# REVERSE GENERALIZED TRAPEZOID TYPE WEIGHTED INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. Assume that $f$ is convex on $[a,b]$ and $x\in(a,b)$. If $w$ is integrable on $[a,b]$ with $\int_t^x w(s)\,ds\geq 0$ for a.e. $t\in(a,x)$ and $\int_x^t w(s)\,ds\geq 0$ for a.e. $t\in(x,b)$, then

$$\begin{aligned}
&f'_+(x)\int_x^b (b-t)\,w(t)\,dt-f'_-(x)\int_a^x (t-a)\,w(t)\,dt\\
&\leq\left(\int_x^b w(s)\,ds\right)f(b)+\left(\int_a^x w(s)\,ds\right)f(a)-\int_a^b w(t)\,f(t)\,dt\\
&\leq f'_-(b)\int_x^b (b-t)\,w(t)\,dt-f'_+(a)\int_a^x (t-a)\,w(t)\,dt.
\end{aligned}$$

In particular, if $\int_t^{\frac{a+b}{2}} w(s)\,ds\geq 0$ for a.e. $t\in\left(a,\frac{a+b}{2}\right)$ and $\int_{\frac{a+b}{2}}^t w(s)\,ds\geq 0$ for a.e. $t\in\left(\frac{a+b}{2},b\right)$, then

$$\begin{aligned}
&f'_+\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^b (b-t)\,w(t)\,dt-f'_-\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}} (t-a)\,w(t)\,dt\\
&\leq\left(\int_{\frac{a+b}{2}}^b w(s)\,ds\right)f(b)+\left(\int_a^{\frac{a+b}{2}} w(s)\,ds\right)f(a)-\int_a^b w(t)\,f(t)\,dt\\
&\leq f'_-(b)\int_{\frac{a+b}{2}}^b (b-t)\,w(t)\,dt-f'_+(a)\int_a^{\frac{a+b}{2}} (t-a)\,w(t)\,dt.
\end{aligned}$$

The above inequalities hold for nonnegative weights $w$. Some examples for particular weights are also provided.

## 1. Introduction

We start with the following result concerning two inequalities of trapezoid type for convex functions obtained in [5]:

**Theorem 1.** *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in[a,b]$ one has the inequality*

$$\begin{aligned}
(1.1)\qquad &\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right]\\
&\leq (x-a)\,f(a)+(b-x)\,f(b)-\int_a^b f(t)\,dt\\
&\leq\frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right].
\end{aligned}$$

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*The constant* $\frac{1}{2}$ *is sharp in both inequalities. The second inequality also holds for* $x = a$ *or* $x = b$.

We have a simpler first inequality in the case of differentiability:

**Corollary 1.** *With the assumptions of Lemma 1 and if* $x \in (a, b)$ *is a point of differentiability for* $f$, *then*

$$\left(\frac{a+b}{2} - x\right)(b-a) f'(x) \le (x-a) f(a) + (b-x) f(b) - \int_a^b f(t)\,dt. \tag{1.2}$$

Now, recall that the following inequality, which is well known in the literature as the Hermite-Hadamard inequality for convex functions, holds

$$f\left(\frac{a+b}{2}\right)(b-a) \le \int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}(b-a). \tag{1.3}$$

The following corollary provides some sharp bounds for the trapezoid difference

$$\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt.$$

**Corollary 2.** *Let* $f : [a, b] \to \mathbb{R}$ *be a convex function on* $[a, b]$. *Then we have the inequality*

$$\begin{aligned} 0 &\le \frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a)^2 \\ &\le \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt \\ &\le \frac{1}{8}\left[f'_-(b) - f'_+(a)\right](b-a)^2. \end{aligned} \tag{1.4}$$

*The constant* $\frac{1}{8}$ *is sharp in both inequalities.*

Motivated by the above results, in this paper we provide several upper and lower bounds for the weighted generalized trapezoid rule

$$\left(\int_x^b w(s)\,ds\right) f(b) + \left(\int_a^x w(s)\,ds\right) f(a) - \int_a^b w(t) f(t)\,dt$$

with $x \in (a, b)$ under various assumptions of the weight $w$ integrable on $[a, b]$ and the convex function $f$ on $[a, b]$. The particular case

$$\left(\int_{\frac{a+b}{2}}^b w(s)\,ds\right) f(b) + \left(\int_a^{\frac{a+b}{2}} w(s)\,ds\right) f(a) - \int_a^b w(t) f(t)\,dt$$

is also investigated. Some examples for particular weights are also provided.

## 2. Main Results

Our first result is as following:

**Theorem 2.** *Assume that* $f$ *is convex on* $[a, b]$ *and* $x \in (a, b)$. *If* $w$ *is integrable on* $[a, b]$ *with* $\int_t^x w(s)\,ds \ge 0$ *for a.e.* $t \in (a, x)$ *and* $\int_x^t w(s)\,ds \ge 0$ *for a.e.* $t \in (x, b)$, *then*

$$f'_{+}(x)\int_{x}^{b}(b-t)\,w(t)\,dt-f'_{-}(x)\int_{a}^{x}(t-a)\,w(t)\,dt \tag{2.1}$$
$$\leq\left(\int_{x}^{b}w(s)\,ds\right)f(b)+\left(\int_{a}^{x}w(s)\,ds\right)f(a)-\int_{a}^{b}w(t)\,f(t)\,dt$$
$$\leq f'_{-}(b)\int_{x}^{b}(b-t)\,w(t)\,dt-f'_{+}(a)\int_{a}^{x}(t-a)\,w(t)\,dt.$$

*In particular, if* $\int_{t}^{\frac{a+b}{2}}w(s)\,ds\geq 0$ *for a.e.* $t\in\left(a,\frac{a+b}{2}\right)$ *and* $\int_{\frac{a+b}{2}}^{t}w(s)\,ds\geq 0$ *for a.e.* $t\in\left(\frac{a+b}{2},b\right)$, *then*

$$f'_{+}\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}(b-t)\,w(t)\,dt-f'_{-}\left(\frac{a+b}{2}\right)\int_{a}^{\frac{a+b}{2}}(t-a)\,w(t)\,dt \tag{2.2}$$
$$\leq\left(\int_{\frac{a+b}{2}}^{b}w(s)\,ds\right)f(b)+\left(\int_{a}^{\frac{a+b}{2}}w(s)\,ds\right)f(a)-\int_{a}^{b}w(t)\,f(t)\,dt$$
$$\leq f'_{-}(b)\int_{\frac{a+b}{2}}^{b}(b-t)\,w(t)\,dt-f'_{+}(a)\int_{a}^{\frac{a+b}{2}}(t-a)\,w(t)\,dt.$$

*If* $w(t)\geq 0$ *for a.e.* $t\in(a,b)$, *then (2.1) and (2.2) are satisfied.*

*Proof.* Let $x\in[a,b]$. By using integration by parts, we have

$$\int_{a}^{x}\left(\int_{t}^{x}w(s)\,ds\right)f'(t)\,dt=\left(\int_{t}^{x}w(s)\,ds\right)f(t)\Big|_{a}^{x}+\int_{a}^{x}w(t)\,f(t)\,dt$$
$$=\int_{a}^{x}w(t)\,f(t)\,dt-\left(\int_{a}^{x}w(s)\,ds\right)f(a)$$

and

$$\int_{x}^{b}\left(\int_{x}^{t}w(s)\,ds\right)f'(t)\,dt=\left(\int_{x}^{t}w(s)\,ds\right)f(t)\Big|_{x}^{b}-\int_{x}^{b}w(t)\,f(t)\,dt$$
$$=\left(\int_{x}^{b}w(s)\,ds\right)f(b)-\int_{x}^{b}w(t)\,f(t)\,dt.$$

Then we have the following identity of interest

$$\int_{a}^{b}\left(\int_{x}^{t}w(s)\,ds\right)f'(t)\,dt \tag{2.3}$$
$$=\int_{x}^{b}\left(\int_{x}^{t}w(s)\,ds\right)f'(t)\,dt-\int_{a}^{x}\left(\int_{t}^{x}w(s)\,ds\right)f'(t)\,dt$$
$$=\left(\int_{x}^{b}w(s)\,ds\right)f(b)+\left(\int_{a}^{x}w(s)\,ds\right)f(a)-\int_{a}^{b}w(t)\,f(t)\,dt$$

for $x\in[a,b]$.

