SOME REFINEMENT OF HOLDER'S AND ITS REVERSE INEQUALITY

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Abstract

Holder's inequality, its refinement, and reverse have received considerable attention in the theory of mathematical analysis and differential equations. In this paper, we give some refinements of Holder's inequality and its reverse using a simple analytical technique of algebra and calculus. Our results show many results related to holder's inequality as special cases of the inequalities presented.

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1. Introduction

Both the holder's inequality and Cauchy play an important role in many areas of mathematics. Several authors have studied and obtained the generalization, refinement, sharpening, variation, and application of this inequality in the literature.

A family of inequalities concerning inner products of vectors and functions began with Cauchy [16]. The extension and generalizations later led to inequalities of Schwarz, Minkowski, and Holder. Inequalities appear frequently in algebra, geometry, and analysis; they are powerful mathematical tools that appear across different areas of mathematics, helping mathematicians and scientists describe relationships, establish limits and bounds, and solve a wide variety of problems. Many researchers have worked on generalization of Holder, its reverse, and refinement (see for example [1-15]. The study is of great importance in Mathematical analysis, information theory, theory of elasticity, and others. In order to prove the main results, we need the following lemma.

2. Lemmas

The following two lemmas will be needed throughout the proof of our theorems.

Lemma 2.1 Let $a, b \geq 1 \, and \; \lambda \in (0,1)$ we have

$$s\left(\sqrt{a}-\sqrt{b}\right)^2 + A(\lambda)\log^2\left(\frac{a}{b}\right) \leq \lambda a \quad + \quad (1-\lambda)b - a^{\lambda}b^{1-\lambda}$$

$$\leq (1-s)\left(\sqrt{a}-\sqrt{b}\right)^2 + B(\lambda)\log^2\left(\frac{a}{b}\right),$$

where $s = \min\{\lambda, 1-\lambda\},\ A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{s}{4} \; and \; B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-s}{4}$

Lemma 2.2 Let $0 < a, b \leq 1, \; and \; \lambda \in (0,1)$ we have

$$s\left(\sqrt{a}-\sqrt{b}\right)^2 + A(\lambda)ab\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)\,b - a^{\lambda}b^{1-\lambda}$$

$$\leq (1-s)\left(\sqrt{a}-\sqrt{b}\right)^2 + B(\lambda)ab\log^2\left(\frac{a}{b}\right),$$

where $s, A(\lambda), B(\lambda)$ are given in lemma 2.1

3. Main Results

Theorem 2.1. Let $1 < p < \infty, 1 < q < \infty, 1 \leq r < \infty, with \; \frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If f and g are two positive functions which admits integral on [a,b] for which there exist $\int_a^b f^p(x)dx \; and \int_a^b g^q(x)dx$ finite with $\int_a^b f^p(x)dx > 0, \int_a^b g^q(x)dx$ and

$$1 < \frac{f^p(x)}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q(x)} \leq M, \qquad \forall \;\; x \in [a,b].$$

Then, we have

$$1-\frac{\int_a^b f(x)\left(g(x)\right)^{q\left(1-1/p\right)}dx}{\left(\int_a^b f^p(x)dx\right)^{1/p}\left(\int_a^b g^q(x)dx\right)^{1-1/p}} \le \frac{2}{\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\left[1-\frac{\int_a^b f^{p/2}(x)g^{q/2}(x)dx}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)^{1/2}}\right]$$

$$+\left(\frac{p-1}{2p^2}-\frac{1}{4\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\log^2(M) \tag{3.01}$$

Proof: From lemma 2.1, let $b=1,\ \lambda=\frac{1}{p}, 1-\lambda=1-\frac{1}{p}$, we have

$$\frac{1}{p}a+\left(1-\frac{1}{p}\right)-a^{1/p} \le \left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\left(\sqrt{a}-1\right)^2+\left[\frac{p-1}{2p^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\right]\log^2 M$$

(3.02)

