Trigonometric generated L_p degree of approximation by smooth Picard singular integral operators

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Abstract

In this article we continue the study of smooth Picard singular integral operators that started in [2], see there chapters 10-14. This time the foundation of our research is a trigonometric Taylor's formula. We establish the L_p convergence of our operators to the unit operator with rates via Jackson type inequalities engaging the first L_p modulus of continuity. Of interest here is a residual appearing term. Note that our operators are not positive.

Mathematics Subject Classification (2022): 26A15, 26D15, 41A17, 41A35.

Keywords and phrases: Singular integral, Picard operator, L_p modulus of continuity, trigonometric Taylor formula.

1 Introduction

We are motivated by [1], [2] chapters 10-14, and [3], [4]. We use a trigonometric new Taylor formula from [3], see also [4]. Here we consider some very general operators, the smooth Picard singular integral operators over the real line and we study further their L_p, $p \geq 1$, convergencnce properties quantitatively. We establish related inequalities involving the first L_p, $p \geq 1$, modulus of continuity with respect to L_p, $p \geq 1$, norm. We provide detailed proofs.

RGMIA Res. Rep. Coll. **26** (2023), Art. 55, 13 pp. Received 10/09/23

2 Results

By [3], [4], for $f \in C^2(\mathbb{R})$ and $a, x \in \mathbb{R}$, we have by the trigonometric Taylor formula

$$f(x) - f(a) = f'(a)\sin(x-a) + 2f''(a)\sin^2\left(\frac{x-a}{2}\right) + \tag{1}$$

$$\int_a^x \left[(f''(t) + f(t)) - (f''(a) + f(a))\right]\sin(x-t)\,dt.$$

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, ..., r. \\ 1 - \sum\limits_{j=1}^{r}(-1)^{r-j}\binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{2}$$

that is

$$\sum_{j=0}^{r} \alpha_j = 1. \tag{3}$$

Here we consider both $f, f'' \in L_p(\mathbb{R}) \cap C(\mathbb{R})$, $1 \leq p < \infty$.

For $x \in \mathbb{R}$, $\xi > 0$ we consider the Lebesgue integrals, so called smooth Picard operators

$$P_{r,\xi}(f,x) = \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+jt)\right)e^{-\frac{|t|}{\xi}}\,dt, \tag{4}$$

see [1]; $P_{r,\xi}$ are not in general positive operators, see [2].

We notice by

$$\frac{1}{2\xi}\int_{-\infty}^{\infty} e^{-\frac{|t|}{\xi}}\,dt = 1, \tag{5}$$

that

$$P_{r,\xi}(c,x) = c, \text{ where } c \text{ is a constant,} \tag{6}$$

and

$$P_{r,\xi}(f,x) - f(x) = \frac{1}{2\xi}\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}(f(x+jt) - f(x))\,e^{-\frac{|t|}{\xi}}\,dt\right). \tag{7}$$

Denote by

$$\omega_1(f,h)_p := \sup_{\substack{t\in\mathbb{R} \\ |t|\leq h}} \|f(x+t) - f(x)\|_{p,x}, \tag{8}$$

the first L_p modulus of smoothness of f, $1 \leq p < \infty$.

By (1) we get that

$$f(x+jt)-f(x)=f'(x)\sin(jt)+2f''(x)\sin^2\left(\frac{jt}{2}\right)+$$

$$\int_x^{x+jt}\left[\left(f''(s)+f(s)\right)-\left(f''(x)+f(x)\right)\right]\sin(x+jt-s)\,ds, \tag{9}$$

or better,

$$f(x+jt)-f(x)=f'(x)\sin(jt)+2f''(x)\sin^2\left(\frac{jt}{2}\right)+$$

$$\int_0^{jt}\left[\left(f''(x+z)+f(x+z)\right)-\left(f''(x)+f(x)\right)\right]\sin(jt-z)\,dz. \tag{10}$$

