REVERSE INEQUALITIES FOR CONVEX FUNCTIONS WITH APPLICATIONS TO NORMS AND SEMI-INNER PRODUCTS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we provide several upper bounds for

$$0 \le \frac{1}{2}\left[(b-a) f(x)+(b-x) f(b)+(x-a) f(a)\right]-\int_a^b f(t)\, dt$$

in the case of convex functions f on $[a, b]$ with $x \in [a, b]$ and, in particular,

$$0 \le \frac{1}{2}(b-a)\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\int_a^b f(t)\, dt.$$

Applications for power of norms and semi-inner products in normed spaces are also provided. The particular case of inner product space $(H, \langle\cdot, \cdot\rangle)$ is considered as well. In this case we showed that

$$\begin{aligned} 0 & \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p+\frac{\|y\|^p+\|x\|^p}{2}\right]-\int_0^1\|(1-t) x+t y\|^p dt \\ & \le \frac{p}{32}\left[\left\langle y-x,\|y\|^{p-2} y-\|x\|^{p-2} x\right\rangle\right] \end{aligned}$$

for any x, $y \in H$ whenever $p \ge 2$; otherwise, for $1 \le p < 2$, it holds for x, $y \neq 0$.

1. Introduction

Let X be a real linear space, $x, y \in X$, $x \neq y$ and let $[x, y] := \{(1-\lambda) x+\lambda y,\ \lambda \in[0,1]\}$ be the *segment* generated by x and y. We consider the function $f : [x, y] \to \mathbb{R}$ and the attached function $g(x, y) : [0,1] \to \mathbb{R}$,

$$g(x, y)(t) := f[(1-t) x+t y], t \in[0,1].$$

It is well known that f is convex on $[x, y]$ iff $g(x, y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g_{\pm}'(x, y)(s)=\left(\nabla_{\pm} f[(1-s) x+s y]\right)(y-x)$, $s \in[0,1)$;
(ii) $g_{+}'(x, y)(0)=\left(\nabla_{+} f(x)\right)(y-x)$;
(iii) $g_{-}'(x, y)(1)=\left(\nabla_{-} f(y)\right)(y-x)$;

where $\left(\nabla_{\pm} f(x)\right)(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\left(\nabla_{+} f(x)\right)(y) := \lim_{h \to 0+}\left[\frac{f(x+h y)-f(x)}{h}\right],$$
$$\left(\nabla_{-} f(x)\right)(y) := \lim_{k \to 0-}\left[\frac{f(x+k y)-f(x)}{k}\right],$$

for $x, y \in X$.

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x \in X$ is convex and thus the following limits exist

1991 *Mathematics Subject Classification.* 26D15, 46C05.

Key words and phrases. Convex functions, Integral inequalities, Norms, Semi-inner products.

RGMIA Res. Rep. Coll. **26** (2023), Art. 52, 16 pp. Received 05/09/23

(iv) $\langle x,y\rangle_s := (\bigtriangledown_+ f_0(y))(x) = \lim_{t\to 0+}\left[\frac{\|y+tx\|^2-\|y\|^2}{2t}\right]$;

(v) $\langle x,y\rangle_i := (\bigtriangledown_- f_0(y))(x) = \lim_{s\to 0-}\left[\frac{\|y+sx\|^2-\|y\|^2}{2s}\right]$;

for any $x,y\in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [4]), assuming that $p,q\in\{s,i\}$ and $p\neq q$:

(a) $\langle x,x\rangle_p = \|x\|^2$ for all $x\in X$;
(aa) $\langle \alpha x,\beta y\rangle_p = \alpha\beta\langle x,y\rangle_p$ if $\alpha,\beta\geq 0$ and $x,y\in X$;
(aaa) $\left|\langle x,y\rangle_p\right| \leq \|x\|\|y\|$ for all $x,y\in X$;
(av) $\langle \alpha x+y,x\rangle_p = \alpha\langle x,x\rangle_p + \langle y,x\rangle_p$ if $x,y\in X$ and $\alpha\in\mathbb{R}$;
(v) $\langle -x,y\rangle_p = -\langle x,y\rangle_q$ for all $x,y\in X$;
(va) $\langle x+y,z\rangle_p \leq \|x\|\|z\| + \langle y,z\rangle_p$ for all $x,y,z\in X$;
(vaa) The mapping $\langle\cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p=s$ (or $p=i$);
(vaaa) The normed linear space $(X,\|\cdot\|)$ is smooth at the point $x_0\in X\setminus\{0\}$ if and only if $\langle y,x_0\rangle_s = \langle y,x_0\rangle_i$ for all $y\in X$; in general $\langle y,x\rangle_i \leq \langle y,x\rangle_s$ for all $x,y\in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y,x\rangle_i = \langle y,x\rangle = \langle y,x\rangle_s$ for all $x,\ y\in X$.

The function

$$f_p(x) = \|x\|^p,\ p\geq 1$$

is also convex and the following limits exist

$$(\bigtriangledown_\pm f_p(y))(x) := \lim_{h\to 0\pm}\frac{\|y+hx\|^p-\|y\|^p}{h} = p\|y\|^{p-2}\langle x,y\rangle_{s(i)}$$

exist for all $x,\ y\in X$ whenever $p\geq 2$, otherwise they exist for any $x\in X$ and nonzero $y\in X$. In particular, if $p=1$, then

$$(\bigtriangledown_\pm f_1(y))(x) := \lim_{h\to 0\pm}\frac{\|y+hx\|-\|y\|}{h} = \left\langle x,\frac{y}{\|y\|}\right\rangle_{s(i)}$$

exist for $x,\ y\in X$ with $y\neq 0$.

