(A) B\| \leq \frac{RI_0(R)}{R - \|A\|} \| |A|^\alpha B \| \| |A^*|^{1-\alpha} C \|^2.$$

We also have the numerical radius inequalities

$$(5.9) \quad \omega(C^* A \exp(A) B) \leq \frac{1}{2} \frac{RI_0(R)}{R - \|A\|} \left(\| |A|^\alpha B \|^2 + \| |A^*|^{1-\alpha} C \|^2 \right)$$

and

$$(5.10) \quad \begin{aligned} \omega^2(C^* A \exp(A) B) &\leq \frac{1}{2} \left(\frac{RI_0(R)}{R - \|A\|} \right)^2 \\ &\times \left[\| |A|^\alpha B \|^2 \| |A^*|^{1-\alpha} C \|^2 + \omega \left(\| |A^*|^{1-\alpha} C \|^2 \| |A|^\alpha B \|^2 \right) \right]. \end{aligned}$$

Let $r \geq 1/2$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 1$. If $B, C \in \mathcal{B}(H)$ with $|A|^\alpha B \in \mathcal{B}_{2pr}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2qr}(H)$ for $\alpha \in [0, 1]$, then $C^* A \exp(A) B \in \mathcal{B}_{2r}(H)$ and by (4.1)

$$(5.11) \quad \|C^* A \exp(A) B\|_{2r} \leq \frac{RI_0(R)}{R - \|A\|} \| |A|^\alpha B \|_{2pr} \| |A^*|^{1-\alpha} C \|_{2qr}.$$

If $r \geq 1/2$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $pr, qr \geq 1$ and $\| |A|^\alpha B \|^{2pr}, \| |A^*|^{1-\alpha} C \|^{2qr} \in \mathcal{B}_1(H)$, then $C^* A \exp(A) B \in \mathcal{B}_{2r}(H)$ and by (4.17)

$$(5.12) \quad \begin{aligned} \omega_{2r}^{2r}(C^* A \exp(A) B) &\leq \left(\frac{RI_0(R)}{R - \|A\|} \right)^{2r} \operatorname{tr} \left(\frac{1}{p} \| |A|^\alpha B \|^{2pr} + \frac{1}{q} \| |A^*|^{1-\alpha} C \|^{2qr} \right). \end{aligned}$$

By using the power series

$$f(z) := \ln(1 - z)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n} z^n$$

that is convergent on open disk $D(0, 1)$, we can define for all elements A in $\mathcal{B}(H)$ with $\|A\| < 1$,

$$\ln(I - A)^{-1} := \sum_{n=1}^{\infty} \frac{1}{n} A^n.$$

We observe that for $|z| < 1$

$$\left| \ln(1 - z)^{-1} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n} |z|^n = \ln(1 - |z|)^{-1}.$$

Now if we assume that $A, B, C \in \mathcal{B}(H)$ and $\|A\| < R < 1$, then by (2.22) we get

$$(5.13) \quad \begin{aligned} \left| \left\langle C^* A \ln(1 - A)^{-1} Bx, y \right\rangle \right| &\leq \frac{R \ln(1 - R)^{-1}}{R - \|A\|} \\ &\times \left\langle \| |A|^\alpha B \|^2 x, x \right\rangle^{1/2} \left\langle \| |A^*|^{1-\alpha} C \|^2 y, y \right\rangle^{1/2} \end{aligned}$$

for $\alpha \in [0, 1]$, $x, y \in H$.

By taking $B = C = I$ in (5.13) we get for $\|A\| < R < 1$ that

$$(5.14) \quad \begin{aligned} & \left| \left\langle A \ln(1 - A)^{-1} x, y \right\rangle \right| \\ & \leq \frac{R \ln(1 - R)^{-1}}{R - \|A\|} \left\langle |A|^{2\alpha} x, x \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle^{1/2} \end{aligned}$$

for $x, y \in H$. In particular,

$$(5.15) \quad \left| \left\langle A \ln(1 - A)^{-1} x, y \right\rangle \right| \leq \frac{R \ln(1 - R)^{-1}}{R - \|A\|} \langle |A| x, x \rangle^{1/2} \langle |A^*| y, y \rangle^{1/2}$$

for $x, y \in H$.

If A is invertible and take $C = I$, $B = A^{-1}$ in (5.13), then we get for $\|A\| < R < 1$ that

$$(5.16) \quad \begin{aligned} & \left| \left\langle \ln(1 - A)^{-1} x, y \right\rangle \right| \\ & \leq \frac{R \ln(1 - R)^{-1}}{R - \|A\|} \left\langle |A|^{-2(1-\alpha)} x, x \right\rangle^{1/2} \left\langle |A^*|^{2(1-\alpha)} y, y \right\rangle^{1/2} \end{aligned}$$

for $x, y \in H$. In particular,

$$(5.17) \quad \left| \left\langle \ln(1 - A)^{-1} x, y \right\rangle \right| \leq \frac{R \ln(1 - R)^{-1}}{R - \|A\|} \left\langle |A|^{-1} x, x \right\rangle^{1/2} \langle |A^*| y, y \rangle^{1/2}.$$

If $0 < A < R < 1$ and we take $B = A^{-\beta}$, $C = A^{-1+\beta}$, $\beta \in [0, 1]$ in (5.13), then we derive

$$(5.18) \quad \begin{aligned} & \left| \left\langle \ln(1 - A)^{-1} x, y \right\rangle \right| \\ & \leq \frac{R \ln(1 - R)^{-1}}{R - \|A\|} \left\langle A^{2(\alpha-\beta)} x, x \right\rangle^{1/2} \left\langle A^{2(\beta-\alpha)} y, y \right\rangle^{1/2} \end{aligned}$$

for $\alpha \in [0, 1]$, $x, y \in H$.

One can state some norm, numerical radius and p -Schatten norm inequalities for $A \ln(1 - A)^{-1}$, however the details are omitted.

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