Furthermore, it holds

$$\sum_{j=0}^{r}\alpha_j\left[f(x+jt)-f(x)\right]=$$

$$f'(x)\sum_{j=0}^{r}\alpha_j\sin(jt)+2f''(x)\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{jt}{2}\right)+$$

$$\sum_{j=0}^{r}\alpha_j\int_0^{jt}\left[\left(f''(x+z)+f(x+z)\right)-\left(f''(x)+f(x)\right)\right]\sin(jt-z)\,dz, \tag{11}$$

or better

$$\sum_{j=0}^{r}\alpha_j\left[f(x+jt)-f(x)\right]=$$

$$f'(x)\sum_{j=0}^{r}\alpha_j\sin(jt)+2f''(x)\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{jt}{2}\right)+$$

$$\sum_{j=0}^{r}\alpha_j j\int_0^{t}\left[\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right]\sin j(t-w)\,dw. \tag{12}$$

Call

$$R:=R(t):=$$

$$\sum_{j=0}^{r}\alpha_j j\int_0^{t}\left[\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right]\sin j(t-w)\,dw, \tag{13}$$

$\forall\ t\in\mathbb{R}$.

We notice that

$$\Delta(x) := P_{r,\xi}(f,x) - f(x) - f'(x)\sum_{j=0}^{r}\alpha_j \frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt\right) \tag{14}$$

$$-2f''(x)\sum_{j=0}^{r}\alpha_j \frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt\right)$$

$$= \frac{1}{2\xi}\int_{-\infty}^{\infty}R(t)^{-\frac{|t|}{\xi}}\,dt.$$

Next we simplify (14).

We observe that

$$\int_{-\infty}^{\infty}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt = \int_{-\infty}^{0}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt + \int_{0}^{\infty}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt. \tag{15}$$

Notice $-\infty \leq t \leq 0 \Rightarrow \infty \geq -t \geq 0$. So that

$$-\int_{-\infty}^{0}\sin(j(-(-t)))\,e^{-\frac{|t|}{\xi}}d(-t) = -\int_{-\infty}^{0}(-\sin(j(-t)))\,e^{-\frac{|t|}{\xi}}d(-t) = \tag{16}$$

$$\int_{-\infty}^{0}(\sin j(-t))\,e^{-\frac{|t|}{\xi}}d(-t) = \int_{\infty}^{0}\sin j(t)\,e^{-\frac{|t|}{\xi}}dt = -\int_{0}^{\infty}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt.$$

Therefore, it is

$$\int_{-\infty}^{\infty}\sin(jt)\,e^{-\frac{|t|}{\xi}}dt = 0, \quad j = 0,1,...,r. \tag{17}$$

Furthermore, we have that

$$\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt = 2\int_{0}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{t}{\xi}}dt, \quad j = 0,1,...,r. \tag{18}$$

The last follows by

$$\int_{-\infty}^{0}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt = -\int_{-\infty}^{0}\left(-\sin\left(\frac{j(-t)}{2}\right)\right)^2 e^{-\frac{|-t|}{\xi}}d(-t) \overset{(z=-t)}{=} \tag{19}$$

$$-\int_{\infty}^{0}\sin^2\left(\frac{jz}{2}\right)e^{-\frac{|z|}{\xi}}dz = \int_{0}^{\infty}\sin^2\left(\frac{jz}{2}\right)e^{-\frac{z}{\xi}}dz.$$

Next, we calculate

$$\int_{0}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt = \xi\int_{0}^{\infty}\sin^2\left(\left(\frac{j\xi}{2}\right)\frac{t}{\xi}\right)e^{-\frac{t}{\xi}}d\frac{t}{\xi} =$$