In [6] Kikianty et al. obtained among others the following norm inequalities

$$\begin{aligned}(1.1)\qquad 0 &\leq \frac{1}{8}p\left\|\frac{x+y}{2}\right\|^{p-2}\left[\left\langle y-x,\frac{x+y}{2}\right\rangle_s - \left\langle y-x,\frac{x+y}{2}\right\rangle_i\right]\\ &\leq \frac{\|y\|^p+\|x\|^p}{2} - \int_0^1\|(1-t)x+ty\|^p\,dt\\ &\leq \frac{1}{8}p\left[\|y\|^{p-2}\langle y-x,y\rangle_i - \|x\|^{p-2}\langle y-x,x\rangle_s\right]\end{aligned}$$

and

$$0 \leq \frac{1}{8}p\left\|\frac{x+y}{2}\right\|^{p-2}\left[\left\langle y-x,\frac{x+y}{2}\right\rangle_s-\left\langle y-x,\frac{x+y}{2}\right\rangle_i\right] \tag{1.2}$$
$$\leq \int_0^1 \left\|(1-t)x+ty\right\|^p dt-\left\|\frac{x+y}{2}\right\|^p$$
$$\leq \frac{1}{8}p\left[\|y\|^{p-2}\langle y-x,y\rangle_i-\|x\|^{p-2}\langle y-x,x\rangle_s\right]$$

that hold for any x, $y \in X$ whenever $p \geq 2$; otherwise, they hold for linearly independent x, $y \in X$. The constant $\frac{1}{8}$ is best in both (1.1) and (1.2).

In this paper we provide several upper bounds for

$$0 \leq \frac{1}{2}\left[(b-a)f(x)+(b-x)f(b)+(x-a)f(a)\right]-\int_a^b f(t)\,dt$$

in the case of convex functions f on $[a,b]$ with $x \in [a,b]$ and, in particular,

$$0 \leq \frac{1}{2}(b-a)\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\int_a^b f(t)\,dt.$$

Applications for power of norms and semi-inner products in normed spaces are also provided. The particular case of inner product space is considered as well.

2. Reverse Inequalities for Convex Functions

Our first result for scalar functions is as follows:

Theorem 1. *Assume that f is convex on $[a,b]$, then*

$$0 \leq \frac{1}{2}\left[(b-a)f(x)+(b-x)f(b)+(x-a)f(a)\right]-\int_a^b f(t)\,dt \tag{2.1}$$
$$\leq \frac{1}{4}\left[(x-a)^2\left(f'_-(x)-f'_+(a)\right)+(b-x)^2\left(f'_-(b)-f'_+(x)\right)\right]$$

for all $x \in (a,b)$.

In particular,

$$0 \leq \frac{1}{2}(b-a)\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\int_a^b f(t)\,dt \tag{2.2}$$
$$\leq \frac{1}{16}(b-a)^2\left\{f'_-(b)-f'_+(a)-\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\right\}$$
$$\leq \frac{1}{16}(b-a)^2\left[f'_-(b)-f'_+(a)\right].$$

Proof. Integrating by parts we have the Montgomery identity

$$(b-a)f(x)-\int_a^b f(t)\,dt=\int_a^x (t-a)f'(t)\,dt+\int_x^b (t-b)f'(t)\,dt \tag{2.3}$$

for all $x \in [a,b]$.

We recall Čebyšev's inequality for two non-decreasing (non-increasing) functions g, h on $[c,d]$ which states that

$$\frac{1}{d-c}\int_c^d g(t)h(t)\,dt \geq \frac{1}{d-c}\int_c^d g(t)\,dt\frac{1}{d-c}\int_c^d h(t)\,dt. \tag{2.4}$$

Since f is convex, then $f'_{\pm}(t)$ exist for all $t \in (a,b)$ and $f'(t)$ is equal with $f'_{\pm}(t)$ for a.e. $t \in (a,b)$. By Čebyšev's inequality we have

$$0 \leq \int_a^x (t-a) f'(t)\,dt - \frac{1}{x-a}\int_a^x (t-a)\,dt \int_a^x f'(t)\,dt = \int_a^x (t-a) f'(t)\,dt - \frac{x-a}{2}[f(x)-f(a)] \tag{2.5}$$

and

$$0 \leq \int_x^b (t-b) f'(t)\,dt - \frac{1}{b-x}\int_x^b (t-b)\,dt \int_x^b f'(t)\,dt = \int_x^b (t-b) f'(t)\,dt + \frac{b-x}{2}[f(b)-f(x)] \tag{2.6}$$

for $x \in [a,b]$.

If we add (2.5) with (2.6), then we get by (2.3)

$$\begin{aligned}
0 &\leq \int_a^x (t-a) f'(t)\,dt - \frac{x-a}{2}[f(x)-f(a)] \\
&+ \int_x^b (t-b) f'(t)\,dt + \frac{b-x}{2}[f(b)-f(x)] \\
&= (b-a) f(x) - \int_a^b f(t)\,dt \\
&+ \frac{b-x}{2}[f(b)-f(x)] - \frac{x-a}{2}[f(x)-f(a)] \\
&= (b-a) f(x) - \int_a^b f(t)\,dt - \frac{b-a}{2} f(x) \\
&+ \frac{b-x}{2} f(b) + \frac{x-a}{2} f(a) \\
&= \frac{1}{2}[(b-a) f(x) + (b-x) f(b) + (x-a) f(a)] - \int_a^b f(t)\,dt
\end{aligned}$$

for $x \in [a,b]$, which proves the first part of (2.1).

We also recall Grüss' inequality [8]

$$\left|\frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt - \frac{1}{b-a}\int_a^b g(t)\,dt \frac{1}{b-a}\int_a^b h(t)\,dt\right| \leq \frac{1}{4}(M-m)(N-n), \tag{2.7}$$

that holds for integrable functions g, h such that $m \leq g \leq M$ and $n \leq h \leq N$ on $[a,b]$, where m, M, n, N are constants.