Substituting $a=\frac{f^p(x)}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q(x)}>1$ into (3.02) , we get

$$\frac{1}{p}\frac{f^{p}(x)}{\int_a^b f^{p}(x)dx}\frac{\int_a^b g^{q}(x)dx}{g^{q}(x)} + \left(1-\frac{1}{p}\right) - \left(\frac{f^{p}(x)}{\int_a^b f^{p}(x)dx}\frac{\int_a^b g^{q}(x)dx}{g^{q}(x)}\right)^{1/p}$$

$$\leq \left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\left[\frac{f^{p}(x)}{\int_a^b f^{p}(x)dx}.\frac{\int_a^b g^{q}(x)dx}{g^{q}(x)} - 2\frac{f^{p/2}(x)}{\left(\int_a^b f^{p}(x)dx\right)^{1/2}}\frac{\left(\int_a^b g^{q}(x)dx\right)^{1/2}}{g^{q/2}(x)} + 1\right]$$

$$+\left[\frac{p-1}{2p^{2}} - \frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\right]\log^{2}(M) \tag{3.03}$$

Simplifying (3.03) completely, we have

$$\frac{1}{p}\frac{f^{p}(x)}{\int_a^b f^{p}(x)dx} + \frac{g^{q}(x)}{\int_a^b g^{q}(x)dx} - \frac{1}{p}\frac{g^{q}(x)}{\int_a^b g^{q}(x)dx} - \frac{f(x)}{\left(\int_a^b f^{p}(x)dx\right)^{1-1/p}}$$

$$\leq \left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\left[\frac{f^{p}(x)}{\int_a^b f^{p}(x)dx} - 2\frac{f^{p/2}(x)g^{q/2}(x)}{\left(\int_a^b f^{p}(x)dx\right)^{1/2}\left(\int_a^b g^{q}(x)dx\right)^{1/2}} + \frac{g^{q}(x)}{\int_a^b g^{q}(x)dx}\right]$$

$$+\left[\frac{p-1}{2p^{2}} - \frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\right]\log^{2}(M)\frac{g^{q}(x)}{\int_a^b g^{q}(x)dx} \tag{3.04}$$

By integrating inequality (3.04), we obtain

$$\frac{1}{p}+1-\frac{1}{p}-\frac{\int_a^b f(x)\left(g(x)\right)^{2\left(1-\frac{1}{p}\right)}dx}{\left(\int_a^b f^p(x)dx\right)^{1/p}\left(\int_a^b g^q(x)dx\right)^{1-\frac{1}{p}}} \leq \left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\left[2-2\frac{\int_a^b f^{p/2}(x)\, g^{q/2}(x)dx}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)}\right]$$

$$+\left[\frac{p-1}{2p^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\right]\log^2(M) \qquad (3.05)$$

Using the fact that, $1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}=\frac{1}{\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}$, *in* (3.5) *to get*

(3.06)

$$1-\frac{\int_a^b f(x)\left(g(x)\right)^{q\left(1-\frac{1}{p}\right)}dx}{\left(\int_a^b f^p(x)dx\right)^{1/p}\left(\int_a^b g^q(x)dx\right)^{1-1/p}} \leq \frac{2}{\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\left[1-\frac{\int_a^b f^{p/2}(x)\, g^{q/2}(x)dx}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)^{1/2}}\right]$$

$$+\left(\frac{p-1}{2p^2}-\frac{1}{4\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\log^2(M) \qquad (3.07)$$

This complete the proof

Theorem 2.2: Let $1<p<\infty,\ 1<q<\infty, 1\leq r<\infty, with\ \frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If f and g are two positive functions which admits integral on [a,b] for which there exit $\int_a^b f^p(x)dx\ and\ \int_a^b g^q(x)dx$ finite with $\int_a^b f^p(x)dx>0, \int_a^b g^q(x)dx>0$ and

$$m < \frac{f^{p}(x)}{\int_a^b f^{p}(x)dx}\frac{\int_a^b g^{q}(x)dx}{g^{q}(x)} < 1, \qquad \forall\ x \in [a,b].$$

Then we have

$$1-\frac{\int_a^b f(x)(g(x))^{q\left(1-\frac{1}{p}\right)}dx}{\left(\int_a^b f^{p}(x)dx\right)^{1/p}\left(\int_a^b g^{q}(x)dx\right)^{1-\frac{1}{p}}} \le \frac{2}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\left[1-\frac{\int_a^b f^{p/2}(x)g^{q/2}(x)dx}{\left(\int_a^b f^{p}(x)dx\right)^{1/2}\left(\int_a^b g^{q}(x)dx\right)^{1/2}}\right]$$

$$+\left(\frac{q-1}{2q^2}-\frac{1}{4\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\log^{2}\left(\frac{1}{m}\right) \tag{3.08}$$

