Trigonometric generated rate of convergence of smooth Picard singular integral operators

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Abstract

In this article we continue the study of smooth Picard singular integral operators that started in [2], see there chapters 10-14. This time the foundation of our research is a trigonometric Taylor's formula. We establish the convergence of our operators to the unit operator with rates via Jackson type inequalities engaging the first modulus of continuity. Of interest here is a residual appearing term. Note that our operators are not positive. Our results are pointwise and uniform.

Mathematics Subject Classification (2022): 26A15, 26D15, 41A17, 41A35.

Keywords and phrases: Singular integral, Picard operator, modulus of continuity, trigonometric Taylor formula.

1. Introduction

We are motivated by [1], [2] chapters 10-14, and [3], [4]. We use a trigonometric new Taylor formula from [3], see also [4]. Here we consider some very general operators, the smooth Picard singular integral operators over the real line and we study further their convergencnce properties quantitatively. We establish related inequalities involving the first modulus of continuity with respect to uniform norm and the estimates are pointwise and uniform. We provide a detailed proof.

2. Results

By [3], [4], for $f \in C^2(\mathbb{R})$ and $a, x \in \mathbb{R}$, we have by trigonometric Taylor formula

$$f(x) - f(a) = f'(a)\sin(x-a) + 2f''(a)\sin^2\left(\frac{x-a}{2}\right) + \tag{1}$$

RGMIA Res. Rep. Coll. **26** (2023), Art. 54, 8 pp. Received 09/09/23

$$\int_a^x \left[\left(f''(t)+f(t)\right)-\left(f''(a)+f(a)\right)\right]\sin(x-t)\,dt.$$

For $r\in\mathbb{N}$ and $n\in\mathbb{Z}^+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j=1,...,r. \\ 1-\sum\limits_{j=1}^{r}(-1)^{r-j}\binom{r}{j} j^{-n}, & j=0, \end{cases} \tag{2}$$

that is

$$\sum_{j=0}^{r}\alpha_j = 1. \tag{3}$$

$C_U(\mathbb{R})$ denotes the space of uniformly continuous functions on $\mathbb{R}$, and $C_B(\mathbb{R})$ denotes the space of bounded continuous functions on $\mathbb{R}$.

Here we consider both $f, f''\in C_U(\mathbb{R})\cup C_B(\mathbb{R})$.

For $x\in\mathbb{R}$, $\xi>0$ we consider the Lebesgue integrals, so called smooth Picard operators

$$P_{r,\xi}(f,x) = \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+jt)\right)e^{-\frac{|t|}{\xi}}\,dt, \tag{4}$$

see [1]; $P_{r,\xi}$ are not in general positive operators, see [2].

We notice by

$$\frac{1}{2\xi}\int_{-\infty}^{\infty} e^{-\frac{|t|}{\xi}}\,dt = 1, \tag{5}$$

that

$$P_{r,\xi}(c,x) = c, \text{ where } c \text{ is a constant}, \tag{6}$$

and

$$P_{r,\xi}(f,x)-f(x) = \frac{1}{2\xi}\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}\left(f(x+jt)-f(x)\right)e^{-\frac{|t|}{\xi}}\,dt\right). \tag{7}$$

Denote by

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in\mathbb{R} \\ |x-y|\le\delta}} |f(x)-f(y)|, \quad \delta>0, \tag{8}$$

the first modulus of continuity of f.

We set

$$\Delta(x) := P_{r,\xi}(f,x)-f(x)-f''(x)\left(\sum_{j=0}^{r}\frac{\alpha_j j^2}{j^2\xi^2+1}\right)\xi^2; \quad \xi>0,\ x\in\mathbb{R}. \tag{9}$$

We present our uniform approximation result.

Theorem 1 *It holds*

$$|\Delta(x)| \leq \|\Delta(x)\|_{\infty} \leq \xi^{2}\omega_{1}\left(f''+f,\xi\right)\left(\sum_{j=0}^{r}|\alpha_{j}|\,j^{2}\,(j+1)\right) =: A;\quad \xi>0,\ x\in\mathbb{R}. \tag{10}$$

And $\|\Delta(x)\|_{\infty} \to 0$, *as* $\xi \to 0$. *If* $f''(x)=0$, *then* $|P_{r,\xi}(f,x)-f(x)| \leq A$.