Using (2.7) for $g(t) = t-a$, $h(t) = f'(t)$ on $[a,x]$ we get

$$\begin{aligned}
0 &\leq \int_a^x (t-a) f'(t)\,dt - \frac{1}{x-a}\int_a^x (t-a)\,dt \int_a^x f'(t)\,dt \\
&\leq \frac{1}{4}(x-a)^2 \left(f'_-(x) - f'_+(a)\right)
\end{aligned}$$

while for $g(t) = t - b$, $h(t) = f'(t)$ on $[x, b]$

$$\begin{aligned} 0 &\leq \int_x^b (t-b) f'(t)\,dt - \frac{1}{b-x}\int_x^b (t-b)\,dt \int_x^b f'(t)\,dt \\ &\leq \frac{1}{4}(b-x)^2 \left(f'_-(b) - f'_+(x)\right). \end{aligned}$$

If we add these two inequalities and use (2.3) we deduce the second part of (2.1). □

In fact, we can improve the upper bounds in Theorem 1 and give other bounds as well as follows:

Theorem 2. *Assume that f is convex on $[a,b]$, then*

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[(b-a) f(x) + (b-x) f(b) + (x-a) f(a)\right] - \int_a^b f(t)\,dt \\ &\leq \frac{1}{2}(x-a)\int_a^x \left| f'(t) - \frac{f(x)-f(a)}{x-a}\right| dt \\ &+ \frac{1}{2}(b-x)\int_x^b \left| f'(t) - \frac{f(b)-f(x)}{b-x}\right| dt \end{aligned} \tag{2.8}$$

for all $x \in (a,b)$.

In particular,

$$\begin{aligned} 0 &\leq \frac{1}{2}(b-a)\left[f\left(\frac{a+b}{2}\right) + \frac{f(b)+f(a)}{2}\right] - \int_a^b f(t)\,dt \\ &\leq \frac{1}{4}(b-a)\left[\int_a^{\frac{a+b}{2}} \left| f'(t) - 2\left(\frac{f\left(\frac{a+b}{2}\right) - f(a)}{b-a}\right)\right| dt \right. \\ &\left. + \int_{\frac{a+b}{2}}^b \left| f'(t) - 2\left(\frac{f(b) - f\left(\frac{a+b}{2}\right)}{b-a}\right)\right| dt\right]. \end{aligned} \tag{2.9}$$

Also we have

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[(b-a) f(x) + (b-x) f(b) + (x-a) f(a)\right] - \int_a^b f(t)\,dt \\ &\leq \frac{1}{8}\left[\left(f'_-(x) - f'_+(a)\right)(x-a)^2 + \left(f'_-(b) - f'_+(x)\right)(b-x)^2\right] \end{aligned} \tag{2.10}$$

for all $x \in (a,b)$.

In particular,

$$\begin{aligned} 0 &\leq \frac{1}{2}(b-a)\left[f\left(\frac{a+b}{2}\right) + \frac{f(b)+f(a)}{2}\right] - \int_a^b f(t)\,dt \\ &\leq \frac{1}{32}(b-a)^2\left\{f'_-(b) - f'_+(a) - \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right]\right\} \\ &\leq \frac{1}{32}(b-a)^2\left[f'_-(b) - f'_+(a)\right]. \end{aligned} \tag{2.11}$$

Proof. In [3] Cheng and Sun obtained the following Gruss type inequality

$$\left|\frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt-\frac{1}{b-a}\int_a^b g(t)\,dt\frac{1}{b-a}\int_a^b h(t)\,dt\right| \le \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|dt \tag{2.12}$$

where $m\le g\le M$ on $[a,b]$, where m, M are constants and h is integrable on $[a,b]$.

The constant $1/2$ is best in (2.12) as shown by Cerone and Dragomir in [1] where a general version for Lebesgue integral and measurable spaces was also given.

Using (2.7) for $g(t)=t-a$, $h(t)=f'(t)$ on $[a,x]$ we get

$$\begin{aligned}0 &\le \int_a^x (t-a)\,f'(t)\,dt-\frac{1}{x-a}\int_a^x (t-a)\,dt\int_a^x f'(t)\,dt\\ &\le \frac{1}{2}(x-a)\int_a^x\left|f'(t)-\frac{1}{x-a}\int_a^x f'(s)\,ds\right|dt\\ &= \frac{1}{2}(x-a)\int_a^x\left|f'(t)-\frac{f(x)-f(a)}{x-a}\right|dt\end{aligned}$$

while for $g(t)=t-b$, $h(t)=f'(t)$ on $[x,b]$

$$\begin{aligned}0 &\le \int_x^b (t-b)\,f'(t)\,dt-\frac{1}{b-x}\int_x^b (t-b)\,dt\int_x^b f'(t)\,dt\\ &\le \frac{1}{2}(b-x)\int_x^b\left|f'(t)-\frac{1}{b-x}\int_x^b f'(s)\,ds\right|dt\\ &= \frac{1}{2}(b-x)\int_x^b\left|f'(t)-\frac{f(b)-f(x)}{b-x}\right|dt\end{aligned}$$

If we add these two inequalities and use (2.3) we deduce the second part of (2.1).

Using (2.7) for $h(t)=t-a$, $g(t)=f'(t)$ on $[a,x]$ we get

$$\begin{aligned}0 &\le \int_a^x (t-a)\,f'(t)\,dt-\frac{1}{x-a}\int_a^x (t-a)\,dt\int_a^x f'(t)\,dt\\ &\le \frac{1}{2}\left(f'_-(x)-f'_+(a)\right)\int_a^x\left|t-a-\frac{1}{x-a}\int_a^x (t-a)\,ds\right|dt\\ &= \frac{1}{2}\left(f'_-(x)-f'_+(a)\right)\int_a^x\left|t-\frac{x+a}{2}\right|dt\\ &= \frac{1}{8}\left(f'_-(x)-f'_+(a)\right)(x-a)^2\end{aligned}$$

while for $h(t)=t-b$, $g(t)=f'(t)$ on $[x,b]$

$$\begin{aligned}0&\le\int_x^b(t-b)\,f'(t)\,dt-\frac{1}{b-x}\int_x^b(t-b)\,dt\int_x^b f'(t)\,dt\\&\le\frac{1}{2}\left(f'_-(b)-f'_+(x)\right)\int_x^b\left|t-b-\frac{1}{b-x}\int_x^b(s-b)\,ds\right|dt\\&=\frac{1}{2}\left(f'_-(b)-f'_+(x)\right)\int_x^b\left|t-\frac{x+b}{2}\right|dt\\&=\frac{1}{8}\left(f'_-(b)-f'_+(x)\right)(b-x)^2.\end{aligned}$$

If we add these two inequalities and use (2.3) we deduce the second part of (2.10). □