(call $\frac{t}{\xi} =: x$ and $\frac{j\xi}{2} =: a$)

$$\xi\int_{0}^{\infty}\sin^2(ax)\,e^{-x}dx = \tag{20}$$

(by Wolfram Alpha Computational Inteligence)

$$\xi\left(\frac{2a^2}{4a^2+1}\right)=\frac{j^2\xi^3}{2\left(j^2\xi^2+1\right)}.$$

Thus, we find that

$$\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt=\frac{j^2\xi^3}{j^2\xi^2+1},\quad j=0,1,...,r. \tag{21}$$

Consequently, it holds ($\xi>0$)

$$\frac{1}{2\xi}\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt=\frac{j^2\xi^2}{2\left(j^2\xi^2+1\right)}\to 0,\text{ as }\xi\to 0,\quad j=0,1,...,r. \tag{22}$$

Eventually we obtain

$$-2f''(x)\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{|t|}{\xi}}dt\right)=-f''(x)\left(\sum_{j=0}^{r}\alpha_j\frac{j^2}{\left(j^2\xi^2+1\right)}\right)\xi^2. \tag{23}$$

Consequently, based on the above, it is

$$\Delta(x)=P_{r,\xi}(f,x)-f(x)-f''(x)\left(\sum_{j=0}^{r}\alpha_j\frac{j^2}{\left(j^2\xi^2+1\right)}\right)\xi^2,\quad \xi>0. \tag{24}$$

We present our first result, L_p approximation, $p>1$.

Theorem 1 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\xi>0$, and both $f,f''\in L_p(\mathbb{R})\cap C(\mathbb{R})$, $x\in\mathbb{R}$. Then*

$$\|\Delta(x)\|_p=\left\|P_{r,\xi}(f)-f-f''\left(\sum_{j=0}^{r}\alpha_j\frac{j^2}{\left(j^2\xi^2+1\right)}\right)\xi^2\right\|_p\leq$$

$$\left(\frac{4(r+1)}{q(q+1)}\right)^{\frac{1}{q}}\left(\frac{2}{p}\right)^{2+\frac{1}{p}}\omega_1\left(f''+f,\xi\right)_p\xi^2$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left[\Gamma(qp-q+p+1)+\frac{(2j)^p}{p^p(p+1)}\Gamma(qp-q+2p+1)\right]\right\}^{\frac{1}{p}}\to 0, \tag{25}$$

as $\xi\to 0$.

Above Γ stands for the gamma function.

Proof. Let

$$I := \int_0^t \left[\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right]\sin j(t-w)\,dw,\ \ \forall\ t\in\mathbb{R}. \tag{26}$$

For $t<0$, we have that

$$|I| = \left|\int_t^0 \left[\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right]\sin j(t-w)\,dw\right| \leq$$

$$\int_t^0 \left|\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right| \left|\sin j(t-w)\right| dw \leq \tag{27}$$

$$j\int_t^0 \left|\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right|(w-t)\,dw =$$

$$-j\int_t^0 \left|\left(f''(x-j(-w))+f(x-j(-w))\right)-\left(f''(x)+f(x)\right)\right|(-t-(-w))\,d(-w)$$

$(t\leq w\leq 0 \Rightarrow -t\geq -w =: \theta\geq 0)$

$$= -j\int_{-t}^0 \left|\left(f''(x-j\theta)+f(x-j\theta)\right)-\left(f''(x)+f(x)\right)\right|(-t-\theta)\,d\theta =$$

$$j\int_0^{-t} \left|\left(f''(x-j\theta)+f(x-j\theta)\right)-\left(f''(x)+f(x)\right)\right|(-t-\theta)\,d\theta = \tag{28}$$

$$j\int_0^{|t|} \left|\left(f''(x+sign(t)\,j\theta)+f(x+sign(t)\,j\theta)\right)-\left(f''(x)+f(x)\right)\right|(|t|-\theta)\,d\theta.$$

So, we have proved that

$$|I| \leq j\int_0^{|t|} \left|\left(f''(x+sign(t)\,j\theta)+f(x+sign(t)\,j\theta)\right)-\left(f''(x)+f(x)\right)\right|(|t|-\theta)\,d\theta, \tag{29}$$