Proof: From lemma 2.2 let $\lambda = \frac{1}{q}, 1-\lambda = 1-\frac{1}{q}$ *and* $a = 1$ *we get*

$$\frac{1}{q}+\left(1-\frac{1}{q}\right)b-b^{1/p} \le \left[1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right]\left(1-\sqrt{b}\right)^2+\left[\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right]\log^{2}\left(\frac{1}{m}\right) \tag{3.09}$$

Substituting $b = \frac{f^{p}(x)}{\int_a^b f^{p}(x)dx}\frac{\int_a^b g^{q}(x)dx}{g^{q}(x)} > 1$ into (3.09) we get

$$\frac{1}{q}+\left(1-\frac{1}{q}\right)\left(\frac{f^{p}(x)}{\int_a^b f^{p}(x)}\cdot\frac{\int_a^b g^{q}(x)dx}{g^{q}(x)}\right)-\left(\frac{f^{p}(x)}{\int_a^b f^{p}(x)dx}\cdot\frac{\int_a^b g^{q}(x)dx}{g^{q}(x)}\right)^{1/p}$$

$$\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left[1-\left(\frac{f^{p}(x)}{\int_a^b f^{p}(x)dx}\cdot\frac{\int_a^b g^{q}(x)dx}{g^{q}(c)}\right)^{1/2}\right]^2$$

$$+\left[\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right]\log^{2}\left(\frac{1}{m}\right)$$

(3.10)

Simplifying equation (3.10) to obtain

$$\frac{1}{q}\frac{g^q(x)}{\int_a^b g^q(x)dx}+\frac{f^p(x)}{\int_a^b f^p(x)dx}-\frac{1}{q}\frac{f^p(x)}{\int_a^b f^p(x)dx}-\frac{f(x)}{\left(\int_a^b f^p(x)dx\right)^{1/p}}\cdot\frac{\{g(x)\}^{q\left(1-\frac{1}{p}\right)}}{\left(\int_a^b g^q(x)dx\right)^{1-\frac{1}{p}}}$$

$$\leq\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left[\frac{g^q(x)}{\int_a^b g^q(x)dx}-\frac{2f^{p/2}(x)\,g^{q/2}(x)}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)^{1/2}}+\frac{f^p(x)}{\int_a^b f^p(x)dx}\right]$$

$$+\frac{g^q(x)}{\int_a^b g^q(x)dx}\left[\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right]\log^2\left(\frac{1}{m}\right)$$

(3.11)

On integrating inequality (3.11),then (3.11) becomes

$$\frac{1}{q}+1-\frac{1}{q}-\frac{\int_a^b f(x)\,[g(x)]^{2\left(1-\frac{1}{p}\right)}dx}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)^{1-\frac{1}{p}}}\leq\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left[1-\frac{2f^{p/2}(x)\,g^{q/2}(x)dx}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)^{1/2}}+1\right]$$

$$+\left[\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right]\log^2\left(\frac{1}{m}\right)$$

(3.12)

Using the fact that, $1-\frac{1}{\max\left[\frac{1}{q},\frac{q-1}{q}\right]}=\frac{1}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}$, in (2.12) we have (3.13)

$$1-\frac{\int_a^b f(x)\left(g(x)\right)^{q\left(1-\frac{1}{p}\right)}dx}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)^{1-\frac{1}{p}}}\le\frac{2}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\left[1-\frac{f^{p/2}(x)g^{q/2}(x)}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)^{1/2}}\right]$$

$$+\left[\frac{q-1}{2q^2}-\frac{1}{4\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right]\log^2\left(\frac{1}{m}\right) \qquad (3.14)$$

This completes the proof.

Theorem 2.3. Let $1<p<\infty, 1<q<\infty, 1\le r<\infty, with\, \frac{1}{p}+\frac{1}{q}=\frac{1}{r}.$ If *f* and *g* are two positive functions which admit integral on [a, b] for which there exist