Remark 1. *The inequalities (2.10) and (2.11) were obtained in a different way in* [5] *where the constants* $\frac{1}{8}$ *and* $\frac{1}{32}$ *have also been established to be best.*

The following inequality obtained by Ostrowski in 1970, [10] is as follows:

$$\left|\frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt-\frac{1}{b-a}\int_a^b g(t)\,dt\frac{1}{b-a}\int_a^b h(t)\,dt\right|\le\frac{1}{8}(b-a)(N-n)\left\|g'\right\|_\infty, \tag{2.13}$$

Now, if we take $g(t)=t-a$, $h(t)=f'(t)$ on $[a,x]$ we get

$$\begin{aligned}0&\le\int_a^x(t-a)\,f'(t)\,dt-\frac{1}{x-a}\int_a^x(t-a)\,dt\int_a^x f'(t)\,dt\\&\le\frac{1}{8}(x-a)^2\left(f'_-(x)-f'_+(a)\right)\end{aligned}$$

while for $g(t)=t-b$, $h(t)=f'(t)$ on $[x,b]$ we get

$$\begin{aligned}0&\le\int_x^b(t-b)\,f'(t)\,dt-\frac{1}{b-x}\int_x^b(t-b)\,dt\int_x^b f'(t)\,dt\\&\le\frac{1}{8}(b-x)^2\left(f'_-(b)-f'_+(x)\right).\end{aligned}$$

If we add these two inequalities and use (2.3) then we also obtain (2.10) having a third different proof of this inequality firstly obtained in [5].

The dual choice $h(t)=t-a$, $g(t)=f'(t)$ on $[a,x]$ gives

$$\begin{aligned}0&\le\int_a^x(t-a)\,f'(t)\,dt-\frac{1}{x-a}\int_a^x(t-a)\,dt\int_a^x f'(t)\,dt\\&\le\frac{1}{8}(x-a)^3\left\|f''(t)\right\|_{\infty,[a,x]},\end{aligned}$$

while for $h(t)=t-b$, $g(t)=f'(t)$ on $[x,b]$ we get

$$\begin{aligned}0&\le\int_x^b(t-b)\,f'(t)\,dt-\frac{1}{b-x}\int_x^b(t-b)\,dt\int_x^b f'(t)\,dt\\&\le\frac{1}{8}(b-x)^3\left\|f''(t)\right\|_{\infty,[x,b]},\end{aligned}$$

which by addition and the use of (2.3) gives for $x \in (a, b)$ that

$$0 \leq \frac{1}{2} [(b-a) f(x) + (b-x) f(b) + (x-a) f(a)] - \int_a^b f(t)\, dt \tag{2.14}$$
$$\leq \frac{1}{8} \left[(x-a)^3 \|f''(t)\|_{\infty,[a,x]} + (b-x)^3 \|f''(t)\|_{\infty,[x,b]} \right]$$
$$\leq \frac{1}{8} \left[(x-a)^3 + (b-x)^3 \right] \|f''(t)\|_{\infty,[a,b]},$$

provided that f is convex and twice differentiable on (a, b)

The constant $\frac{1}{8}$ can be improved as follows:

Theorem 3. *Assume that f is convex and twice differentiable on (a, b) with $L_\infty [a, b]$, then*

$$0 \leq \frac{1}{2} [(b-a) f(x) + (b-x) f(b) + (x-a) f(a)] - \int_a^b f(t)\, dt \tag{2.15}$$
$$\leq \frac{1}{12} \left[(x-a)^3 \|f''(t)\|_{\infty,[a,x]} + (b-x)^3 \|f''(t)\|_{\infty,[x,b]} \right]$$
$$\leq \frac{1}{4} (b-a) \left[\frac{1}{12} (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|f''(t)\|_{\infty,[a,b]},$$

for all $x \in (a, b)$.

In particular,

$$0 \leq \frac{1}{2} (b-a) \left[f\left(\frac{a+b}{2} \right) + \frac{f(b) + f(a)}{2} \right] - \int_a^b f(t)\, dt \tag{2.16}$$
$$\leq \frac{1}{96} \left[\|f''(t)\|_{\infty,\left[a,\frac{a+b}{2}\right]} + \|f''(t)\|_{\infty,\left[\frac{a+b}{2},b\right]} \right] (b-a)^3$$
$$\leq \frac{1}{48} (b-a)^3 \|f''(t)\|_{\infty,[a,b]}.$$

Proof. The following result obtained by Čebyšev in 1882, [2], states that

$$\left| \frac{1}{b-a} \int_a^b g(t) h(t)\, dt - \frac{1}{b-a} \int_a^b g(t)\, dt \frac{1}{b-a} \int_a^b h(t)\, dt \right| \tag{2.17}$$
$$\leq \frac{1}{12} \|h'\|_\infty \|g'\|_\infty (b-a)^2,$$

provided that f', g' exist and are continuous on $[a, b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

If we take $h(t) = t - a$, $g(t) = f'(t)$ on $[a, x]$ and $h(t) = t - b$, $g(t) = f'(t)$ on $[x, b]$ we get, by a similar argument as above, that

$$0 \leq \frac{1}{2} [(b-a) f(x) + (b-x) f(b) + (x-a) f(a)] - \int_a^b f(t)\, dt \tag{2.18}$$
$$\leq \frac{1}{12} \left[(x-a)^3 \|f''(t)\|_{\infty,[a,x]} + (b-x)^3 \|f''(t)\|_{\infty,[x,b]} \right]$$
$$\leq \frac{1}{12} \left[(x-a)^3 + (b-x)^3 \right] \|f''(t)\|_{\infty,[a,b]}$$

for all $x \in (a, b)$.