Proof. By (1) we get that

$$f(x+jt)-f(x) = f'(x)\sin(jt) + 2f''(x)\sin^{2}\left(\frac{jt}{2}\right) +$$

$$\int_{x}^{x+jt}\left[\left(f''(s)+f(s)\right)-\left(f''(x)+f(x)\right)\right]\sin(x+jt-s)\,ds, \tag{11}$$

or better

$$f(x+jt)-f(x) = f'(x)\sin(jt) + 2f''(x)\sin^{2}\left(\frac{jt}{2}\right) +$$

$$\int_{0}^{jt}\left[\left(f''(x+z)+f(x+z)\right)-\left(f''(x)+f(x)\right)\right]\sin(jt-z)\,dz. \tag{12}$$

Furthermore, it holds

$$\sum_{j=0}^{r}\alpha_{j}\left[f(x+jt)-f(x)\right] =$$

$$f'(x)\sum_{j=0}^{r}\alpha_{j}\sin(jt) + 2f''(x)\sum_{j=0}^{r}\alpha_{j}\sin^{2}\left(\frac{jt}{2}\right) +$$

$$\sum_{j=0}^{r}\alpha_{j}\int_{0}^{jt}\left[\left(f''(x+z)+f(x+z)\right)-\left(f''(x)+f(x)\right)\right]\sin(jt-z)\,dz, \tag{13}$$

or better

$$\sum_{j=0}^{r}\alpha_{j}\left[f(x+jt)-f(x)\right] =$$

$$f'(x)\sum_{j=0}^{r}\alpha_{j}\sin(jt) + 2f''(x)\sum_{j=0}^{r}\alpha_{j}\sin^{2}\left(\frac{jt}{2}\right) +$$

$$\sum_{j=0}^{r}\alpha_{j}j\int_{0}^{t}\left[\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right]\sin j(t-w)\,dw. \tag{14}$$

Call

$$R := R(t) := \sum_{j=0}^{r} \alpha_j j \int_0^t \left[\left(f''(x+jw) + f(x+jw) \right) - \left(f''(x) + f(x) \right) \right] \sin j(t-w)\, dw, \quad \forall\, t \in \mathbb{R}. \tag{15}$$

Then, for $t \geq 0$,

$$|R| \leq \sum_{j=0}^{r} |\alpha_j|\, j \int_0^t \left| \left(f''(x+jw) + f(x+jw) \right) - \left(f''(x) + f(x) \right) \right| |\sin j(t-w)|\, dw \leq$$

$$\sum_{j=0}^{r} |\alpha_j|\, j \int_0^t \omega_1 \left(f'' + f, jw \right) j (t-w)\, dw =$$

$(\xi > 0)$

$$\sum_{j=0}^{r} |\alpha_j|\, j^2 \int_0^t \omega_1 \left(f'' + f, \frac{\xi j w}{\xi} \right) (t-w)\, dw \leq \tag{16}$$

$$\sum_{j=0}^{r} |\alpha_j|\, j^2 \omega_1 \left(f'' + f, \xi \right) \int_0^t \left(1 + \frac{jw}{\xi} \right) (t-w)\, dw =$$

$$\omega_1 \left(f'' + f, \xi \right) \sum_{j=0}^{r} |\alpha_j|\, j^2 \left[\int_0^t (t-w)\, dw + \frac{j}{\xi} \int_0^t w^{2-1} (t-w)^{2-1}\, dw \right] =$$

$$\omega_1 \left(f'' + f, \xi \right) \sum_{j=0}^{r} |\alpha_j|\, j^2 \left[\left. \frac{(t-w)^2}{2} \right|_t^0 + \frac{j}{\xi} \frac{1}{6} t^3 \right] =$$

$$\omega_1 \left(f'' + f, \xi \right) \sum_{j=0}^{r} |\alpha_j|\, j^2 \left[\frac{t^2}{2} + j \frac{t^3}{6\xi} \right].$$

Hence $(t \geq 0)$

$$|R| \leq \omega_1 \left(f'' + f, \xi \right) \sum_{j=0}^{r} |\alpha_j|\, j^2 \left[\frac{t^2}{2} + j \frac{t^3}{6\xi} \right]. \tag{17}$$

Let now $t < 0$, then

$$|R| \leq \sum_{j=0}^{r} |\alpha_j|\, j \left| \int_0^t \left[\left(f''(x+jw) + f(x+jw) \right) - \left(f''(x) + f(x) \right) \right] \sin j(t-w)\, dw \right| \leq$$

$$\sum_{j=0}^{r} |\alpha_j|\, j \int_t^0 \left| \left(f''(x+jw) + f(x+jw) \right) - \left(f''(x) + f(x) \right) \right| |\sin j(t-w)|\, dw \leq$$

$$\sum_{j=0}^{r} |\alpha_j|\, j \int_t^0 \omega_1 \left(f'' + f, -jw \right) j (w-t)\, dw =$$

$$\sum_{j=0}^{r}|\alpha_j|\,j\int_t^0\omega_1\left(f''+f,-jw\frac{\xi}{\xi}\right)j\,(w-t)\,dw=$$

$$\omega_1\,(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|\,j^2\int_t^0\left(1-\frac{j}{\xi}w\right)(w-t)\,dw= \tag{18}$$

$$\omega_1\,(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\int_t^0(w-t)\,dw+\frac{j}{\xi}\int_t^0(0-w)^{2-1}(w-t)^{2-1}\,dw\right]=$$

$$\omega_1\,(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\left.\frac{(w-t)^2}{2}\right|_t^0+\frac{j}{\xi}\frac{1}{6}(-t)^3\right]=$$

$$\omega_1\,(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\frac{t^2}{2}+j\frac{(-t)^3}{6\xi}\right].$$