Since $f$ is convex on $[a,b]$, then for $x\in(a,b)$, $f'_{+}(x)\leq f'(t)\leq f'_{-}(b)$ for a.e. $t\in(x,b)$, which implies, by the fact that $\int_x^t w(s)\,ds\geq 0$ for a.e. $t\in(x,b)$, that

$$f'_{+}(x)\int_x^b\left(\int_x^t w(s)\,ds\right)dt \leq \int_x^b\left(\int_x^t w(s)\,ds\right)f'(t)\,dt \leq f'_{-}(b)\int_x^b\left(\int_x^t w(s)\,ds\right)dt. \tag{2.4}$$

Since

$$\int_x^b\left(\int_x^t w(s)\,ds\right)dt = \left(\int_x^t w(s)\,ds\right)t\Big|_x^b - \int_x^b w(t)\,tdt = \left(\int_x^b w(s)\,ds\right)b - \int_x^b w(t)\,tdt = \int_x^b (b-t)\,w(t)\,dt$$

hence by (2.4) we get

$$f'_{+}(x)\int_x^b (b-t)\,w(t)\,dt \leq \int_x^b\left(\int_x^t w(s)\,ds\right)f'(t)\,dt \leq f'_{-}(b)\int_x^b (b-t)\,w(t)\,dtt. \tag{2.5}$$

By the convexity of $f$ on $[a,b]$, we also have $f'_{+}(a)\leq f'(t)\leq f'_{-}(x)$ for a.e. $t\in(a,x)$, which implies, by the fact that $\int_t^x w(s)\,ds\geq 0$ for a.e. $t\in(a,x)$, that

$$f'_{+}(a)\int_a^x\left(\int_t^x w(s)\,ds\right)dt \leq \int_a^x\left(\int_t^x w(s)\,ds\right)f'(t)\,dt \leq f'_{-}(x)\int_a^x\left(\int_t^x w(s)\,ds\right)dt. \tag{2.6}$$

Since

$$\int_a^x\left(\int_t^x w(s)\,ds\right)dt = \left(\int_t^x w(s)\,ds\right)t\Big|_a^x + \int_a^x w(t)\,tdt = \int_a^x w(t)\,tdt - a\int_a^x w(s)\,ds = \int_a^x (t-a)\,w(t)\,dt$$

and by (2.6) we get

$$f'_{+}(a)\int_a^x (t-a)\,w(t)\,dt \leq \int_a^x\left(\int_t^x w(s)\,ds\right)f'(t)\,dt \leq f'_{-}(x)\int_a^x (t-a)\,w(t)\,dt,$$

namely

$$-f'_{-}(x)\int_a^x (t-a)\,w(t)\,dt \leq -\int_a^x\left(\int_t^x w(s)\,ds\right)f'(t)\,dt \leq -f'_{+}(a)\int_a^x (t-a)\,w(t)\,dt. \tag{2.7}$$

Now, if we add (2.5) to (2.7) we get

$$\begin{aligned} & f'_+ (x) \int_x^b (b-t)\, w(t)\, dt - f'_- (x) \int_a^x (t-a)\, w(t)\, dt \\ & \leq \int_x^b \left( \int_x^t w(s)\, ds \right) f'(t)\, dt - \int_a^x \left( \int_t^x w(s)\, ds \right) f'(t)\, dt \\ & \leq f'_- (b) \int_x^b (b-t)\, w(t)\, dt - f'_+ (a) \int_a^x (t-a)\, w(t)\, dt \end{aligned}$$

and by (2.3) we obtain the desired inequality (2.1). □

**Remark 1.** *Observe that for $x \in (a,b)$,*

$$\int_x^b (b-t)\, dt = \frac{1}{2} (b-x)^2, \quad \int_a^x (t-a)\, dt = \frac{1}{2} (x-a)^2$$

*and by (2.1) and (2.2) for $w \equiv 1$ we obtain the results (1.1) and (1.4) from* [5] *mentioned in the introduction.*

**Corollary 3.** *With the assumptions of Theorem 2 and if $x \in (a,b)$ is a point of differentiability for $f$, then*

$$\begin{aligned} & f'(x) \left[ \int_x^b (b-t)\, w(t)\, dt - \int_a^x (t-a)\, w(t)\, dt \right] \\ & \leq \left( \int_x^b w(s)\, ds \right) f(b) + \left( \int_a^x w(s)\, ds \right) f(a) - \int_a^b w(t)\, f(t)\, dt \\ & \leq f'_- (b) \int_x^b (b-t)\, w(t)\, dt - f'_+ (a) \int_a^x (t-a)\, w(t)\, dt. \end{aligned} \tag{2.8}$$

*If $f$ is differentiable in $\frac{a+b}{2}$, then*

$$\begin{aligned} & f'\left( \frac{a+b}{2} \right) \left[ \int_{\frac{a+b}{2}}^b (b-t)\, w(t)\, dt - \int_a^{\frac{a+b}{2}} (t-a)\, w(t)\, dt \right] \\ & \leq \left( \int_{\frac{a+b}{2}}^b w(s)\, ds \right) f(b) + \left( \int_a^{\frac{a+b}{2}} w(s)\, ds \right) f(a) - \int_a^b w(t)\, f(t)\, dt \\ & \leq f'_- (b) \int_{\frac{a+b}{2}}^b (b-t)\, w(t)\, dt - f'_+ (a) \int_a^{\frac{a+b}{2}} (t-a)\, w(t)\, dt. \end{aligned} \tag{2.9}$$

**Remark 2.** *We observe that if $f'\left(\frac{a+b}{2}\right) \geq (\leq)\, 0$ and*

$$\int_{\frac{a+b}{2}}^b (b-t)\, w(t)\, dt \geq (\leq) \int_a^{\frac{a+b}{2}} (t-a)\, w(t)\, dt,$$

*then we have*

$$\int_a^b w(t)\, f(t)\, dt \leq \left( \int_{\frac{a+b}{2}}^b w(s)\, ds \right) f(b) + \left( \int_a^{\frac{a+b}{2}} w(s)\, ds \right) f(a).$$

*Moreover, if $w$ is symmetrical, i.e. $w(a+b-t) = w(t)$ for all $t \in [a,b]$, then*

$$\int_{\frac{a+b}{2}}^b (b-t)\, w(t)\, dt = \int_{\frac{a+b}{2}}^b (b-t)\, w(a+b-t)\, dt.$$

*If we make the change of variable* $s=a+b-t$, *then* $dt=-ds$ *and*

$$\int_{\frac{a+b}{2}}^{b}(b-t)\,w(a+b-t)\,dt=-\int_{\frac{a+b}{2}}^{a}(s-a)\,w(s)\,ds=\int_{a}^{\frac{a+b}{2}}(s-a)\,w(s)\,ds$$