$\forall\ t\in\mathbb{R}$,

and by (13),

$$|R(t)| \leq \sum_{j=0}^{r} |\alpha_j|\, j^2$$

$$\int_0^{|t|} \left|\left(f''(x+jsign(t)\,\theta)+f(x+jsign(t)\,\theta)\right)-\left(f''(x)+f(x)\right)\right|(|t|-\theta)\,d\theta, \tag{30}$$

$\forall\ t\in\mathbb{R}$.

By (14) we have

$$\Delta(x) = \frac{1}{2\xi}\int_{-\infty}^{\infty} R(t)\,e^{-\frac{|t|}{\xi}}\,dt. \tag{31}$$

Hence it holds $(p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1)$,

$$\int_{-\infty}^{\infty} |\Delta(x)|^p \, dx = \frac{1}{(2\xi)^p} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} R(t) \, e^{-\frac{|t|}{\xi}} dt \right|^p dx \leq$$

$$\frac{1}{(2\xi)^p} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)| \, e^{-\frac{|t|}{\xi}} dt \right)^p dx = \tag{32}$$

$$\frac{1}{(2\xi)^p} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)| \, e^{-\frac{|t|}{2\xi}} e^{-\frac{|t|}{2\xi}} dt \right)^p dx \leq$$

$$\frac{1}{(2\xi)^p} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)|^p \, e^{-\frac{|pt|}{2\xi}} dt \right) \left(\int_{-\infty}^{\infty} e^{-\frac{|qt|}{2\xi}} dt \right)^{\frac{p}{q}} dx =$$

$$\frac{1}{(2\xi)^p} \left(\frac{4\xi}{q} \right)^{\frac{p}{q}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)|^p \, e^{-\frac{|pt|}{2\xi}} dt \right) dx =$$

$$\frac{2^{p-2}\xi^{-1}}{q^{p-1}} \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)|^p \, e^{-\frac{|pt|}{2\xi}} dt \right) dx \right) =: (*) .$$

But, we need to treat

$$|R(t)| \leq \sum_{j=0}^{r} |\alpha_j| \, j^2 \tag{33}$$

$$\left(\int_0^{|t|} |(f''(x + j sign(t)\theta) + f(x + j sign(t)\theta)) - (f''(x) + f(x))|^p \, d\theta \right)^{\frac{1}{p}}$$

$$\left(\int_0^{|t|} (|t| - \theta)^q \, d\theta \right)^{\frac{1}{q}} =$$

$$\sum_{j=0}^{r} |\alpha_j| \, j^2 \left(\int_0^{|t|} |(f''(x + j sign(t)\theta) + f(x + j sign(t)\theta)) - (f''(x) + f(x))|^p \, d\theta \right)^{\frac{1}{p}}$$

$$\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} =$$

$$\left\{ \sum_{j=0}^{r} \left(\left(|\alpha_j| \, j^2 \right)^p \right)^{\frac{1}{p}} \right.$$

$$\left. \left(\int_0^{|t|} |(f''(x + j sign(t)\theta) + f(x + j sign(t)\theta)) - (f''(x) + f(x))|^p \, d\theta \right)^{\frac{1}{p}} \right\}$$

$$\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \leq \tag{34}$$

$(0 < \frac{1}{p} < 1)$

$$\left\{ (r+1)^{\frac{1}{q}} \right.$$

$$\left. \left(\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \int_0^{|t|} |(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))|^p d\theta \right)^{\frac{1}{p}} \right\}$$

$$\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}.$$

Hence, we find that

$$|R(t)|^p \leq \left(\frac{r+1}{q+1} \right)^{\frac{p}{q}} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \right.$$

$$\left. \left(\int_0^{|t|} |(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))|^p d\theta \right) \right] |t|^{\frac{(q+1)p}{q}}. \tag{35}$$