We found that $(t<0)$

$$|R|\leq\omega_1\,(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\frac{t^2}{2}+j\frac{(-t)^3}{6\xi}\right]. \tag{19}$$

Consequently, for $t\in\mathbb{R}$, we obtain

$$|R\,(t)|\leq\omega_1\,(f''+f,\xi)\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\frac{t^2}{2}+\frac{j}{6\xi}|t|^3\right],\quad \xi>0. \tag{20}$$

So, we have

$$\sum_{j=0}^{r}\alpha_j\left[f\,(x+jt)-f\,(x)\right]-f'\,(x)\sum_{j=0}^{r}\alpha_j\sin\,(jt)-2f''\,(x)\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{jt}{2}\right)=R\,(t)\,. \tag{21}$$

Therefore, it holds

$$\Delta_1\,(x):=P_{r,\xi}\,(f,x)-f\,(x)-f'\,(x)\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin\,(jt)\,e^{-\frac{|t|}{\xi}}\,dt\right)$$

$$-2f''\,(x)\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}\,dt\right)= \tag{22}$$

$$\frac{1}{2\xi}\int_{-\infty}^{\infty}R\,(t)\,e^{-\frac{|t|}{\xi}}\,dt.$$

Hence we get

$$|\Delta_1\,(x)|\leq\frac{1}{2\xi}\int_{-\infty}^{\infty}|R\,(t)|\,e^{-\frac{|t|}{\xi}}\,dt\leq$$

$$\frac{1}{2\xi}\int_{-\infty}^{\infty}\left[\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\frac{t^2}{2}+\frac{j}{6\xi}\left|t\right|^3\right]\right]e^{-\frac{|t|}{\xi}}dt \tag{23}$$

(we use

$$\int_{-\infty}^{\infty}t^k e^{-\frac{|t|}{\xi}}dt=\begin{cases}0, & k \text{ odd},\\ 2k!\xi^{k+1}, & k \text{ even}\end{cases} \tag{24}$$

)

$$=\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\frac{1}{2\xi}\int_{-\infty}^{\infty}\left[\frac{t^2}{2}+\frac{j}{6\xi}\left|t\right|^3\right]e^{-\frac{|t|}{\xi}}dt$$

$$=\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\frac{1}{4\xi}\int_{-\infty}^{\infty}t^2e^{-\frac{|t|}{\xi}}dt+\frac{j}{12\xi^2}\int_{-\infty}^{\infty}\left|t\right|^3e^{-\frac{|t|}{\xi}}dt\right] \tag{25}$$

$$=\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\frac{1}{4\xi}4\xi^3+\frac{j\xi^4}{6\xi^2}\int_0^{\infty}\left(\frac{t}{\xi}\right)^3e^{-\frac{|t|}{\xi}}d\frac{t}{\xi}\right]$$

$$=\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\xi^2+\xi^2\frac{j}{6}\int_0^{\infty}z^3e^{-z}dz\right]$$

$$=\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\xi^2+\xi^2\frac{j}{6}6\right]=\omega_1\left(f''+f,\xi\right)\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left[\xi^2+\xi^2 j\right]$$

$$=\omega_1\left(f''+f,\xi\right)\xi^2\left(\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left(1+j\right)\right).$$

We have proved that

$$\left|\Delta_1\left(x\right)\right|\leq\xi^2\omega_1\left(f''+f,\xi\right)\left(\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left(j+1\right)\right);\quad \xi>0,\ x\in\mathbb{R}. \tag{26}$$

Notice that $\Delta_1\left(x\right)\to 0$, as $\xi\to 0$.

Next we simplify left hand side (22).