Since

$$(x-a)^3+(b-x)^3=(b-a)\left[\frac{1}{4}(b-a)^2+3\left(x-\frac{a+b}{2}\right)^2\right],$$

hence by (2.18) we derive (2.15). □

The case of *euclidean norms*

$$\|u\|_{2,[a,b]}:=\left(\int_a^b |u(t)|^2\,dt\right)^{1/2}$$

of the derivative was considered by A. Lupaş in [9] in which he proved that

$$\left|\frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt-\frac{1}{b-a}\int_a^b g(t)\,dt\frac{1}{b-a}\int_a^b h(t)\,dt\right| \le \frac{1}{\pi^2}\|h'\|_{2,[a,b]}\|g'\|_{2,[a,b]}(b-a), \tag{2.19}$$

provided that h, g are absolutely continuous and $h', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Finally, we have:

Theorem 4. *Assume that f is convex and twice differentiable on (a,b) with $f'' \in L_2[a,b]$, then*

$$\begin{aligned} 0 &\le \frac{1}{2}\left[(b-a)f(x)+(b-x)f(b)+(x-a)f(a)\right]-\int_a^b f(t)\,dt \\ &\le \frac{1}{\pi^2}\left[\|f''\|_{2,[a,x]}(x-a)^{3/2}+\|f''\|_{2,[x,b]}(b-x)^{3/2}\right] \\ &\le \frac{1}{\pi^2}\|f''\|_{2,[a,b]}\left[\frac{1}{4}(b-a)^2+3\left(x-\frac{a+b}{2}\right)^2\right]^{1/2}(b-a)^{1/2} \end{aligned} \tag{2.20}$$

for all $x\in(a,b)$.

In particular,

$$\begin{aligned} 0 &\le \frac{1}{2}(b-a)\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\int_a^b f(t)\,dt \\ &\le \frac{\sqrt{2}}{4\pi^2}\left[\|f''\|_{2,\left[a,\frac{a+b}{2}\right]}+\|f''\|_{2,\left[\frac{a+b}{2},b\right]}\right](b-a)^{3/2} \\ &\le \frac{1}{2\pi^2}\|f''\|_{2,[a,b]}(b-a)^{3/2}. \end{aligned} \tag{2.21}$$

Proof. If we use (2.19) we have

$$\begin{aligned} 0 &\le \int_a^x (t-a)f'(t)\,dt-\frac{1}{x-a}\int_a^x (t-a)\,dt\int_a^x f'(t)\,dt \\ &\le \frac{1}{\pi^2}\left(\int_a^x dt\right)^{1/2}\|f''\|_{2,[a,x]}(x-a)^2=\frac{1}{\pi^2}\|f''\|_{2,[a,x]}(x-a)^{3/2} \end{aligned}$$

and

$$\begin{aligned}0 &\leq \int_x^b (t-b) f'(t)\,dt - \frac{1}{b-x}\int_x^b (t-b)\,dt \int_x^b f'(t)\,dt \\ &\leq \frac{1}{\pi^2}\left(\int_x^b dt\right)^{1/2} \|f''\|_{2,[x,b]} (b-x)^2 = \frac{1}{\pi^2}\|f''\|_{2,[x,b]}(b-x)^{3/2}\end{aligned}$$

for all $x \in (a,b)$.

If we add these inequalities, then we get by (2.3) that

$$\begin{aligned}0 &\leq \frac{1}{2}\left[(b-a) f(x) + (b-x) f(b) + (x-a) f(a)\right] - \int_a^b f(t)\,dt \\ &\leq \frac{1}{\pi^2}\left[\|f''\|_{2,[a,x]}(x-a)^{3/2} + \|f''\|_{2,[x,b]}(b-x)^{3/2}\right] \\ &\leq \frac{1}{\pi^2}\left(\|f''\|_{2,[a,x]}^2 + \|f''\|_{2,[x,b]}^2\right)^{1/2}\left[\left((x-a)^{3/2}\right)^2 + \left((b-x)^{3/2}\right)^2\right]^{1/2} \\ &= \frac{1}{\pi^2}\|f''\|_{2,[a,b]}\left[(x-a)^3 + (b-x)^3\right]^{1/2} \\ &= \frac{1}{\pi^2}\|f''\|_{2,[a,b]}\left[\frac{1}{4}(b-a)^2 + 3\left(x - \frac{a+b}{2}\right)^2\right]^{1/2}(b-a)^{1/2},\end{aligned}$$

which proves (2.20). □

3. Applications for Semi-Inner Products

If $\varphi : [0,1] \to \mathbb{R}$ is convex, then by taking $a = 0$ and $b = 1$ in the previous section, then we get for $x = s \in [0,1]$ that

$$\begin{aligned}0 &\leq \frac{1}{2}\left[\varphi(s) + (1-s)\varphi(1) + s\varphi(0)\right] - \int_0^1 \varphi(t)\,dt \\ &\leq \frac{1}{8}\left[\left(\varphi'_-(s) - \varphi'_+(0)\right)s^2 + \left(\varphi'_-(1) - \varphi'_+(s)\right)(1-s)^2\right]\end{aligned} \tag{3.1}$$

for all $s \in (0,1)$. In particular,

$$\begin{aligned}0 &\leq \frac{1}{2}\left[\varphi\left(\frac{1}{2}\right) + \frac{\varphi(1)+\varphi(0)}{2}\right] - \int_0^1 \varphi(t)\,dt \\ &\leq \frac{1}{32}\left\{\varphi'_-(1) - \varphi'_+(0) - \left[\varphi'_+\left(\frac{1}{2}\right) - \varphi'_-\left(\frac{1}{2}\right)\right]\right\} \\ &\leq \frac{1}{32}\left[\varphi'_-(1) - \varphi'_+(0)\right].\end{aligned} \tag{3.2}$$

Also, if we φ is twice differentiable with $\varphi'' \in L_\infty[0,1]$, we have

$$\begin{aligned}0 &\leq \frac{1}{2}\left[\varphi(s) + (1-s)\varphi(1) + s\varphi(0)\right] - \int_0^1 \varphi(t)\,dt \\ &\leq \frac{1}{12}\left[s^3\|\varphi''(t)\|_{\infty,[0,s]} + (1-s)^3\|\varphi''(t)\|_{\infty,[s,1]}\right] \\ &\leq \frac{1}{4}\left[\frac{1}{12} + \left(s - \frac{1}{2}\right)^2\right]\|\varphi''(t)\|_{\infty,[0,1]},\end{aligned} \tag{3.3}$$

for all $s\in(0,1)$. In particular,

$$0\leq\frac{1}{2}\left[\varphi\left(\frac{1}{2}\right)+\frac{\varphi(1)+\varphi(0)}{2}\right]-\int_0^1\varphi(t)\,dt \tag{3.4}$$
$$\leq\frac{1}{96}\left[\left\|\varphi''(t)\right\|_{\infty,\left[0,\frac{1}{2}\right]}+\left\|\varphi''(t)\right\|_{\infty,\left[\frac{1}{2},1\right]}\right]\leq\frac{1}{48}\left\|\varphi''(t)\right\|_{\infty,[0,1]}.$$