*and*

$$\int_{\frac{a+b}{2}}^{b}(b-t)\,w(t)\,dt-\int_{a}^{\frac{a+b}{2}}(t-a)\,w(t)\,dt=0.$$

*Since, in this situation,*

$$\int_{\frac{a+b}{2}}^{b}w(s)\,ds=\int_{a}^{\frac{a+b}{2}}w(s)\,ds=\frac{1}{2}\int_{a}^{b}w(s)\,ds,$$

*then by (2.9) we get*

$$\begin{aligned}0&\le\frac{f(b)+f(a)}{2}-\frac{\int_a^b w(t)\,f(t)\,dt}{\int_a^b w(s)\,ds}\\&\le\left[f'_-(b)-f'_+(a)\right]\frac{1}{\int_a^b w(s)\,ds}\int_a^{\frac{a+b}{2}}(t-a)\,w(t)\,dt.\end{aligned}\tag{2.10}$$

*The first inequality in (2.10) is the second Féjer's inequality* [4] *for convex functions and symmetrical weights. The second inequality in (2.10) provides a reverse.*

## 3. Related Results

We recall *Čebyšev's inequality* for two non-decreasing (non-increasing) functions $g$, $h$ on $[c,d]$, which states that

$$\frac{1}{d-c}\int_c^d g(t)\,h(t)\,dt\ge\frac{1}{d-c}\int_c^d g(t)\,dt\frac{1}{d-c}\int_c^d h(t)\,dt.\tag{3.1}$$

If the functions have opposite monotonicities, the inequality reverses in (3.1).

We also recall *Grüss' inequality* [6]

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt-\frac{1}{b-a}\int_a^b g(t)\,dt\frac{1}{b-a}\int_a^b h(t)\,dt\right|\\&\le\frac{1}{4}(M-m)(N-n),\end{aligned}\tag{3.2}$$

that holds for integrable functions $g$, $h$ such that $m\le g\le M$ and $n\le h\le N$ on $[a,b]$, where $m$, $M$, $n$, $N$ are constants.

**Theorem 3.** *Let $f$ be a convex function on $[a,b]$, $w$ integrable and nonnegative on $[a,b]$ and $x \in (a,b)$. Then*

$$\begin{aligned}
(3.3)\qquad & \frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt - \frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt \\
&\le \left(\int_x^b w(s)\,ds\right) f(b) + \left(\int_a^x w(s)\,ds\right) f(a) - \int_a^b w(t)\,f(t)\,dt \\
&\le \frac{1}{4}(b-x)\left[f'_-(b) - f'_+(x)\right]\int_x^b w(s)\,ds \\
&+ \frac{1}{4}(x-a)\left[f'_-(x) - f'_+(a)\right]\int_a^x w(s)\,ds \\
&+ \frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt - \frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt.
\end{aligned}$$

*In particular,*

$$\begin{aligned}
(3.4)\qquad & 2\left[\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\,w(t)\,dt \right.\\
&\left. -\frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,w(t)\,dt\right] \\
&\le \left(\int_{\frac{a+b}{2}}^b w(s)\,ds\right) f(b) + \left(\int_a^{\frac{a+b}{2}} w(s)\,ds\right) f(a) - \int_a^b w(t)\,f(t)\,dt \\
&\le \frac{1}{8}(b-a)\left[f'_-(b) - f'_+\left(\frac{a+b}{2}\right)\right]\int_{\frac{a+b}{2}}^b w(s)\,ds \\
&+ \frac{1}{8}(b-a)\left[f'_-\left(\frac{a+b}{2}\right) - f'_+(a)\right]\int_a^{\frac{a+b}{2}} w(s)\,ds \\
&+ 2\left[\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\,w(t)\,dt \right.\\
&\left. -\frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,w(t)\,dt\right].
\end{aligned}$$

*Proof.* Since $f$ is convex and $w$ is nonnegative, then $f'$ and $\int_x^t w(s)\,ds$ are nondecreasing on $[x,b]$, $f'_+(x) \le f'(t) \le f'_-(b)$, $0 \le \int_x^t w(s)\,ds \le \int_x^b w(s)\,ds$ and by Čebyšev and Gruss' inequalities we have

$$\begin{aligned}
0 &\le \int_x^b \left(\int_x^t w(s)\,ds\right) f'(t)\,dt - \frac{\int_x^b f'(t)\,dt}{b-x}\int_x^b \left(\int_x^t w(s)\,ds\right) dt \\
&\le \frac{1}{4}(b-x)\left[f'_-(b) - f'_+(x)\right]\int_x^b w(s)\,ds,
\end{aligned}$$

namely

$$
(3.5)\qquad 0\le\int_x^b\left(\int_x^t w(s)\,ds\right)f'(t)\,dt-\frac{f(b)-f(x)}{b-x}\int_x^b(b-t)\,w(t)\,dt
\le\frac{1}{4}(b-x)\left[f'_-(b)-f'_+(x)\right]\int_x^b w(s)\,ds.
$$

Since $-\int_t^x w(s)\,ds$ and $f'(t)$ are monotonic nondecreasing on $[a,x]$, $f'_+(a)\le f'(t)\le f'_-(x)$, $-\int_a^x w(s)\,ds\le-\int_t^x w(s)\,ds\le 0$, then by Čebyšev and Gruss' inequalities we have

$$
0\le-\int_a^x\left(\int_t^x w(s)\,ds\right)f'(t)\,dt+\frac{\int_a^x f'(t)\,dt}{x-a}\int_a^x\left(\int_t^x w(s)\,ds\right)dt
\le\frac{1}{4}(x-a)\left[f'_-(x)-f'_+(a)\right]\int_a^x w(s)\,ds,
$$

namely

$$
(3.6)\qquad 0\le-\int_a^x\left(\int_t^x w(s)\,ds\right)f'(t)\,dt+\frac{f(x)-f(a)}{x-a}\int_a^x(t-a)\,w(t)\,dt
\le\frac{1}{4}(x-a)\left[f'_-(x)-f'_+(a)\right]\int_a^x w(s)\,ds.
$$

If we add the inequalities (3.5) and (3.6) we get

$$
\begin{aligned}
0&\le\int_x^b\left(\int_x^t w(s)\,ds\right)f'(t)\,dt-\int_a^x\left(\int_t^x w(s)\,ds\right)f'(t)\,dt\\
&-\frac{f(b)-f(x)}{b-x}\int_x^b(b-t)\,w(t)\,dt+\frac{f(x)-f(a)}{x-a}\int_a^x(t-a)\,w(t)\,dt\\
&\le\frac{1}{4}(b-x)\left[f'_-(b)-f'_+(x)\right]\int_x^b w(s)\,ds\\
&+\frac{1}{4}(x-a)\left[f'_-(x)-f'_+(a)\right]\int_a^x w(s)\,ds
\end{aligned}
$$

and by (2.3) we get

$$
\begin{aligned}
0&\le\left(\int_x^b w(s)\,ds\right)f(b)+\left(\int_a^x w(s)\,ds\right)f(a)-\int_a^b w(t)\,f(t)\,dt\\
&-\frac{f(b)-f(x)}{b-x}\int_x^b(b-t)\,w(t)\,dt+\frac{f(x)-f(a)}{x-a}\int_a^x(t-a)\,w(t)\,dt\\
&\le\frac{1}{4}(b-x)\left[f'_-(b)-f'_+(x)\right]\int_x^b w(s)\,ds\\
&+\frac{1}{4}(x-a)\left[f'_-(x)-f'_+(a)\right]\int_a^x w(s)\,ds,
\end{aligned}
$$