$\int_a^b f^p(x)dx \; and \int_a^b g^q(x)dx \; finite\, with \int_a^b f^p(x)dx>0, and \int_a^b g^q(x)dx>0$

$$m<\frac{f^p(x)}{\int_a^b f^p(x)dx}\;\frac{\int_a^b g^q(x)dx}{g^q(x)}<1$$

$$1-\frac{\int_a^b f(x)(g(x)^{q\left(1-1/p\right)}dx}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)^{1-1/p}}\le\frac{2}{\min\left(\frac{1}{q},\frac{q-1}{q}\right)}\left[1-\frac{2f^{p/2}(x)g^{q/2}(x)dx}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)^{1/2}}\right]$$

$$+\left(\frac{q-1}{2q^2}-\frac{1}{4\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\log^2\left(\frac{1}{m}\right) \qquad (3.15)$$

Proof: From lemma 2.2 let $\lambda=\frac{1}{q}, 1-\lambda=1-\frac{1}{q}\; and\; a=1\; we\; have$

$$\frac{1}{q}+\left(1-\frac{1}{q}\right)b-b^{1/p}\leq\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left(1-\sqrt{b}\right)^2+\left(\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right)\log^2\left(\frac{1}{m}\right)$$

Substituting $b=\frac{f^p(x)}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q(x)}>1$ into (3.15) we get

$$\frac{1}{q}+\left(1-\frac{1}{q}\right)\frac{f^p(x)}{\int_a^p f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q(x)}-\left(\frac{f^p(x)}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q(x)}\right)^{1/p}$$

$$\leq\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left(1-\left(\frac{f^p(x)}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q(x)}\right)^{1/2}\right)^2+\left(\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)\right)\log^2\left(\frac{1}{m}\right)$$

(3.16)

Simplifying (3.16) completely to have

$$\left(\frac{1}{q`}\frac{g^q(x)}{\int_a^b g^q(x)dx}+\frac{f^p(x)}{\int_a^b f^p(x)dx}-\frac{1}{q}\frac{f^p(x)}{\int_a^b f^p(x)dx}-\frac{f(x)}{\left(\int_a^b f^p(x)dx\right)^{1/p}}\frac{(g(x))^{q(1-1/p)}}{\left(\int_a^b g^q(x)dx\right)^{1-1/p}}\right)$$

$$\leq\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left[\frac{g^{q}(x)}{\int_a^b g^{q}(x)dx}-\frac{2f^{p/2}(x)g^{q/2}(x)}{\left(\int_a^b f^{p}(x)dx\right)^{1/2}\left(\int_a^b g^{q}(x)dx\right)^{1/2}}+\frac{f^{p}(x)}{\int_a^b f^{p}(x)dx}\right]$$

$$+\frac{g^{q}(x)}{\int_a^b g^{q}(x)dx}\left(\frac{q-1}{2q^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{q},\frac{q-1}{q}\right)}\right)\right)\log^{2}\left(\frac{1}{m}\right) \qquad (3.16)$$

On integrating equation (3.16) with respect to x, we get

$$\frac{1}{q}+1-\frac{1}{q}-\frac{\int_a^b f(x)(g(x))^{q(1-1/p)}dx}{\left(\int_a^b f^{p}(x)dx\right)^{1/2}\left(\int_a^b g^{q}(x)dx\right)^{1-1/p}}\leq\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)$$

$$\left[1-\frac{2f^{p/2}(x)g^{q/2}(x)dx}{\left(\int_a^b f^{p}(x)dx\right)^{1/2}\left(\int_a^b g^{q}(x)dx\right)^{1/2}}+1\right]+\left(\frac{q-1}{2q^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right)\log^{2}\left(\frac{1}{m}\right) \qquad (3.17)$$

Using the fact that, $1-\frac{1}{\max\left[\frac{1}{q},\frac{q-1}{q}\right]}=\frac{1}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}$, in (3.17) we have (3.18)

$$1-\frac{\int_a^b f(x)(g(x))^{q\left(1-1/p\right)}dx}{\left(\int_a^b f^{p}(x)dx\right)^{1/2}\left(\int_a^b g^{q}(x)dx\right)^{1-1/p}}\leq\frac{2}{\min\left(\frac{1}{q},\frac{q-1}{q}\right)}\left[1-\frac{2f^{p/2}(x)g^{q/2}(x)dx}{\left(\int_a^b f^{p}(x)dx\right)^{1/2}\left(\int_a^b g^{q}(x)dx\right)^{1/2}}\right]$$

$$+\left(\frac{q-1}{2q^2}-\frac{1}{4\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\log^2\left(\frac{1}{m}\right) \tag{3.19}$$

This completes the proof.