We observe that

$$\int_{-\infty}^{\infty}\sin\left(jt\right)e^{-\frac{|t|}{\xi}}dt=\int_{-\infty}^{0}\sin\left(jt\right)e^{-\frac{|t|}{\xi}}dt+\int_{0}^{\infty}\sin\left(jt\right)e^{-\frac{|t|}{\xi}}dt. \tag{27}$$

Notice $-\infty\leq t\leq 0\Rightarrow\infty\geq -t\geq 0$. So that

$$-\int_{-\infty}^{0}\sin\left(j\left(-\left(-t\right)\right)\right)e^{-\frac{|t|}{\xi}}d\left(-t\right)=-\int_{-\infty}^{0}\left(-\sin\left(j\left(-t\right)\right)\right)e^{-\frac{|t|}{\xi}}d\left(-t\right)= \tag{28}$$

$$\int_{-\infty}^{0}\sin j\left(-t\right)e^{-\frac{|t|}{\xi}}d\left(-t\right)=\int_{\infty}^{0}\sin j\left(t\right)e^{-\frac{|t|}{\xi}}dt=-\int_{0}^{\infty}\sin\left(jt\right)e^{-\frac{|t|}{\xi}}dt.$$

Therefore, it is

$$\int_{-\infty}^{\infty} \sin(jt)\, e^{-\frac{|t|}{\xi}} dt = 0, \quad j = 0, 1, ..., r. \tag{29}$$

Furthermore, we have that

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) e^{-\frac{|t|}{\xi}} dt = 2\int_{0}^{\infty} \sin^2\left(\frac{jt}{2}\right) e^{-\frac{t}{\xi}} dt, \quad j = 0, 1, ..., r. \tag{30}$$

The last follows by

$$\int_{-\infty}^{0} \sin^2\left(\frac{jt}{2}\right) e^{-\frac{|t|}{\xi}} dt = -\int_{-\infty}^{0} \left(-\sin\left(\frac{j(-t)}{2}\right)\right)^2 e^{-\frac{|-t|}{\xi}} d(-t) \overset{(z=-t)}{=} \tag{31}$$

$$-\int_{\infty}^{0} \sin^2\left(\frac{jz}{2}\right) e^{-\frac{|z|}{\xi}} dz = \int_{0}^{\infty} \sin^2\left(\frac{jz}{2}\right) e^{-\frac{z}{\xi}} dz.$$

Next, we calculate

$$\int_{0}^{\infty} \sin^2\left(\frac{jt}{2}\right) e^{-\frac{|t|}{\xi}} dt = \xi\int_{0}^{\infty} \sin^2\left(\left(\frac{j\xi}{2}\right)\frac{t}{\xi}\right) e^{-\frac{t}{\xi}} d\frac{t}{\xi} =$$

(call $\frac{t}{\xi} =: x$ and $\frac{j\xi}{2} =: a_1$)

$$\xi\int_{0}^{\infty} \sin^2(a_1 x)\, e^{-x} dx = \tag{32}$$

(by Wolfram Alpha Computational Inteligence)

$$\xi\left(\frac{2a_1^2}{4a_1^2+1}\right) = \frac{j^2\xi^3}{2\left(j^2\xi^2+1\right)}.$$

Thus

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) e^{-\frac{|t|}{\xi}} dt = \frac{j^2\xi^3}{j^2\xi^2+1}, \quad j = 0, 1, ..., r. \tag{33}$$

Consequently, it holds

$$\frac{1}{2\xi}\int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) e^{-\frac{|t|}{\xi}} dt = \frac{j^2\xi^2}{2\left(j^2\xi^2+1\right)} \to 0, \text{ as } \xi \to 0, \quad j = 0, 1, ..., r. \tag{34}$$

Finally we obtain

$$-2f''(x)\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) e^{-\frac{|t|}{\xi}} dt\right) = -f''(x)\left(\sum_{j=0}^{r}\alpha_j\frac{j^2}{\left(j^2\xi^2+1\right)}\right)\xi^2. \tag{35}$$

Clearly now it is $\Delta_1(x) = \Delta(x)$.

The proof of the theorem is finally completed. ■

We finish with

Corollary 2 *It follows ($\xi > 0$, $x \in \mathbb{R}$)*

$$\left\| P_{r,\xi}(f,x) - f(x) \right\|_\infty \leq \omega_1 \left(f'' + f.\xi \right) \left(\sum_{j=0}^{r} |\alpha_j| \, j^2 (j+1) \right) \xi^2 + \tag{36}$$

$$\left\| f''(x) \right\|_\infty \left(\sum_{j=0}^{r} |\alpha_j| \frac{j^2}{1 + j^2 \xi^2} \right) \xi^2 \to 0, \quad \text{as } \xi \to 0.$$

Proof. Easy, by (9) and (10). ∎

References

[1] G.A. Anastassiou, *Basic convergence with rates of smooth Picard singular integral operators*, J. Computational Analysis and Appl., 8(4) (2006), 313-334.

[2] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, New York, Chapter 10, 2011.

[3] G.A. Anastassiou, *Opial and Ostrowski type inequalities based on trigonometric and hyperbolic type Taylor formulae*, submitted, 2023.

[4] Ali Hasan Ali, Zsolt Pales, *Taylor-type expansions in terms of exponential polynomials*, Mathematical Inequalities and Applications, 25(4) (2022), 1123-1141.