which is equivalent to (3.3). □

**Remark 3.** *If $w$ is symmetrical on $[a,b]$, then from (3.4) we derive*

$$0 \leq 4\left[\frac{\frac{f(b)+f(a)}{2}-f\left(\frac{a+b}{2}\right)}{b-a}\right]\int_{a}^{\frac{a+b}{2}}(t-a)\,w(t)\,dt \tag{3.7}$$
$$\leq \frac{f(b)+f(a)}{2}\left(\int_{a}^{b}w(s)\,ds\right)-\int_{a}^{b}w(t)\,f(t)\,dt$$
$$\leq \frac{1}{16}(b-a)\left[f_{-}'(b)-f_{+}'\left(\frac{a+b}{2}\right)+f_{-}'\left(\frac{a+b}{2}\right)-f_{+}'(a)\right]\int_{a}^{b}w(s)\,ds$$
$$+4\left[\frac{\frac{f(b)+f(a)}{2}-f\left(\frac{a+b}{2}\right)}{b-a}\right]\int_{a}^{\frac{a+b}{2}}(t-a)\,w(t)\,dt.$$

**Remark 4.** *For $w \equiv 1$ we obtain*

$$\frac{1}{2}\left[f(b)-f(x)\right](b-x)-\frac{1}{2}(x-a)\left[f(x)-f(a)\right] \tag{3.8}$$
$$\leq (b-x)\,f(b)+(x-a)\,f(a)-\int_{a}^{b}w(t)\,f(t)\,dt$$
$$\leq \frac{1}{4}(b-x)^{2}\left[f_{-}'(b)-f_{+}'(x)\right]$$
$$+\frac{1}{4}(x-a)^{2}\left[f_{-}'(x)-f_{+}'(a)\right]$$
$$+\frac{1}{2}\left[f(b)-f(x)\right](b-x)-\frac{1}{2}(x-a)\left[f(x)-f(a)\right].$$

*for $x \in (a,b)$. In particular,*

$$0 \leq \frac{1}{2}(b-a)\left[\frac{f(b)+f(a)}{2}-f\left(\frac{a+b}{2}\right)\right] \tag{3.9}$$
$$\leq (b-a)\frac{f(b)+f(a)}{2}-\int_{a}^{b}f(t)\,dt$$
$$\leq \frac{1}{16}(b-a)^{2}\left[f_{-}'(b)-f_{+}'\left(\frac{a+b}{2}\right)+f_{-}'\left(\frac{a+b}{2}\right)-f_{+}'(a)\right]$$
$$+\frac{1}{2}(b-a)\left[\frac{f(b)+f(a)}{2}-f\left(\frac{a+b}{2}\right)\right].$$

*From (3.8) we obtain*

$$0 \leq \frac{1}{2}\left[(b-a)\,f(x)+(b-x)\,f(b)+(x-a)\,f(a)\right]-\int_{a}^{b}f(t)\,dt \tag{3.10}$$
$$\leq \frac{1}{4}(b-x)^{2}\left[f_{-}'(b)-f_{+}'(x)\right]+\frac{1}{4}(x-a)^{2}\left[f_{-}'(x)-f_{+}'(a)\right],$$

*while from (3.9) we also get that*

$$0 \leq \frac{1}{2}(b-a)\left[\frac{f(b)+f(a)}{2}+f\left(\frac{a+b}{2}\right)\right]-\int_{a}^{b}f(t)\,dt \tag{3.11}$$
$$\leq \frac{1}{16}(b-a)^{2}\left[f_{-}'(b)-f_{+}'\left(\frac{a+b}{2}\right)+f_{-}'\left(\frac{a+b}{2}\right)-f_{+}'(a)\right]$$
$$\leq \frac{1}{16}(b-a)^{2}\left[f_{-}'(b)-f_{+}'(a)\right].$$

We also have:

**Theorem 4.** *Let $f$ be a convex function on $[a,b]$, $w$ integrable and nonnegative on $[a,b]$ and $x \in (a,b)$. Then*

$$\begin{aligned} & \frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt - \frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt \\ & \leq \left(\int_x^b w(s)\,ds\right) f(b) + \left(\int_a^x w(s)\,ds\right) f(a) - \int_a^b w(t)\,f(t)\,dt \\ & \leq \frac{1}{2}\int_x^b w(s)\,ds \int_x^b \left| f'(t) - \frac{f(b)-f(x)}{b-x}\right| dt \\ & + \frac{1}{2}\int_a^x w(s)\,ds \int_a^x \left| f'(t) - \frac{f(x)-f(a)}{x-a}\right| dt \\ & + \frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt - \frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt. \end{aligned} \tag{3.12}$$

*In particular,*

$$\begin{aligned} & 2\left[\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\,w(t)\,dt \right. \\ & \left. - \frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,w(t)\,dt\right] \\ & \leq \left(\int_{\frac{a+b}{2}}^b w(s)\,ds\right) f(b) + \left(\int_a^{\frac{a+b}{2}} w(s)\,ds\right) f(a) - \int_a^b w(t)\,f(t)\,dt \\ & \leq \frac{1}{2}\int_{\frac{a+b}{2}}^b w(s)\,ds \int_{\frac{a+b}{2}}^b \left| f'(t) - 2\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\right| dt \\ & + \frac{1}{2}\int_a^{\frac{a+b}{2}} w(s)\,ds \int_a^{\frac{a+b}{2}} \left| f'(t) - 2\frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\right| dt \\ & + 2\left[\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\,w(t)\,dt \right. \\ & \left. - \frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,w(t)\,dt\right]. \end{aligned} \tag{3.13}$$

*Proof.* In [3] Cheng and Sun obtained the following Gruss type inequality

$$\begin{aligned} & \left| \frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt - \frac{1}{b-a}\int_a^b g(t)\,dt \frac{1}{b-a}\int_a^b h(t)\,dt \right| \\ & \leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds \right| dt \end{aligned} \tag{3.14}$$

where $m \leq g \leq M$ on $[a,b]$, where $m$, $M$ are constants and $h$ is integrable on $[a,b]$.

The constant $1/2$ is best in (3.14) as shown by Cerone and Dragomir in [1] where a general version for Lebesgue integral and measurable spaces was also given.