Moreover, if $\varphi''\in L_2[0,1]$, then

$$0\leq\frac{1}{2}\left[\varphi(s)+(1-s)\varphi(1)+s\varphi(0)\right]-\int_0^1\varphi(t)\,dt \tag{3.5}$$
$$\leq\frac{1}{\pi^2}\left[s^{3/2}\left\|\varphi''\right\|_{2,[0,s]}+(1-s)^{3/2}\left\|\varphi''\right\|_{2,[s,1]}\right]$$
$$\leq\frac{1}{\pi^2}\left\|\varphi''\right\|_{2,[0,1]}\left[\frac{1}{4}+3\left(s-\frac{1}{2}\right)^2\right]^{1/2}$$

for all $s\in(0,1)$. In particular,

$$0\leq\frac{1}{2}\left[\varphi\left(\frac{1}{2}\right)+\frac{\varphi(1)+\varphi(0)}{2}\right]-\int_0^1\varphi(t)\,dt \tag{3.6}$$
$$\leq\frac{\sqrt{2}}{4\pi^2}\left[\left\|\varphi''\right\|_{2,\left[0,\frac{1}{2}\right]}+\left\|\varphi''\right\|_{2,\left[\frac{1}{2},1\right]}\right]\leq\frac{1}{2\pi^2}\left\|\varphi''\right\|_{2,[0,1]}.$$

Now, let f be a convex function on the convex set C in the linear space X. If we write the inequalities (3.1) and (3.2) for

$$\varphi(t)=g(x,y)(t):=f\left[(1-t)x+ty\right],\ t\in[0,1]$$

where x, $y\in C$ $x\neq y$, then we get

$$0\leq\frac{1}{2}\left[f\left[(1-s)x+sy\right]+(1-s)f(y)+sf(x)\right]-\int_0^1 f\left[(1-t)x+ty\right]dt \tag{3.7}$$
$$\leq\frac{1}{8}\Big[\left(\left(\nabla_-f\left[(1-s)x+sy\right]\right)(y-x)-\left(\nabla_+f(x)\right)(y-x)\right)s^2$$
$$+\left(\left(\nabla_-f(y)\right)(y-x)-\left(\nabla_+f\left[(1-s)x+sy\right]\right)(y-x)\right)(1-s)^2\Big]$$

for all $s\in(0,1)$. In particular,

$$0\leq\frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+\frac{f(y)+f(x)}{2}\right]-\int_0^1 f\left[(1-t)x+ty\right]dt \tag{3.8}$$
$$\leq\frac{1}{32}\left[\left(\nabla_-f(y)\right)(y-x)-\left(\nabla_+f(x)\right)(y-x)\right]$$
$$-\frac{1}{32}\left[\left(\nabla_+f\left(\frac{x+y}{2}\right)\right)(y-x)-\left(\nabla_-f\left(\frac{x+y}{2}\right)\right)\right]$$
$$\leq\frac{1}{32}\left[\left(\nabla_-f(y)\right)(y-x)-\left(\nabla_+f(x)\right)(y-x)\right].$$

The following result for norms and semi-inner products holds:

Proposition 1. *Let $(X, \|\cdot\|)$ is a normed linear space, x, $y \in X$, $s \in (0,1)$ and $p \geq 1$. Then*

$$0 \leq \frac{1}{2}\left[\|(1-s)x+sy\|^p+(1-s)\|y\|^p+s\|x\|^p\right]-\int_0^1\|(1-t)x+ty\|^p\,dt \tag{3.9}$$
$$\leq \frac{p}{8}\left(\|(1-s)x+sy\|^{p-2}\langle y-x,(1-s)x+sy\rangle_i-\|x\|^{p-2}\langle y-x,x\rangle_s\right)s^2$$
$$+\frac{p}{8}\left(\|y\|^{p-2}\langle y-x,y\rangle_i-\|(1-s)x+sy\|^{p-2}\langle y-x,(1-s)x+sy\rangle_s\right)(1-s)^2$$

holds for any $x, y \in X$ whenever $p \geq 2$; otherwise, it holds for linearly independent x, $y \in X$.

For $p = 2$ we get from (3.9) that

$$0 \leq \frac{1}{2}\left[\|(1-s)x+sy\|^2+(1-s)\|y\|^2+s\|x\|^2\right] \tag{3.10}$$
$$-\int_0^1\|(1-t)x+ty\|^2\,dt$$
$$\leq \frac{1}{4}\left(\langle y-x,(1-s)x+sy\rangle_i-\langle y-x,x\rangle_s\right)s^2$$
$$+\frac{1}{4}\left(\langle y-x,y\rangle_i-\langle y-x,(1-s)x+sy\rangle_s\right)(1-s)^2$$

For $p = 1$ we have

$$0 \leq \frac{1}{2}\left[\|(1-s)x+sy\|+(1-s)\|y\|+s\|x\|\right] \tag{3.11}$$
$$-\int_0^1\|(1-t)x+ty\|\,dt$$
$$\leq \frac{1}{8}\left(\left\langle y-x,\frac{(1-s)x+sy}{\|(1-s)x+sy\|}\right\rangle_i-\left\langle y-x,\frac{x}{\|x\|}\right\rangle_s\right)s^2$$
$$+\frac{1}{8}\left(\left\langle y-x,\frac{y}{\|y\|}\right\rangle_i-\left\langle y-x,\frac{(1-s)x+sy}{\|(1-s)x+sy\|}\right\rangle_s\right)(1-s)^2$$

for linearly independent x, $y \in X$.