Therefore, we get

$$(*) \leq \left(\frac{2^{p-2}\xi^{-1}}{q^{p-1}} \left(\frac{r+1}{q+1} \right)^{\frac{p}{q}} \right) \left\{ \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \right. \right. \right.$$

$$\left. \int_0^{|t|} |(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))|^p d\theta \right]$$

$$\left. \left. |t|^{\frac{(q+1)p}{q}} e^{-\frac{|pt|}{2\xi}} dt \right) dx \right\} \tag{36}$$

(set $c_1 = \frac{2^{p-2}\xi^{-1}}{q^{p-1}} \left(\frac{r+1}{q+1} \right)^{\frac{p}{q}}$)

$$= c_1 \left\{ \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \right. \right. \right.$$

$$\left. \left. \int_0^{|t|} |(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))|^p d\theta \right] dx \right)$$

$$\left. |t|^{\frac{(q+1)p}{q}} e^{-\frac{|pt|}{2\xi}} dt \right\}$$

$$= c_1 \left\{ \int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \right. \right. \tag{37}$$

$$\int_0^{|t|}\left(\int_{-\infty}^{\infty}\left|\left(f''\left(x+jsign\left(t\right)\theta\right)+f\left(x+jsign\left(t\right)\theta\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right|^p dx\right)d\theta\Bigg]$$

$$\left.|t|^{\frac{(q+1)p}{q}}\,e^{-\frac{|pt|}{2\xi}}dt\right\}\leq$$

$(\xi>0)$

$$c_1\left\{\int_{-\infty}^{\infty}\left[\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}\left(\int_0^{|t|}\omega_1\left(f''+f,\xi\frac{j\theta}{\xi}\right)_p^p d\theta\right)\right]\right.$$

$$\left.|t|^{\frac{(q+1)p}{q}}\,e^{-\frac{|pt|}{2\xi}}dt\right\}\leq$$

$$\omega_1\left(f''+f,\xi\right)_p^p c_1\left\{\int_{-\infty}^{\infty}\left[\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}\left(\int_0^{|t|}\left(1+\frac{j}{\xi}\theta\right)^p d\theta\right)\right]\right.$$

$$\left.|t|^{(q+1)(p-1)}\,e^{-\frac{|pt|}{2\xi}}dt\right\}\leq \qquad (38)$$

$$c_1\omega_1\left(f''+f,\xi\right)_p^p\left\{\int_{-\infty}^{\infty}\left[\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}2^{p-1}\left(\int_0^{|t|}\left(1+\frac{j^p}{\xi^p}\theta^p\right)d\theta\right)\right]\right.$$

$$\left.|t|^{(q+1)(p-1)}\,e^{-\frac{|pt|}{2\xi}}dt\right\}=$$

$$2^{p-1}c_1\omega_1\left(f''+f,\xi\right)_p^p$$

$$\left\{\int_{-\infty}^{\infty}\left[\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}\left(|t|+\frac{j^p}{\xi^p}\frac{|t|^{p+1}}{p+1}\right)\right]|t|^{(q+1)(p-1)}\,e^{-\frac{|pt|}{2\xi}}dt\right\}=$$

$$2^{p-1}c_1\omega_1\left(f''+f,\xi\right)_p^p$$

$$\left\{\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}\int_{-\infty}^{\infty}\left(|t|+\frac{j^p}{\xi^p}\frac{|t|^{p+1}}{p+1}\right)|t|^{(q+1)(p-1)}\,e^{-\frac{|pt|}{2\xi}}dt\right\}= \qquad (39)$$

$$2^{p}c_1\omega_1\left(f''+f,\xi\right)_p^p$$

$$\left\{\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}\int_0^{\infty}\left(t+\frac{j^p}{\xi^p\left(p+1\right)}t^{p+1}\right)t^{(q+1)(p-1)}e^{-\frac{pt}{2\xi}}dt\right\}=$$