If we use (3.14) for $g(t) = \int_x^t w(s)\,ds$ and $h(t) = f'(t)$ on the interval $[x,b]$, then we get

$$\begin{aligned}
0 &\le \int_x^b \left(\int_x^t w(s)\,ds\right) f'(t)\,dt - \frac{f(b)-f(x)}{b-x}\int_x^b \left(\int_x^t w(s)\,ds\right) dt \\
&\le \frac{1}{2}\int_x^b w(s)\,ds \int_x^b \left| f'(t) - \frac{1}{b-x}\int_x^b f'(s)\,ds \right| dt \\
&= \frac{1}{2}\int_x^b w(s)\,ds \int_x^b \left| f'(t) - \frac{f(b)-f(x)}{b-x} \right| dt.
\end{aligned}$$

Also, by (3.14) for $g(t) = -\int_t^x w(s)\,ds$ and $h(t) = f'(t)$ on the interval $[a,x]$, we get

$$\begin{aligned}
0 &\le -\int_a^x \left(\int_t^x w(s)\,ds\right) f'(t)\,dt + \frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt \\
&\le \frac{1}{2}\int_a^x w(s)\,ds \int_a^x \left| f'(t) - \frac{1}{x-a}\int_a^x f'(s)\,ds \right| dt \\
&= \frac{1}{2}\int_a^x w(s)\,ds \int_a^x \left| f'(t) - \frac{f(x)-f(a)}{x-a} \right| dt.
\end{aligned}$$

If we add these inequalities, then we get

$$\begin{aligned}
0 &\le \int_x^b \left(\int_x^t w(s)\,ds\right) f'(t)\,dt - \int_a^x \left(\int_t^x w(s)\,ds\right) f'(t)\,dt \\
&- \frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt + \frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt \\
&\le \frac{1}{2}\int_x^b w(s)\,ds \int_x^b \left| f'(t) - \frac{f(b)-f(x)}{b-x} \right| dt \\
&+ \frac{1}{2}\int_a^x w(s)\,ds \int_a^x \left| f'(t) - \frac{f(x)-f(a)}{x-a} \right| dt
\end{aligned}$$

and by (2.3) we get (3.12). □

**Remark 5.** *If $w$ is symmetrical on $[a,b]$, then by (3.13) we get*

$$\begin{aligned}
0 &\le 4\left[\frac{\frac{f(b)+f(a)}{2} - f\left(\frac{a+b}{2}\right)}{b-a}\right]\int_a^{\frac{a+b}{2}} (t-a)\,w(t)\,dt \qquad (3.15)\\
&\le \frac{f(b)+f(a)}{2}\left(\int_a^b w(s)\,ds\right) - \int_a^b w(t)\,f(t)\,dt \\
&\le \frac{1}{4}\int_a^b w(s)\,ds \left[\int_{\frac{a+b}{2}}^b \left| f'(t) - 2\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a} \right| dt \right. \\
&\left. + \int_a^{\frac{a+b}{2}} \left| f'(t) - 2\frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a} \right| dt \right] \\
&+ 4\left[\frac{\frac{f(b)+f(a)}{2} - f\left(\frac{a+b}{2}\right)}{b-a}\right]\int_a^{\frac{a+b}{2}} (t-a)\,w(t)\,dt.
\end{aligned}$$

**Remark 6.** *For $w \equiv 1$ we obtain by (3.12) that*

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[(b-a) f(x)+(b-x) f(b)+(x-a) f(a)\right]-\int_{a}^{b} f(t)\, dt \\ &\leq \frac{1}{2} \int_{x}^{b}\left|f'(t)(b-x)-f(b)+f(x)\right| dt \\ &+\frac{1}{2} \int_{a}^{x}\left|(x-a) f'(t)-f(x)+f(a)\right| dt \end{aligned} \tag{3.16}$$

*for $x \in (a, b)$. In particular,*

$$\begin{aligned} 0 &\leq \frac{1}{2}(b-a)\left[\frac{f(b)+f(a)}{2}+f\left(\frac{a+b}{2}\right)\right]-\int_{a}^{b} f(t)\, dt \\ &\leq \frac{1}{2} \int_{\frac{a+b}{2}}^{b}\left|\frac{1}{2} f'(t)(b-a)-f(b)+f\left(\frac{a+b}{2}\right)\right| dt \\ &+\frac{1}{2} \int_{a}^{\frac{a+b}{2}}\left|\frac{1}{2}(b-a) f'(t)-f\left(\frac{a+b}{2}\right)+f(a)\right| dt . \end{aligned} \tag{3.17}$$

*Observe that*

$$f'(t)(b-x)-f(b)+f(x)=\int_{x}^{b}\left(f'(t)-f'(s)\right) ds$$

*and*

$$(x-a) f'(t)-f(x)+f(a)=\int_{a}^{x}\left(f'(t)-f'(s)\right) ds,$$

*which implies that*

$$\begin{aligned} \int_{x}^{b}\left|f'(t)(b-x)-f(b)+f(x)\right| dt &\leq \int_{x}^{b}\left|\int_{x}^{b}\left(f'(t)-f'(s)\right) ds\right| dt \\ &\leq \int_{x}^{b} \int_{x}^{b}\left|f'(t)-f'(s)\right| ds dt \end{aligned}$$

*and*

$$\begin{aligned} \int_{a}^{x}\left|(x-a) f'(t)-f(x)+f(a)\right| dt &\leq \int_{a}^{x}\left|\int_{a}^{x}\left(f'(t)-f'(s)\right) ds\right| dt \\ &\leq \int_{a}^{x} \int_{a}^{x}\left|f'(t)-f'(s)\right| ds dt . \end{aligned}$$

*By (3.16) we then get*

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[(b-a) f(x)+(b-x) f(b)+(x-a) f(a)\right]-\int_{a}^{b} f(t)\, dt \\ &\leq \frac{1}{2}\left[\int_{x}^{b} \int_{x}^{b}\left|f'(t)-f'(s)\right| ds dt+\int_{a}^{x} \int_{a}^{x}\left|f'(t)-f'(s)\right| ds dt\right] \end{aligned} \tag{3.18}$$

*for* $x \in (a,b)$. *In particular,*

$$0 \leq \frac{1}{2}(b-a)\left[\frac{f(b)+f(a)}{2}+f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,dt \tag{3.19}$$
$$\leq \frac{1}{2}\left[\int_{\frac{a+b}{2}}^b \int_{\frac{a+b}{2}}^b |f'(t)-f'(s)|\,dsdt+\int_a^{\frac{a+b}{2}}\int_a^{\frac{a+b}{2}}|f'(t)-f'(s)|\,dsdt\right].$$

Also we have:

**Theorem 5.** *Let $f$ be a convex function on $[a,b]$, $w$ integrable and nonnegative on $[a,b]$ and $x \in (a,b)$. Then*

$$\frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt-\frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt \tag{3.20}$$
$$\leq \left(\int_x^b w(s)\,ds\right)f(b)+\left(\int_a^x w(s)\,ds\right)f(a)-\int_a^b w(t)\,f(t)\,dt$$
$$\leq \frac{1}{2}\frac{f'_-(b)-f'_+(x)}{b-x}\int_x^b\int_x^b\left|\int_s^t w(u)\,du\right|dsdt$$
$$+\frac{1}{2}\frac{f'_-(x)-f'_+(a)}{x-a}\int_a^x\int_a^x\left|\int_t^s w(u)\,du\right|dsdt$$
$$+\frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt-\frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt.$$

*In particular,*

$$2\left[\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\,w(t)\,dt\right. \tag{3.21}$$
$$\left.-\frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\,w(t)\,dt\right]$$
$$\leq \left(\int_{\frac{a+b}{2}}^b w(s)\,ds\right)f(b)+\left(\int_a^{\frac{a+b}{2}} w(s)\,ds\right)f(a)-\int_a^b w(t)\,f(t)\,dt$$
$$\leq \frac{f'_-(b)-f'_+\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^b\int_{\frac{a+b}{2}}^b\left|\int_s^t w(u)\,du\right|dsdt$$
$$+\frac{f'_-\left(\frac{a+b}{2}\right)-f'_+(a)}{b-a}\int_a^{\frac{a+b}{2}}\int_a^{\frac{a+b}{2}}\left|\int_t^s w(u)\,du\right|dsdt$$
$$+2\left[\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\,w(t)\,dt\right.$$
$$\left.-\frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\,w(t)\,dt\right].$$