Theorem 2.4. Let $1<p<\infty, 1<q<\infty, 1\le r<\infty$ *with* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If f and g are two positive functions which admit integral on [a,b] for which there exist $\int_a^b f^p(x)dx$ and $\int_a^b g^q(x)dx$ are finite with $\int_a^b f^p(x)dx>0, \int_a^b g^q(x)dx>0,$ *then*

$$m<k\frac{f^p(x)}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q(x)}<1 \quad x\in[a,b], k>0$$

Proof: Taking lemma 2.1,let $a=1, \lambda=\frac{1}{q}, 1-\lambda=1-\frac{1}{q}$ we have

$$\frac{1}{q}+\left(1-\frac{1}{q}\right)b-b^{1/p}\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{1}\right\}}\right)\left(1-\sqrt{b}\right)^2+\left(\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right)\log^2\left(\frac{1}{m}\right)$$

(3.20)

Substituting $b=\frac{f^p(x)}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q(x)}>1$ into (3.20) we get

$$\frac{1}{q}+\left(1-\frac{1}{q}\right)\ \frac{kf^{p}(x)}{\int_a^b f^{p}(x)dx}\frac{\int_a^b g^{q}(x)dx}{g^{q}(x)}-\left(\frac{k\ f^{p}(x)}{\int_a^b f^{p}(x)dx}\frac{\int_a^b g^{q}(x)dx}{g^{q}(x)}\right)^{1/p}$$

$$\leq\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left(1-\left(\frac{kf^{p}(x)}{\int_a^b f^{p}(x)dx}\frac{\int_a^b g^{q}(x)}{g^{q}(x)}\right)^{1/2}\right)^{2}+\left(\frac{q-1}{2q^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right)\log^{2}\left(\frac{1}{m}\right)$$

(3.21)

Simplifying equation(3.21) completely to have

$$\left[\frac{1}{q}\frac{g^{q}(x)}{\int_a^b g^{q}(x)dx}+\frac{k\ f^{p}(x)}{\int_a^b f^{p}(x)dx}-\frac{1}{q}\frac{f^{p}(x)}{\int_a^b f^{p}(x)dx}-\frac{\sqrt[p]{k}\quad f(x)}{\left(\int_a^b f^{p}(x)dx\right)^{1/p}}\frac{\left(g(x)\right)^{q\left(1-\frac{1}{p}\right)}}{\left(\int_a^b g^{q}(x)dx\right)^{1-1/p}}\right]$$

$$\leq\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left(\frac{g^{q}(x)}{\int_a^b g^{q}(x)dx}-\frac{2\sqrt{k}\,f^{p/2}(x)}{\left(\int_a^b f^{p}(x)dx\right)^{1/2}}\frac{g^{q/2}(x)}{\left(\int_a^b g^{q}(x)dx\right)^{1/2}}+\frac{g^{q}(x)}{\int_a^b g^{q}(x)dx}\right)$$

$$+\frac{g^{q}(x)}{\int_a^b g^{q}(x)dx}\left[\frac{q-1}{2q^{2}}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{q},\frac{q-1}{q}\right)}\right)\right]\log^{2}\left(\frac{1}{m}\right) \qquad (3.22)$$

On integrating equation (3.22) with respect to *x*, we have

$$\frac{1}{q}+k-\frac{1}{q}-\frac{\sqrt[p]{k}\quad f(x)}{\left(\int_a^b f^p(x)dx\right)^{1/p}}\frac{\left(g(x)\right)^{q(1-\frac{1}{p})}dx}{\left(\int_a^b g^q(x)dx\right)^{1-1/p}}\le\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\left[1-\frac{2\sqrt{k}\ f^{p/2}(x)g^{q/2}(x)}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)^{1/2}}+1\right]$$

$$+\left(\frac{q-1}{2q^2}-\frac{1}{4}\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\right)\log^2\left(\frac{1}{m}\right) \tag{3.23}$$

Using the fact that $\left(1-\frac{1}{\max\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)=\frac{1}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}$ in equation (3.23) we have

$$k-\frac{\sqrt[p]{k}\quad f(x)}{\left(\int_a^b f^p(x)dx\right)^{1/p}}\frac{\left(g(x)\right)^{q\left(1-1/p\right)}dx}{\left(\int_a^b g^q(x)dx\right)^{1-\frac{1}{p}}}\le\frac{2}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\left[1-\frac{\sqrt{k}\ f^{p/2}(x)g^{q/2}(x)dx}{\left(\int_a^b f^p(x)dx\right)^{1/2}\left(\int_a^b g^q(x)dx\right)^{1/2}}\right]$$

$$+\left(\frac{q-1}{2q^2}-\frac{1}{\min\left\{\frac{1}{q},\frac{q-1}{q}\right\}}\right)\log^2\left(\frac{1}{m}\right) \tag{3.24}$$

This completes the proof.