$$2^{p}c_1\omega_1\left(f''+f,\xi\right)_p^p$$

$$\left\{\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}\left[\int_0^{\infty}t^{(q+1)(p-1)+1}e^{-\frac{pt}{2\xi}}dt+\frac{j^p}{\xi^p\left(p+1\right)}\int_0^{\infty}t^{(q+1)(p-1)+p+1}e^{-\frac{pt}{2\xi}}dt\right]\right\}=$$

$$(40)$$

$$2^p c_1 \omega_1 \left(f'' + f, \xi\right)_p^p \left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\xi^{qp-q+p+1} \int_0^\infty \left(\frac{t}{\xi}\right)^{(q+1)(p-1)+1} e^{-\left(\frac{p}{2}\right)\frac{t}{\xi}} d\frac{t}{\xi} \right.\right.$$

$$\left.\left. + \frac{j^p}{\xi^p (p+1)} \xi^{qp-q+2p+1} \int_0^\infty \left(\frac{t}{\xi}\right)^{qp-q+2p} e^{-\left(\frac{p}{2}\right)\frac{t}{\xi}} d\frac{t}{\xi} \right] \right\} =$$

(above it is $(q+1)(p-1)+1 = qp-q+p > 0$, $(q+1)(p-1)+p+1 = qp-q+2p > 0$)

$$2^p c_1 \omega_1 \left(f'' + f, \xi\right)_p^p \xi^{qp-q+p+1} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left\{ \int_0^\infty x^{qp-q+p} e^{-\left(\frac{p}{2}\right)x} dx \right.\right. \tag{41}$$

$$\left.\left. + \frac{j^p}{(p+1)} \int_0^\infty x^{qp-q+2p} e^{-\left(\frac{p}{2}\right)x} dx \right] \right\} =$$

(next we use, for $a, b > 0$ that it holds $\int_0^\infty x^a e^{-bx} dx = b^{-a-1} \Gamma(a+1)$, by Wolfram Alpha computational intelligence)

$$2^p c_1 \omega_1 \left(f'' + f, \xi\right)_p^p \xi^{qp-q+p+1} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left\{ \left(\frac{p}{2}\right)^{-qp+q-p-1} \Gamma(qp-q+p+1) \right.\right.$$

$$\left.\left. + \frac{j^p}{(p+1)} \left(\frac{p}{2}\right)^{-qp+q-2p-1} \Gamma(qp-q+2p+1) \right] \right\} =$$

$$\frac{2^{2(p-1)}}{q^{p-1}} \left(\frac{r+1}{q+1}\right)^{\frac{p}{q}} \omega_1 \left(f'' + f, \xi\right)_p^p \xi^{qp-q+p} \tag{42}$$

$$\left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left\{ \left(\frac{p}{2}\right)^{-qp+q-p-1} \left(\Gamma(qp-q+p+1) + \frac{j^p}{(p+1)} \left(\frac{p}{2}\right)^{-p} \Gamma(qp-q+2p+1) \right) \right\} \right].$$

So for $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\xi > 0$, we have proved:

$$\int_{-\infty}^{\infty} |\Delta(x)|^p dx \leq \frac{2^{2(p-1)}}{q^{p-1}} \left(\frac{r+1}{q+1}\right)^{\frac{p}{q}} \omega_1 \left(f'' + f, \xi\right)_p^p \xi^{qp-q+p} \left(\frac{p}{2}\right)^{-qp+q-p-1}$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\Gamma(qp-q+p+1) + \frac{(2j)^p}{p^p (p+1)} \Gamma(qp-q+2p+1) \right] \right\}. \tag{43}$$

The proof of the theorem is now completed. ■

Next comes the L_1 related approximation.