*Proof.* If we use (3.14) for $h(t)=\int_x^t w(s)\,ds$ and $g(t)=f'(t)$ on the interval $[x,b]$, then we get

$$
\begin{aligned}
0 &\le \int_x^b \left(\int_x^t w(s)\,ds\right) f'(t)\,dt - \frac{f(b)-f(x)}{b-x}\int_x^b \left(\int_x^t w(s)\,ds\right) dt \\
&\le \frac{1}{2}\left[f'_-(b)\right]-f'_+(x)\int_x^b \left|\int_x^t w(s)\,ds - \frac{1}{b-x}\int_x^b\left(\int_x^s w(u)\,du\right) w(s)\,ds\right| dt \\
&= \frac{1}{2}\frac{f'_-(b)-f'_+(x)}{b-x}\int_x^b\left|\int_x^b\left(\int_x^t w(u)\,du-\int_x^s w(u)\,du\right)ds\right| dt \\
&\le \frac{1}{2}\frac{f'_-(b)-f'_+(x)}{b-x}\int_x^b\int_x^b\left|\int_x^t w(u)\,du-\int_x^s w(u)\,du\right| dsdt.
\end{aligned}
$$

Also, by (3.14) for $h(t)=-\int_t^x w(s)\,ds$ and $g(t)=f'(t)$ on the interval $[a,x]$, we obtain

$$
\begin{aligned}
0 &\le -\int_a^x\left(\int_t^x w(s)\,ds\right) f'(t)\,dt + \frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt \\
&\le \frac{1}{2}\left[f'_-(x)-f'_+(a)\right]\int_a^x\left|\int_t^x w(u)\,du - \frac{1}{x-a}\int_a^x\int_s^x w(u)\,duds\right| dt \\
&= \frac{1}{2}\frac{f'_-(x)-f'_+(a)}{x-a}\int_a^x\left|\int_a^x\left(\int_t^x w(u)\,du-\int_s^x w(u)\,du\right)ds\right| dt. \\
&\le \frac{1}{2}\frac{f'_-(x)-f'_+(a)}{x-a}\int_a^x\int_a^x\left|\int_t^x w(u)\,du-\int_s^x w(u)\,du\right| dsdt.
\end{aligned}
$$

If we add these inequalities, then we get

$$
\begin{aligned}
0 &\le \int_x^b\left(\int_x^t w(s)\,ds\right) f'(t)\,dt - \int_a^x\left(\int_t^x w(s)\,ds\right) f'(t)\,dt \\
&\quad - \frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt + \frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt \\
&\le \frac{1}{2}\frac{f'_-(b)-f'_+(x)}{b-x}\int_x^b\int_x^b\left|\int_s^t w(u)\,du\right| dsdt \\
&\quad + \frac{1}{2}\frac{f'_-(x)-f'_+(a)}{x-a}\int_a^x\int_a^x\left|\int_t^s w(u)\,du\right| dsdt
\end{aligned}
$$

and by (2.3) we obtain (3.20). □

**Remark 7.** *Recall that for $c<d$ we have*

$$\int_c^d\int_c^d |s-t|\,dtds = \frac{1}{3}(d-c)^3.$$

*Therefore*

$$\int_x^b\int_x^b |t-s|\,dsdt = \frac{1}{3}(b-x)^3 \quad \textit{and} \quad \int_a^x\int_a^x |s-t|\,dsdt = \frac{1}{3}(x-a)^3.$$

*By utilizing (3.20) for $w \equiv 1$ we get*

$$0 \leq \frac{1}{2}\left[(b-a) f(x)+(b-x) f(b)+(x-a) f(a)\right]-\int_a^b f(t)\, dt \tag{3.22}$$
$$\leq \frac{1}{6}\left\{\left[f_-'(b)-f_+'(x)\right](b-x)^2+\left[f_-'(x)-f_+'(a)\right](x-a)^2\right\}$$

*for all $x \in (a,b)$.*

*In particular,*

$$0 \leq \frac{1}{2}(b-a)\left[\frac{f(b)+f(a)}{2}+f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\, dt \tag{3.23}$$
$$\leq \frac{1}{24}(b-a)^2\left\{f_-'(b)-f_+'(a)-\left[f_+'(x)-f_-'(x)\right]\right\}$$
$$\leq \frac{1}{24}(b-a)^2\left[f_-'(b)-f_+'(a)\right].$$

## 4. More Reverse Inequalities

The following inequality obtained by Ostrowski in 1970, [7] is as follows:

$$\left|\frac{1}{b-a}\int_a^b g(t)\, h(t)\, dt-\frac{1}{b-a}\int_a^b g(t)\, dt \frac{1}{b-a}\int_a^b h(t)\, dt\right| \tag{4.1}$$
$$\leq \frac{1}{8}(b-a)(N-n)\left\|g'\right\|_\infty.$$

By utilizing this inequality we can state:

**Theorem 6.** *Let $f$ be a convex function on $[a,b]$, $w$ bounded and nonnegative on $[a,b]$ and $x \in (a,b)$. Then*

$$\frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\, w(t)\, dt-\frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\, w(t)\, dt \tag{4.2}$$
$$\leq \left(\int_x^b w(s)\, ds\right) f(b)+\left(\int_a^x w(s)\, ds\right) f(a)-\int_a^b w(t)\, f(t)\, dt$$
$$\leq \frac{1}{8}\left[(b-x)^2\left(f_-'(b)-f_+'(x)\right)+(x-a)^2\left(f_-'(x)-f_+'(a)\right)\right]\|w\|_{\infty,[a,b]}$$
$$+\frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\, w(t)\, dt-\frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\, w(t)\, dt.$$

*In particular,*

$$2\left[\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\, w(t)\, dt\right. \tag{4.3}$$
$$\left.-\frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\, w(t)\, dt\right]$$

$$\leq \left(\int_{\frac{a+b}{2}}^{b} w(s)\,ds\right) f(b) + \left(\int_{a}^{\frac{a+b}{2}} w(s)\,ds\right) f(a) - \int_a^b w(t) f(t)\,dt$$
$$\leq \frac{1}{32}(b-a)\left[f'_-(b) - f'_+(a) - \left(f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right)\right]\|w\|_{\infty,[a,b]}$$
$$+2\left[\frac{f(b) - f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^{b}(b-t)\,w(t)\,dt\right.$$
$$\left.-\frac{f\left(\frac{a+b}{2}\right) - f(a)}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\,w(t)\,dt\right].$$

*Proof.* If we use (4.1) for $g(t) = \int_x^t w(s)\,ds$ and $h(t) = f'(t)$ on the interval $[x,b]$, then we get

$$0 \leq \int_x^b \left(\int_x^t w(s)\,ds\right) f'(t)\,dt - \frac{f(b) - f(x)}{b-x}\int_x^b \left(\int_x^t w(s)\,ds\right) dt$$
$$\leq \frac{1}{8}(b-x)^2\left(f'_-(b) - f'_+(x)\right)\|w\|_{\infty,[x,b]}.$$

Also, by (4.1) for $g(t) = -\int_t^x w(s)\,ds$ and $h(t) = f'(t)$ on the interval $[a,x]$, we get

$$0 \leq -\int_a^x \left(\int_t^x w(s)\,ds\right) f'(t)\,dt + \frac{f(x) - f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt$$
$$\leq \frac{1}{8}(x-a)^2\left(f'_-(x) - f'_+(a)\right)\|w\|_{\infty,[a,x]}.$$