Theorem 2.5. Let $1<p<\infty, 1<q<\infty, 1\le r<\infty, with\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If f and g are two positive function $f\in L^P$, $g\in L^q$ $with\ \|f\|_p>0, \|g\|>0$ for which there exist

$$1<\frac{f^p}{\|f\|_p^p}\frac{\|g\|_q^q}{g^q}\le M, \qquad \forall\ x\in[a,b], M>0$$

Proof: Taking in theorem 2.2, $b=1, \lambda=\frac{1}{p}, 1-\lambda=1-\frac{1}{p}$, we will obtain

$$\frac{1}{p}a+\left(1-\frac{1}{p}\right)-a^{1/p} \le \left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\left(\sqrt{a}-1\right)^2+\left(\frac{p-1}{2p^2}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\right)\log M$$

(3.25)

Putting $a=\frac{f^p}{\|f\|_p^p}\cdot\frac{\|g\|_q^q}{g^q}>1$ we will have

$$\frac{1}{p}\cdot\frac{f^p}{\|f\|}\cdot\frac{\|g\|_q^q}{g^q}+\left(1-\frac{1}{p}\right)-\left(\frac{f^p}{\|f\|_p^p}\cdot\frac{\|g\|_q^q}{g^q}\right)^{1/p} \le \left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\left[\left(\frac{f^p}{\|f\|_p^p}\cdot\frac{\|g\|_q^q}{g^q}\right)^{1/2}-1\right]^2$$

$$+\left(\frac{p-1}{2p^2}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\right)\log^2(M) \qquad (3.26)$$

Simplifying (3.26) completely we get

$$\left(\frac{1}{p}\frac{f^p}{\|f\|_P^P}+\frac{g^q}{\|g\|_q^q}-\frac{1}{p}\frac{g^q}{\|g\|_q^q}-\frac{f}{\left(\|f\|_p\right)}\cdot\frac{g^{q\left(1-1/p\right)}}{\left(\|g\|_q^q\right)^{1-1/p}}\right) \le \left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\left(\frac{f^p}{\|f\|_p^p}-\frac{2f^{p/2}g^{q/2}}{\|f\|_p^{p/2}\|g\|_g^{q/2}}+\frac{g^q}{\|g\|_q^q}\right)$$

$$+\left[\frac{p-1}{2p^2}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\right]\log^2(M) \qquad (3.27)$$

On Integrating both sides we have;

$$\frac{1}{p}+1-\frac{1}{p}-\frac{f\ \ g^{q\left(1-\frac{1}{p}\right)}}{\|f\|_p\left(\|g\|_q^q\right)^{1-1/p}} \le \left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\left[1-\frac{2\int_\Omega f^{p/2} g^{q/2}\,d\mu}{\|f\|_p^{p/2}\,\|g\|_q^{q/2}}+1\right]$$

$$+\left(\frac{p-1}{2p^2}-\frac{1}{4}\left(1-\frac{1}{\max\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\right)\log^2(M) \tag{3.28}$$

Using the fact that $\left(1-\frac{1}{\max\left\{\frac{1}{p},\frac{p-1}{p}\right\}}\right)=\frac{1}{\min\left\{\frac{1}{p},\frac{p-1}{p}\right\}}$ in equation (3.28) we have

$$1-\frac{f\ \ g^{q\left(1-1/p\right)}}{\|f\|_p\left(\|g\|_q^q\right)^{1-1/p}} \le \frac{2}{\min\left(\frac{1}{p},\frac{p-1}{p}\right)}\left[1-\frac{\int_\Omega f^{p/2} g^{q/2}\,d\mu}{\|f\|_p^{q/2}\,\|g\|_q^{q/2}}\right]+\left(\frac{p-1}{2p^2}-\frac{1}{4\min\left(\frac{1}{p},\frac{p-1}{p}\right)}\right)\log^2(M)$$

.

This complete the proof.

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