Theorem 2 *Let $\xi > 0$, both $f, f'' \in L_1(\mathbb{R}) \cap C(\mathbb{R})$, $x \in \mathbb{R}$. Then*

$$\|\Delta(x)\|_1 \leq \left[\sum_{j=0}^{r} |\alpha_j| j^2 (2+3j) \right] \omega_1 \left(f'' + f, \xi\right)_1 \xi^2 \to 0, \quad \text{as } \xi \to 0. \tag{44}$$

Proof. We have that

$$|\Delta(x)| = \frac{1}{2\xi}\left|\int_{-\infty}^{\infty} R(t)\, e^{-\frac{|t|}{\xi}} dt\right| \leq \frac{1}{2\xi}\int_{-\infty}^{\infty} |R(t)|\, e^{-\frac{|t|}{\xi}} dt \leq$$

$$\frac{1}{2\xi}\sum_{j=0}^{r} |\alpha_j|\, j^2 \int_{-\infty}^{\infty}\left(\int_0^{|t|} |(f''(x+jsign(t)\theta) + f(x+jsign(t)\theta)) - \right. \tag{45}$$

$$\left.(f''(x) + f(x))|\,(|t|-\theta)\, d\theta\right) e^{-\frac{|t|}{\xi}} dt.$$

Thus, we get

$$\|\Delta(x)\|_1 = \int_{-\infty}^{\infty} |\Delta(x)|\, dx \leq$$

$$\frac{1}{2\xi}\sum_{j=0}^{r} |\alpha_j|\, j^2 \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|} |(f''(x+jsign(t)\theta) + f(x+jsign(t)\theta)) - \right.\right.\right. \tag{46}$$

$$\left.\left.\left.(f''(x) + f(x))|\,(|t|-\theta)\, d\theta\right) e^{-\frac{|t|}{\xi}} dt\right) dx\right) \leq$$

$$\frac{1}{2\xi}\sum_{j=0}^{r} |\alpha_j|\, j^2 \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(|t|\int_0^{|t|} |(f''(x+jsign(t)\theta) + f(x+jsign(t)\theta)) - \right.\right.\right.$$

$$\left.\left.\left.(f''(x) + f(x))|\, d\theta\right) e^{-\frac{|t|}{\xi}} dt\right) dx\right) =$$

$$\frac{1}{2\xi}\sum_{j=0}^{r} |\alpha_j|\, j^2 \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(|t|\int_0^{|t|} |(f''(x+jsign(t)\theta) + f(x+jsign(t)\theta)) - \right.\right.\right.$$

$$\left.\left.\left.(f''(x) + f(x))|\, d\theta\right) e^{-\frac{|t|}{\xi}} dx\right) dt\right) = \tag{47}$$

$$\frac{1}{2\xi}\sum_{j=0}^{r} |\alpha_j|\, j^2 \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|} |(f''(x+jsign(t)\theta) + f(x+jsign(t)\theta)) - \right.\right.\right.$$

$$\left.\left.(f''(x) + f(x))|\, d\theta)\, dx\right)|t|\, e^{-\frac{|t|}{\xi}} dt\right) =$$

$$\frac{1}{2\xi}\sum_{j=0}^{r} |\alpha_j|\, j^2 \left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(\int_{-\infty}^{\infty} |(f''(x+jsign(t)\theta) + f(x+jsign(t)\theta)) - \right.\right.\right.$$

$$\left.\left.(f''(x) + f(x))|\, dx)\, d\theta\right)|t|\, e^{-\frac{|t|}{\xi}} dt\right) \leq$$

$$\frac{1}{2\xi}\sum_{j=0}^{r} |\alpha_j|\, j^2 \left(\int_{-\infty}^{\infty}\left(\int_0^{|t|} \omega_1(f''+f, j\theta)_1\, d\theta\right)|t|\, e^{-\frac{|t|}{\xi}} dt\right) \leq$$