If we add these inequalities, then we get

$$0 \leq \int_x^b \left(\int_x^t w(s)\,ds\right) f'(t)\,dt - \int_a^x \left(\int_t^x w(s)\,ds\right) f'(t)\,dt$$
$$-\frac{f(b) - f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt + \frac{f(x) - f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt$$
$$\leq \frac{1}{8}(b-x)^2\left(f'_-(b) - f'_+(x)\right)\|w\|_{\infty,[x,b]}$$
$$+\frac{1}{8}(x-a)^2\left(f'_-(x) - f'_+(a)\right)\|w\|_{\infty,[a,x]}$$
$$\leq \frac{1}{8}\left[(b-x)^2\left(f'_-(b) - f'_+(x)\right) + (x-a)^2\left(f'_-(x) - f'_+(a)\right)\right]\|w\|_{\infty,[a,b]}$$

and by (2.3) we obtain (4.2). □

**Remark 8.** *By utilizing (4.2) for $w \equiv 1$ we get*

$$0 \leq \frac{1}{2}\left[(b-a)f(x) + (b-x)f(b) + (x-a)f(a)\right] - \int_a^b f(t)\,dt \tag{4.4}$$
$$\leq \frac{1}{8}\left\{\left[f'_-(b) - f'_+(x)\right](b-x)^2 + \left[f'_-(x) - f'_+(a)\right](x-a)^2\right\}$$

*for all* $x \in (a,b)$. *In particular,*

$$0 \leq \frac{1}{2}(b-a)\left[\frac{f(b)+f(a)}{2}+f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,dt \tag{4.5}$$
$$\leq \frac{1}{32}(b-a)^2\left\{f'_-(b)-f'_+(a)-\left[f'_+(x)-f'_-(x)\right]\right\}$$
$$\leq \frac{1}{32}(b-a)^2\left[f'_-(b)-f'_+(a)\right].$$

*These inequalities are better than the ones in Remark 7.*

As a dual result we also have:

**Theorem 7.** *Let* $f$ *be a twice differentiable convex function on* $[a,b]$, $w$ *integrable and nonnegative on* $[a,b]$ *and* $x \in (a,b)$. *Then*

$$\frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt-\frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt \tag{4.6}$$
$$\leq \left(\int_x^b w(s)\,ds\right)f(b)+\left(\int_a^x w(s)\,ds\right)f(a)-\int_a^b w(t)\,f(t)\,dt$$
$$\leq \frac{1}{8}\left[(b-x)^2\left(\int_x^b w(s)\,ds\right)+(x-a)^2\left(\int_a^x w(s)\,ds\right)\right]\|f''\|_{\infty,[a,b]}$$
$$+\frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt-\frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt.$$

*In particular,*

$$2\left[\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\,w(t)\,dt\right. \tag{4.7}$$
$$\left.-\frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,w(t)\,dt\right]$$
$$\leq \left(\int_{\frac{a+b}{2}}^b w(s)\,ds\right)f(b)+\left(\int_a^{\frac{a+b}{2}} w(s)\,ds\right)f(a)-\int_a^b w(t)\,f(t)\,dt$$
$$\leq \frac{1}{32}(b-a)^2\left(\int_a^b w(s)\,ds\right)\|f''\|_{\infty,[a,b]}$$
$$+2\left[\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\,w(t)\,dt\right.$$
$$\left.-\frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,w(t)\,dt\right].$$

*Proof.* If we use (4.1) for $h(t)=\int_x^t w(s)\,ds$ and $g(t)=f'(t)$ on the interval $[x,b]$, then we get

$$\begin{aligned}0&\leq\int_x^b\left(\int_x^t w(s)\,ds\right)f'(t)\,dt-\frac{f(b)-f(x)}{b-x}\int_x^b\left(\int_x^t w(s)\,ds\right)dt\\&\leq\frac{1}{8}(b-x)^2\left(\int_x^b w(s)\,ds\right)\|f''\|_{\infty,[x,b]}\,.\end{aligned}$$

Also, by (4.1) for $h(t)=-\int_t^x w(s)\,ds$ and $g(t)=f'(t)$ on the interval $[a,x]$, we get

$$\begin{aligned}0&\leq-\int_a^x\left(\int_t^x w(s)\,ds\right)f'(t)\,dt+\frac{f(x)-f(a)}{x-a}\int_a^x(t-a)\,w(t)\,dt\\&\leq\frac{1}{8}(x-a)^2\left(\int_a^x w(s)\,ds\right)\|f''\|_{\infty,[a,x]}\,.\end{aligned}$$

If we add these inequalities, then we get

$$\begin{aligned}0&\leq\int_x^b\left(\int_x^t w(s)\,ds\right)f'(t)\,dt-\int_a^x\left(\int_t^x w(s)\,ds\right)f'(t)\,dt\\&\quad-\frac{f(b)-f(x)}{b-x}\int_x^b(b-t)\,w(t)\,dt+\frac{f(x)-f(a)}{x-a}\int_a^x(t-a)\,w(t)\,dt\\&\leq\frac{1}{8}(b-x)^2\left(\int_x^b w(s)\,ds\right)\|f''\|_{\infty,[x,b]}+\frac{1}{8}(x-a)^2\left(\int_a^x w(s)\,ds\right)\|f''\|_{\infty,[a,x]}\\&\leq\frac{1}{8}\left[(b-x)^2\left(\int_x^b w(s)\,ds\right)+(x-a)^2\left(\int_a^x w(s)\,ds\right)\right]\|f''\|_{\infty,[a,b]}\end{aligned}$$

and by (2.3) we obtain (4.6). □

**Remark 9.** *If $w$ is symmetrical on $[a,b]$, then by (4.7) we get*

$$\begin{aligned}0&\leq4\left[\frac{\frac{f(b)+f(a)}{2}-f\left(\frac{a+b}{2}\right)}{b-a}\right]\int_a^{\frac{a+b}{2}}(t-a)\,w(t)\,dt\\&\leq\frac{f(b)+f(a)}{2}\left(\int_a^b w(s)\,ds\right)-\int_a^b w(t)\,f(t)\,dt\\&\leq\frac{1}{32}(b-a)^2\left(\int_a^b w(s)\,ds\right)\|f''\|_{\infty,[a,b]}\\&\quad+4\left[\frac{\frac{f(b)+f(a)}{2}-f\left(\frac{a+b}{2}\right)}{b-a}\right]\int_a^{\frac{a+b}{2}}(t-a)\,w(t)\,dt.\end{aligned}\tag{4.8}$$

**Remark 10.** *By utilizing (4.6) for $w\equiv1$ we get*

$$\begin{aligned}0&\leq\frac{1}{2}\left[(b-a)f(x)+(b-x)f(b)+(x-a)f(a)\right]-\int_a^b f(t)\,dt\\&\leq\frac{1}{8}\left[(b-x)^3+(x-a)^3\right]\|f''\|_{\infty,[a,b]}\end{aligned}\tag{4.9}$$

*for all $x\in(a,b)$.*

*Since*

$$(x-a)^3+(b-x)^3=(b-a)\left[\frac{1}{4}(b-a)^2+3\left(x-\frac{a+b}{2}\right)^2\right],$$

*then (4.9) can be written as*

$$\begin{aligned}(4.10)\qquad 0 &\leq \frac{1}{2}\left[(b-a)f(x)+(b-x)f(b)+(x-a)f(a)\right]-\int_a^b f(t)\,dt\\ &\leq \frac{1}{8}(b-a)\left[\frac{1}{4}(b-a)^2+3\left(x-\frac{a+b}{2}\right)^2\right]\|f''\|_{\infty,[a,b]}.\end{aligned}$$