The proof follows by the inequality (3.7) applied for the convex function $f_p(x) = \|x\|^p$, $p \geq 1$.

Corollary 1. *With the assumptions of Proposition 1,*

$$0 \leq \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p+\frac{\|y\|^p+\|x\|^p}{2}\right]-\int_0^1\|(1-t)x+ty\|^p\,dt \tag{3.12}$$
$$\leq \frac{p}{32}\left[\|y\|^{p-2}\langle y-x,y\rangle_i-\|x\|^{p-2}\langle y-x,x\rangle_s\right]$$
$$-\frac{p}{32}\left\|\frac{x+y}{2}\right\|^{p-2}\left[\left\langle y-x,\frac{x+y}{2}\right\rangle_s-\left\langle y-x,\frac{x+y}{2}\right\rangle_i\right]$$
$$\leq \frac{p}{32}\left[\|y\|^{p-2}\langle y-x,y\rangle_i-\|x\|^{p-2}\langle y-x,x\rangle_s\right]$$

holds for any $x, y \in X$ whenever $p \geq 2$; otherwise, it holds for linearly independent x, $y \in X$.

For $p=2$ we derive

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^2 + \frac{\|y\|^2+\|x\|^2}{2}\right] - \int_0^1 \|(1-t)x+ty\|^2 dt \tag{3.13}$$
$$\le \frac{1}{16}\left[\langle y-x,y\rangle_i - \langle y-x,x\rangle_s\right]$$
$$- \frac{1}{16}\left[\left\langle y-x,\frac{x+y}{2}\right\rangle_s - \left\langle y-x,\frac{x+y}{2}\right\rangle_i\right]$$
$$\le \frac{1}{16}\left[\langle y-x,y\rangle_i - \langle y-x,x\rangle_s\right].$$

For $p=1$ we have

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\| + \frac{\|y\|+\|x\|}{2}\right] - \int_0^1 \|(1-t)x+ty\| dt \tag{3.14}$$
$$\le \frac{1}{32}\left[\left\langle y-x,\frac{y}{\|y\|}\right\rangle_i - \left\langle y-x,\frac{x}{\|x\|}\right\rangle_s\right]$$
$$- \frac{1}{32}\left[\left\langle y-x,\frac{x+y}{\|x+y\|}\right\rangle_s - \left\langle y-x,\frac{x+y}{\|x+y\|}\right\rangle_i\right]$$
$$\le \frac{1}{32}\left[\left\langle y-x,\frac{y}{\|y\|}\right\rangle_i - \left\langle y-x,\frac{x}{\|x\|}\right\rangle_s\right]$$

for linearly independent $x,\ y \in X$.

If $(H, \langle\cdot,\cdot\rangle)$ is an *inner product space*, then

$$\langle x,y\rangle_s = \langle x,y\rangle_i = \operatorname{Re}\langle x,y\rangle \text{ for all } x,\ y \in H.$$

From (3.12) we then get

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p + \frac{\|y\|^p+\|x\|^p}{2}\right] - \int_0^1 \|(1-t)x+ty\|^p dt \tag{3.15}$$
$$\le \frac{p}{32}\left[\left\langle y-x,\|y\|^{p-2}y - \|x\|^{p-2}x\right\rangle\right]$$

for any $x,\ y\in H$ whenever $p\ge 2$; otherwise, it holds for $x,\ y\neq 0$.

For $p=1$ we get

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\| + \frac{\|y\|+\|x\|}{2}\right] - \int_0^1 \|(1-t)x+ty\| dt \tag{3.16}$$
$$\le \frac{1}{32}\left[\left\langle y-x,\frac{y}{\|y\|} - \frac{x}{\|x\|}\right\rangle\right]$$

for any $x,\ y\in X$, $x,\ y\neq 0$.

Consider the twice differentiable and nonnegative function f and for $q\ge 1$, we put $\varphi(t) := [f(t)]^q$. Then

$$\varphi'(t) := q[f(t)]^{q-1} f'(t)$$

and

$$\varphi''(t) := q[f(t)]^{q-2}\left[(q-1)[f'(t)]^2 + f(t)f''(t)\right]. \tag{3.17}$$

Let $(H, \langle\cdot,\cdot\rangle)$ be an *inner product space*, $x,y\in H$ and define

$$f_{x,y,2}(t) := \|(1-t)x+ty\|^2,\ t\in[0,1].$$

Then

$$f'_{x,y,2}(t) := 2\operatorname{Re}\langle y-x,(1-t)x+ty\rangle,$$

and

$$f''_{x,y,2}(t) := 2\|y-x\|^2, \ t\in(0,1).$$

Consider $p\geq 1$ and put

$$\varphi_{x,y,p}(t) := \|(1-t)x+ty\|^p = [f_{x,y,2}(t)]^{p/2}.$$

By using (3.17) we derive for $q=p/2$ that

$$\begin{aligned}\varphi''_{x,y,p}(t) &= p/2\left[\|(1-t)x+ty\|^2\right]^{p/2-2}\\ &\times\Big[4(p/2-1)[\operatorname{Re}\langle y-x,(1-t)x+ty\rangle]^2\\ &+2\|(1-t)x+ty\|^2\|y-x\|^2\Big]\\ &= p\|(1-t)x+ty\|^{p-4}\Big[(p-2)[\operatorname{Re}\langle y-x,(1-t)x+ty\rangle]^2\\ &+\|(1-t)x+ty\|^2\|y-x\|^2\Big]\end{aligned}$$

which holds for any x, $y\in H$ if $p\geq 4$ and x, y linearly independent if $p\in[1,4)$.

We observe that, by using Schwarz inequality, we get

$$[\operatorname{Re}\langle y-x,(1-t)x+ty\rangle]^2 \leq \|y-x\|^2\|(1-t)x+ty\|^2$$

and then

$$\begin{aligned}\left|\varphi''_{x,y,p}(t)\right| &\leq p\|(1-t)x+ty\|^{p-4}\Big[|p-2|[\operatorname{Re}\langle y-x,(1-t)x+ty\rangle]^2\\ &+\|(1-t)x+ty\|^2\|y-x\|^2\Big]\\ &\leq p\|(1-t)x+ty\|^{p-4}\Big[|p-2|\|y-x\|^2\|(1-t)x+ty\|^2\\ &+\|(1-t)x+ty\|^2\|y-x\|^2\Big]\\ &= p(|p-2|+1)\|y-x\|^2\|(1-t)x+ty\|^{p-2}.\end{aligned}$$

For $p\geq 2$ we obtain

$$\left|\varphi''_{x,y,p}(t)\right| \leq p(p-1)\|y-x\|^2\|(1-t)x+ty\|^{p-2}$$

for any x, $y\in H$.