$$\frac{\omega_1(f''+f,\xi)_1}{2\xi}\sum_{j=0}^{r} |\alpha_j|\, j^2 \left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(1+\frac{j}{\xi}\theta\right) d\theta\right)|t|\, e^{-\frac{|t|}{\xi}} dt\right) =$$

$$\frac{\omega_1 (f'' + f, \xi)_1}{2\xi} \sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_{-\infty}^{\infty} \left[|t| + \frac{j}{\xi} \frac{|t|^2}{2} \right] |t| e^{-\frac{|t|}{\xi}} dt \right) = \tag{48}$$

$$\xi^2 \omega_1 (f'' + f, \xi)_1 \sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_0^{\infty} \left[\frac{t}{\xi} + \frac{j}{2} \left(\frac{t}{\xi} \right)^2 \right] \frac{t}{\xi} e^{-\frac{t}{\xi}} d\frac{t}{\xi} \right) =$$

$$\xi^2 \omega_1 (f'' + f, \xi)_1 \sum_{j=0}^{r} |\alpha_j| j^2 \left[\int_0^{\infty} \left(t + \frac{j}{2} t^2 \right) t e^{-t} dt \right] =$$

$$\xi^2 \omega_1 (f'' + f, \xi)_1 \sum_{j=0}^{r} |\alpha_j| j^2 \left[\int_0^{\infty} t^2 e^{-t} dt + \frac{j}{2} \int_0^{\infty} t^3 e^{-t} dt \right] =$$

$$\xi^2 \omega_1 (f'' + f, \xi)_1 \sum_{j=0}^{r} |\alpha_j| j^2 \left[2 + \frac{j}{2} \cdot 6 \right] =$$

$$\xi^2 \omega_1 (f'' + f, \xi)_1 \sum_{j=0}^{r} |\alpha_j| j^2 [2 + 3j] . \tag{49}$$

Above we used that

$$\int_0^{\infty} t^2 e^{-t} dt = 2 \text{ and } \int_0^{\infty} t^3 e^{-t} dt = 6.$$

The theorem is proved. ■

We finish with the following

Corollary 3 *(to Theorem 1) It holds*

$$\|P_{r,\xi}(f) - f\|_p \le \|f''\|_p \left(\sum_{j=0}^{r} |\alpha_j| \frac{j^2}{(j^2\xi^2 + 1)} \right) \xi^2 +$$

$$\left(\frac{4(r+1)}{q(q+1)} \right)^{\frac{1}{q}} \left(\frac{2}{p} \right)^{2+\frac{1}{p}} \omega_1 (f'' + f, \xi)_p \xi^2 \tag{50}$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\Gamma(qp - q + p + 1) + \frac{(2j)^p}{p^p (p+1)} \Gamma(qp - q + 2p + 1) \right] \right\}^{\frac{1}{p}} \to 0,$$

as $\xi \to 0$.

Corollary 4 *(to Theorem 2) It holds*

$$\|P_{r,\xi}(f) - f\|_1 \le \|f''\|_1 \left(\sum_{j=0}^{r} |\alpha_j| \frac{j^2}{(j^2\xi^2 + 1)} \right) \xi^2 +$$

$$\left[\sum_{j=0}^{r} |\alpha_j| j^2 (2 + 3j) \right] \omega_1 (f'' + f, \xi)_1 \xi^2 \to 0, \quad \text{as } \xi \to 0. \tag{51}$$

References

[1] G.A. Anastassiou, *Basic convergence with rates of smooth Picard singular integral operators*, J. Computational Analysis and Appl., 8(4) (2006), 313-334.

[2] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, New York, Chapter 12, 2011.

[3] G.A. Anastassiou, *Opial and Ostrowski type inequalities based on trigonometric and hyperbolic type Taylor formulae*, submitted, 2023.

[4] Ali Hasan Ali, Zsolt Pales, *Taylor-type expansions in terms of exponential polynomials*, Mathematical Inequalities and Applications, 25(4) (2022), 1123-1141.