*In particular,*

$$\begin{aligned}(4.11)\qquad 0 &\leq \frac{1}{2}(b-a)\left[\frac{f(b)+f(a)}{2}+f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,dt\\ &\leq \frac{1}{32}(b-a)^3\|f''\|_{\infty,[a,b]}.\end{aligned}$$

The following result obtained by Čebyšev in 1882, [2], states that

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b g(t)\,h(t)\,dt-\frac{1}{b-a}\int_a^b g(t)\,dt\frac{1}{b-a}\int_a^b h(t)\,dt\right|\\ &\leq \frac{1}{12}\|h'\|_\infty\|g'\|_\infty(b-a)^2,\end{aligned}$$

provided that $f'$, $g'$ exist and are continuous on $[a,b]$ and $\|f'\|_\infty=\sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

Finally, we have:

**Theorem 8.** *Let $f$ be a twice differentiable convex function on $[a,b]$, $w$ bounded and nonnegative on $[a,b]$ and $x\in(a,b)$. Then*

$$\begin{aligned}(4.12)\qquad &\frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt-\frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt\\ &\leq\left(\int_x^b w(s)\,ds\right)f(b)+\left(\int_a^x w(s)\,ds\right)f(a)-\int_a^b w(t)\,f(t)\,dt\\ &\leq\frac{1}{12}\left[(b-x)^3\|w\|_{\infty,[x,b]}+(x-a)^3\|w\|_{\infty,[a,x]}\right]\|f''\|_{\infty,[a,b]}\\ &+\frac{f(b)-f(x)}{b-x}\int_x^b (b-t)\,w(t)\,dt-\frac{f(x)-f(a)}{x-a}\int_a^x (t-a)\,w(t)\,dt.\end{aligned}$$

*In particular,*

$$(4.13)\quad 2\left[\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^{b}(b-t)\,w(t)\,dt - \frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\int_{a}^{\frac{a+b}{2}}(t-a)\,w(t)\,dt\right]$$
$$\le \left(\int_{\frac{a+b}{2}}^{b} w(s)\,ds\right) f(b) + \left(\int_{a}^{\frac{a+b}{2}} w(s)\,ds\right) f(a) - \int_a^b w(t)\,f(t)\,dt$$
$$\le \frac{1}{96}(b-a)^3\left[\|w\|_{\infty,\left[\frac{a+b}{2},b\right]} + \|w\|_{\infty,\left[a,\frac{a+b}{2}\right]}\right]\|f''\|_{\infty,[a,b]}$$
$$+2\left[\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\int_{\frac{a+b}{2}}^{b}(b-t)\,w(t)\,dt - \frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\int_{a}^{\frac{a+b}{2}}(t-a)\,w(t)\,dt\right].$$

**Remark 11.** *By utilizing (4.6) for $w\equiv 1$ we get*

$$(4.14)\quad 0 \le \frac{1}{2}\left[(b-a)\,f(x) + (b-x)\,f(b) + (x-a)\,f(a)\right] - \int_a^b f(t)\,dt$$
$$\le \frac{1}{12}(b-a)\left[\frac{1}{4}(b-a)^2 + 3\left(x-\frac{a+b}{2}\right)^2\right]\|f''\|_{\infty,[a,b]}$$

*for all $x\in(a,b)$.*

*In particular, we have*

$$(4.15)\quad 0 \le \frac{1}{2}(b-a)\left[\frac{f(b)+f(a)}{2} + f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)\,dt \le \frac{1}{48}(b-a)^3\|f''\|_{\infty,[a,b]}$$

*These inequalities are better than the ones in Remark 10.*

## 5. Some Examples for Other Weights

We consider the weight $w(t)=\left|t-\frac{a+b}{2}\right|$, $t\in[a,b]$. Observe that

$$\int_{\frac{a+b}{2}}^{b}(b-t)\left(t-\frac{a+b}{2}\right)dt = \frac{1}{48}(b-a)^3,$$

$$\int_{a}^{\frac{a+b}{2}}(t-a)\left(\frac{a+b}{2}-t\right)dt = \frac{1}{48}(b-a)^3$$

and

$$\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)dt = \int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)dt = \frac{1}{8}(b-a)^2.$$

Then by (2.2) we obtain

$$0 \le \frac{1}{48}(b-a)^3\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right] \le \frac{1}{8}(b-a)^2[f(b)+f(a)]-\int_a^b\left|t-\frac{a+b}{2}\right|f(t)\,dt \le \frac{1}{48}(b-a)^3\left[f'_-(b)-f'_+(a)\right]. \tag{5.1}$$

From (3.4) we get

$$0 \le \frac{1}{12}(b-a)^2\left[f(b)+f(a)+f\left(\frac{a+b}{2}\right)\right]-\int_a^b\left|t-\frac{a+b}{2}\right|f(t)\,dt \le \frac{1}{384}(b-a)^3\left[f'_-(b)-f'_+\left(\frac{a+b}{2}\right)+f'_-\left(\frac{a+b}{2}\right)-f'_+(a)\right] \le \frac{1}{384}(b-a)^3\left[f'_-(b)-f'_+(a)\right]. \tag{5.2}$$

From (4.7) we obtain

$$0 \le \frac{1}{12}(b-a)^2\left[f(b)+f(a)+f\left(\frac{a+b}{2}\right)\right]-\int_a^b\left|t-\frac{a+b}{2}\right|f(t)\,dt \le \frac{1}{128}(b-a)^4\left\|f''\right\|_{\infty,[a,b]}. \tag{5.3}$$

Now, consider the weight $w(t)=(t-a)(b-t)$, $t\in[a,b]$. Observe that

$$\int_{\frac{a+b}{2}}^b (b-t)^2(t-a)\,dt=\frac{5}{192}(b-a)^4,$$

$$\int_a^{\frac{a+b}{2}} (t-a)^2(b-t)\,dt=\frac{5}{192}(b-a)^4$$

and

$$\int_{\frac{a+b}{2}}^b (t-a)(b-t)\,dt=\int_a^{\frac{a+b}{2}}(t-a)(b-t)\,dt=\frac{1}{12}(b-a)^3.$$

Then by (2.2) we obtain

$$0 \le \frac{5}{192}(b-a)^4\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right] \le \frac{1}{12}(b-a)^3[f(b)+f(a)]-\int_a^b(t-a)(b-t)f(t)\,dt \le \frac{5}{192}(b-a)^4\left[f'_-(b)-f'_+(a)\right]. \tag{5.4}$$

From (3.4) we derive

$$0 \leq \frac{1}{12}(b-a)^3\left[3[f(b)+f(a)]+10f\left(\frac{a+b}{2}\right)\right] - \int_a^b (t-a)(b-t)f(t)\,dt \leq \frac{1}{96}(b-a)^4\left[f'_-(b)-f'_+\left(\frac{a+b}{2}\right)+f'_-\left(\frac{a+b}{2}\right)-f'_+(a)\right] \leq \frac{1}{96}(b-a)^4\left[f'_-(b)-f'_+(a)\right]. \tag{5.5}$$

From (4.7) we get

$$0 \leq \frac{1}{12}(b-a)^3\left[3[f(b)+f(a)]+10f\left(\frac{a+b}{2}\right)\right] - \int_a^b (t-a)(b-t)f(t)\,dt \leq \frac{1}{196}(b-a)^5\|f''\|_{\infty,[a,b]}. \tag{5.6}$$

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