For $p\in[1,2)$ we have

$$\left|\varphi''_{x,y,p}(t)\right| \leq \frac{p(3-p)\|y-x\|^2}{\|(1-t)x+ty\|^{2-p}}$$

for any x, $y\in H$ linearly independent. For $p=1$ we get

$$\left|\varphi''_{x,y,p}(t)\right| \leq \frac{2\|y-x\|^2}{\|(1-t)x+ty\|}.$$

If $p\geq 2$ and since

$$\|(1-t)x+ty\| \leq (1-t)\|x\|+t\|y\| \leq \max\{\|x\|,\|y\|\}$$

for $t \in [0,1]$, then

$$\left|\varphi''_{x,y,p}(t)\right| \le p(p-1)\|y-x\|^2((1-t)\|x\|+t\|y\|)^{p-2} \le p(p-1)\|y-x\|^2\max\left\{\|x\|^{p-2},\|y\|^{p-2}\right\} \quad (3.18)$$

for $t \in [0,1]$.

We then can state:

Proposition 2. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space, $x,y \in H$, $s \in (0,1)$ and $p \ge 2$. Then*

$$0 \le \frac{1}{2}\left[\|(1-s)x+sy\|^p+(1-s)\|y\|^p+s\|x\|^p\right]-\int_0^1\|(1-t)x+ty\|^p\,dt \le \frac{1}{4}\left[\frac{1}{12}+\left(s-\frac{1}{2}\right)^2\right]p(p-1)\|y-x\|^2\max\left\{\|x\|^{p-2},\|y\|^{p-2}\right\}. \quad (3.19)$$

In particular,

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p+\frac{\|y\|^p+\|x\|^p}{2}\right]-\int_0^1\|(1-t)x+ty\|^p\,dt \le \frac{1}{48}p(p-1)\|y-x\|^2\max\left\{\|x\|^{p-2},\|y\|^{p-2}\right\}.$$

The proof follows by (3.3) applied for the function $\varphi_{x,y,p}$ and by the inequality (3.18).

Proposition 3. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space, $x,y \in H$, $s \in (0,1)$ and $p \ge 2$. Then*

$$0 \le \frac{1}{2}\left[\|(1-s)x+sy\|^p+(1-s)\|y\|^p+s\|x\|^p\right]-\int_0^1\|(1-t)x+ty\|^p\,dt \le \frac{1}{\pi^2}p(p-1)\|y-x\|^2\left[\frac{1}{4}+3\left(s-\frac{1}{2}\right)^2\right]^{1/2}\Psi(x,y;p) \quad (3.20)$$

where

$$\Psi(x,y;p) := \begin{cases} \|x\|^{p-2} & \text{if } \|x\|=\|y\|, \\ \left(\frac{\|x\|^{2p-3}-\|y\|^{2p-3}}{(2p-3)(\|x\|-\|y\|)}\right)^{1/2} & \text{if } \|x\| \ne \|y\|. \end{cases}$$

In particular,

$$0 \le \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^p+\frac{\|y\|^p+\|x\|^p}{2}\right]-\int_0^1\|(1-t)x+ty\|^p\,dt \le \frac{1}{2\pi^2}p(p-1)\|y-x\|^2\Psi(x,y;p). \quad (3.21)$$

Proof. From (3.18) we get

$$\begin{aligned}
&\left(\int_0^1 \left|\varphi''_{x,y,p}(t)\right|^2 dt\right)^{1/2} \\
&\le p(p-1)\|y-x\|^2 \left(\int_0^1 ((1-t)\|x\| + t\|y\|)^{2(p-2)} dt\right)^{1/2} \\
&= p(p-1)\|y-x\|^2 \times \begin{cases} \|x\|^{p-2} & \text{if } \|x\| = \|y\| \\ \left(\frac{\|x\|^{2p-3}-\|y\|^{2p-3}}{(2p-3)(\|x\|-\|y\|)}\right)^{1/2} & \text{if } \|x\| \neq \|y\| \end{cases}
\end{aligned}$$

and by inequality (3.5) we obtain (3.20). □

References

[1] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications. *Tamkang J. Math.*, **38** (2007) 37–49; Preprint in *RGMIA Res. Rep. Coll.*, **5** (2002) Art. 14.

[2] P. L. Chebyshev, Sur les expressions approximatives des intègrals dèfinis par les outres prises entre les même limites, *Proc. Math. Soc. Charkov,* **2** (1882), 93-98.

[3] X.-L Cheng and J. Sun, Note on the perturbed trapezoid inequality. *J. Inequal. Pure Appl. Math.* , **3** (2002), Art. 29.

[4] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, 1990.

[5] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Matemeatica Complutense,* **16** (2003), No.2, 373-382.

[6] E. Kikianty, S.S. Dragomir and P. Cerone, Sharp inequalities of Ostrowski type for convex functions defined on linear spaces and application, *Computers and Mathematics with Applications,* **56** (2008) 2235–2246

[7] L. Féjer, ¨Uber die Fourierreihen, II, *Math. Naturwiss, Anz. Ungar. Akad. Wiss.,* **24** (1906), 369-390.

[8] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{y-s}\int_s^y \varphi(s)g(s)ds - \frac{1}{(y-s)^2}\int_s^y \varphi(s)ds \int_s^y g(s)ds$, *Math. Z.* , **39**(1935), 215-226.

[9] A. Lupaş, The best constant in an integral inequality, *Mathematica* (Cluj, Romania), **15** (**38**) (2) (1973), 219-222.

[10] A. M. Ostrowski, On an integral inequality, *Aequat. Math.,* **4** (1970), 358-373.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa