Parametrized and Trigonometric generated quantitative convergence of smooth Picard singular integral operators

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis,
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

In this article we continue the research on the smooth Picard singular integral operators that started in [2], see there chapters 10-14. This time the foundation of our research is a parametrized trigonometric Taylor's formula. We establish the convergence of our operators to the unit operator with rates via Jackson type inequalities engaging the first modulus of continuity. Of interest here is a parametrized residual appearing term. Note that our operators are not positive. Our results are pointwise and uniform.

Mathematics Subject Classification (2022): 26A15, 26D15, 41A17, 41A35.

Keywords and phrases: Singular integral, Picard operator, modulus of continuity, parametrized trigonometric Taylor formula.

1 Introduction

We are motivated by [1], [2] chapters 10-14, and [3], [4]. We use a parametrized trigonometric new Taylor formula from [3], see also [4]. Here we consider some very general operators, the smooth Picard singular integral operators over the real line and we study further their parametrized convergence properties quantitatively. We establish related inequalities involving the first modulus of continuity with respect to uniform norm and the estimates are pointwise and uniform. We provide a detailed proof.

RGMIA Res. Rep. Coll. **26** (2023), Art. 56, 13 pp. Received 10/09/23

2 Results

By [3], [4], for $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right) \neq 0$, and $f \in C^4(\mathbb{R})$, $a, x \in \mathbb{R}$, we have the following general trigonometric Taylor formula:

$$f(x)-f(a)=f'(a)\left(\frac{\beta^3 \sin(\alpha(x-a))-\alpha^3 \sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$
$$f''(a)\left(\frac{\cos(\alpha(x-a))-\cos(\beta(x-a))}{\beta^2-\alpha^2}\right)+$$
$$f'''(a)\left(\frac{\beta \sin(\alpha(x-a))-\alpha \sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+ \tag{1}$$
$$\frac{2\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\left(\beta^2 \sin^2\left(\frac{\alpha(x-a)}{2}\right)-\alpha^2 \sin^2\left(\frac{\beta(x-a)}{2}\right)\right)+$$
$$\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\int_a^x \left[\left(f''''(t)+\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2 f(t)\right)\right.$$
$$\left.-\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2 f(a)\right)\right]$$
$$\left[\beta \sin(\alpha(x-t))-\alpha \sin(\beta(x-t))\right] dt.$$

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j=1,...,r. \\ 1-\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j} j^{-n}, & n=0, \end{cases} \tag{2}$$

that is

$$\sum_{j=0}^{r} \alpha_j = 1. \tag{3}$$

$C_U(\mathbb{R})$ denotes the space of uniformly continuous functions on $\mathbb{R}$, and $C_B(\mathbb{R})$ denotes the space of bounded continuous functions on $\mathbb{R}$.

Here we consider all $f, f', f'', f''', f^{(4)} \in C_U(\mathbb{R}) \cup C_B(\mathbb{R})$.

For $x \in \mathbb{R}$, $\xi > 0$ we consider the Lebesgue integrals, so called smooth Picard operators

$$P_{r,\xi}(f,x)=\frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+jt)\right)e^{-\frac{|t|}{\xi}} dt, \tag{4}$$

see [1], $P_{r,\xi}$ are not in general positive operators, see [2].

We notice by

$$\frac{1}{2\xi}\int_{-\infty}^{\infty} e^{-\frac{|t|}{\xi}} dt = 1, \tag{5}$$

that

$$P_{r,\xi}(c,x)=c, \text{ where } c \text{ is a constant,}$$

and

$$P_{r,\xi}(f,x)-f(x)=\frac{1}{2\xi}\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}\left(f(x+jt)-f(x)\right)e^{-\frac{|t|}{\xi}}dt\right). \quad (6)$$

Denote by

$$\omega_1(f,\delta):=\sup_{\substack{x,y\in\mathbb{R}\\|x-y|\leq\delta}}|f(x)-f(y)|, \quad \delta>0,$$

the first modulus of continuity of f.

We set

$$E(x):=P_{r,\xi}(f,x)-f(x)-f''(x)\left\{\sum_{j=0}^{r}\alpha_j j^2\left(\frac{1}{(\alpha^2j^2\xi^2+1)(\beta^2j^2\xi^2+1)}\right)\right\}\xi^2-$$

$$\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)\left[\sum_{j=0}^{r}\alpha_j j^4\left(\frac{1}{(\alpha^2j^2\xi^2+1)(\beta^2j^2\xi^2+1)}\right)\right]\xi^4; \quad (7)$$

$\xi>0$, $x\in\mathbb{R}$.

We present our main uniform approximation result.

Theorem 1 *It holds*

$$|E(x)|\leq\|E(x)\|_\infty\leq\frac{2\xi^2\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|} \quad (8)$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\,j^2(j+1)\right)=:B;\quad \xi>0,\ x\in\mathbb{R},$$

and $\|E(x)\|_\infty\to0$, *as* $\xi\to0$.

If $f''(x)=f^{(4)}(x)=0$, *then*

$$|P_{r,\xi}(f,x)-f(x)|\leq B.$$

Proof. By (1) we get that

$$f(x+jt)-f(x)=f'(x)\left(\frac{\beta^3\sin(\alpha jt)-\alpha^3\sin(\beta jt)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$

$$f''(x)\left(\frac{\cos(\alpha jt)-\cos(\beta jt)}{\beta^2-\alpha^2}\right)+$$

$$f'''(x)\left(\frac{\beta\sin(\alpha jt)-\alpha\sin(\beta jt)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$

$$\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\left(\beta^2\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sin^2\left(\frac{\beta jt}{2}\right)\right)+$$

$$\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\int_x^{x+jt}\Big[\left(f^{(4)}(s)+\left(\alpha^2+\beta^2\right)f''(s)+\alpha^2\beta^2 f(s)\right) \tag{9}$$

$$-\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\right)\Big]$$

$$\left[\beta\sin(\alpha(x+jt-s))-\alpha\sin(\beta(x+jt-s))\right]ds,$$

or better,

$$f(x+jt)-f(x)=f'(x)\left(\frac{\beta^3\sin(\alpha jt)-\alpha^3\sin(\beta jt)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$

$$f''(x)\left(\frac{\cos(\alpha jt)-\cos(\beta jt)}{\beta^2-\alpha^2}\right)+$$

$$f'''(x)\left(\frac{\beta\sin(\alpha jt)-\alpha\sin(\beta jt)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$

$$\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\left(\beta^2\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sin^2\left(\frac{\beta jt}{2}\right)\right)+$$

$$\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\int_0^{jt}\Big[\left(f^{(4)}(x+z)+\left(\alpha^2+\beta^2\right)f''(x+z)+\alpha^2\beta^2 f(x+z)\right)$$

$$-\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\right)\Big] \tag{10}$$

$$\left[\beta\sin(\alpha(jt-z))-\alpha\sin(\beta(jt-z))\right]dz.$$

Furthermore it holds

$$\sum_{j=0}^{r}\alpha_j\left[f(x+jt)-f(x)\right]=$$

$$\left(\frac{f'(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta^3\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha^3\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]+$$

$$\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\cos(\alpha jt)-\sum_{j=0}^{r}\alpha_j\cos(\beta jt)\right]+$$

$$\left(\frac{f^{(3)}(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]+$$

$$\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\left(\beta^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\beta jt}{2}\right)\right)+ \tag{11}$$

$$\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\sum_{j=0}^{r}\alpha_j\int_0^{jt}\Big[\Big(f^{(4)}(x+z)+\left(\alpha^2+\beta^2\right)f''(x+z)+\alpha^2\beta^2 f(x+z)\Big)$$

$$-\Big(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\Big)\Big]$$

$$\left[\beta\sin\left(\alpha\left(jt-z\right)\right)-\alpha\sin\left(\beta\left(jt-z\right)\right)\right]dz,$$

or better

$$\sum_{j=0}^{r}\alpha_j\left[f(x+jt)-f(x)\right]=$$

$$\left(\frac{f'(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta^3\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha^3\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]+$$

$$\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\cos(\alpha jt)-\sum_{j=0}^{r}\alpha_j\cos(\beta jt)\right]+$$

$$\left(\frac{f^{(3)}(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]+$$

$$\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\left(\beta^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\beta jt}{2}\right)\right)+ \tag{12}$$

$$\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\sum_{j=0}^{r}\alpha_j j\int_0^{t}\Big[\Big(f^{(4)}(x+jw)+\left(\alpha^2+\beta^2\right)f''(x+jw)+\alpha^2\beta^2 f(x+jw)\Big)$$

$$-\Big(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\Big)\Big]$$

$$\left[\beta\sin\left(\alpha j\left(t-w\right)\right)-\alpha\sin\left(\beta j\left(t-w\right)\right)\right]dw.$$

We call

$$R:=R(t):=\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\sum_{j=0}^{r}\alpha_j j$$

$$\int_0^{t}\Big[\Big(f^{(4)}(x+jw)+\left(\alpha^2+\beta^2\right)f''(x+jw)+\alpha^2\beta^2 f(x+jw)\Big) \tag{13}$$

$$-\Big(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\Big)\Big]$$

$$\left[\beta\sin\left(\alpha j\left(t-w\right)\right)-\alpha\sin\left(\beta j\left(t-w\right)\right)\right]dw,\quad\forall\ t\in\mathbb{R}.$$

Then, for $t \geq 0$,

$$|R| \leq \frac{1}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|} \sum_{j=0}^{r} |\alpha_j| \, j$$

$$\int_0^t \left| \left(f^{(4)}(x+jw) + \left(\alpha^2+\beta^2\right) f''(x+jw) + \alpha^2\beta^2 f(x+jw) \right) \right.$$

$$\left. - \left(f^{(4)}(x) + \left(\alpha^2+\beta^2\right) f''(x) + \alpha^2\beta^2 f(x) \right) \right|$$

$$|\beta \sin(\alpha j(t-w)) - \alpha \sin(\beta j(t-w))| \, dw \leq$$

(by $|\sin x| \leq |x|$, $\forall \, x \in \mathbb{R}$)

$$\frac{2|\alpha\beta|}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|} \sum_{j=0}^{r} |\alpha_j| \, j^2 \int_0^t \omega_1 \left(\left(f^{(4)} + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f \right), jw \right) (t-w) \, dw$$

($\xi > 0$)

$$= \frac{2}{\left|\beta^2-\alpha^2\right|} \sum_{j=0}^{r} |\alpha_j| \, j^2 \int_0^t \omega_1 \left(\left(f^{(4)} + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f \right), \frac{\xi j w}{\xi} \right) (t-w) \, dw \leq \tag{14}$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|} \sum_{j=0}^{r} |\alpha_j| \, j^2 \omega_1 \left(\left(f^{(4)} + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f \right), \xi \right)$$

$$\int_0^t \left(1 + \frac{jw}{\xi} \right) (t-w) \, dw =$$

$$\frac{2\omega_1 \left(\left(f^{(4)} + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f \right), \xi \right)}{\left|\beta^2-\alpha^2\right|}$$

$$\sum_{j=0}^{r} |\alpha_j| \, j^2 \left[\int_0^t (t-w) \, dw + \frac{j}{\xi} \int_0^t w^{2-1} (t-w)^{2-1} \, dw \right] =$$

$$\frac{2\omega_1 \left(\left(f^{(4)} + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f \right), \xi \right)}{\left|\beta^2-\alpha^2\right|} \sum_{j=0}^{r} |\alpha_j| \, j^2 \left[\frac{t^2}{2} + j \frac{t^3}{6\xi} \right] = \tag{15}$$

$$\frac{\omega_1 \left(\left(f^{(4)} + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f \right), \xi \right)}{\left|\beta^2-\alpha^2\right|} \sum_{j=0}^{r} |\alpha_j| \, j^2 \left[t^2 + j \frac{t^3}{3\xi} \right].$$

Hence it holds ($t \geq 0$)

$$|R| \leq \frac{\omega_1 \left(\left(f^{(4)} + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f \right), \xi \right)}{\left|\beta^2-\alpha^2\right|} \sum_{j=0}^{r} |\alpha_j| \, j^2 \left[t^2 + \frac{j}{3\xi} t^3 \right], \tag{16}$$

$\xi > 0$.

Let now $t < 0$, then

$$|R| \leq \frac{1}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}\sum_{j=0}^{r}|\alpha_j|\, j$$

$$\int_t^0 \Big|\Big(f^{(4)}(x+jw)+\left(\alpha^2+\beta^2\right)f''(x+jw)+\alpha^2\beta^2 f(x+jw)\Big)$$

$$-\Big(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\Big)\Big|$$

$$|\beta\sin(\alpha j(t-w))-\alpha\sin(\beta j(t-w))|\, dw \leq$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\sum_{j=0}^{r}|\alpha_j|\, j^2\int_t^0 \omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right), -j\frac{w\xi}{\xi}\right)(w-t)\, dw \leq$$

$$\frac{2\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|}$$

$$\sum_{j=0}^{r}|\alpha_j|\, j^2\left(\int_t^0\left(1-\frac{j}{\xi}w\right)(w-t)\, dw\right) = \tag{17}$$

$$\frac{2\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|}$$

$$\sum_{j=0}^{r}|\alpha_j|\, j^2\left[\int_t^0(w-t)\, dw+\frac{j}{\xi}\int_t^0(0-w)^{2-1}(w-t)^{2-1}\, dw\right] =$$

$$\frac{\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|}\sum_{j=0}^{r}|\alpha_j|\, j^2\left[t^2+\frac{j}{3\xi}(-t)^3\right].$$

We found that (for $t < 0$)

$$|R| \leq \frac{\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|}\sum_{j=0}^{r}|\alpha_j|\, j^2\left[t^2+\frac{j}{3\xi}(-t)^3\right]. \tag{18}$$

Consequently, for $t \in \mathbb{R}$, we obtain

$$|R(t)| \leq \frac{\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|}\sum_{j=0}^{r}|\alpha_j|\, j^2\left[t^2+\frac{j}{3\xi}|t|^3\right],\ \ \xi>0. \tag{19}$$

So, we have (by (12))

$$\sum_{j=0}^{r}\alpha_j\left[f(x+jt)-f(x)\right]-$$

$$\left(\frac{f'(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta^3\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha^3\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]-$$

$$\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\cos(\alpha jt)-\sum_{j=0}^{r}\alpha_j\cos(\beta jt)\right]-$$

$$\left(\frac{f^{(3)}(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]- \tag{20}$$

$$\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\left(\beta^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\beta jt}{2}\right)\right)$$

$$=R(t).$$

Therefore, it holds

$$E_1(x):=P_{r,\xi}(f,x)-f(x)-\left(\frac{f'(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)$$

$$\left[\beta^3\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin(\alpha jt)e^{-\frac{|t|}{\xi}}dt\right)-\alpha^3\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin(\beta jt)e^{-\frac{|t|}{\xi}}dt\right)\right]-$$

$$\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)$$

$$\left[\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\cos(\alpha jt)e^{-\frac{|t|}{\xi}}dt\right)-\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\cos(\beta jt)e^{-\frac{|t|}{\xi}}dt\right)\right]-$$

$$\left(\frac{f^{(3)}(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)$$

$$\left[\beta\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin(\alpha jt)e^{-\frac{|t|}{\xi}}dt\right)-\alpha\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin(\beta jt)e^{-\frac{|t|}{\xi}}dt\right)\right]- \tag{21}$$

$$\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)$$

$$\left[\beta^2\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)e^{-\frac{|t|}{\xi}}dt\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\beta jt}{2}\right)e^{-\frac{|t|}{\xi}}dt\right)\right]$$

$$=\frac{1}{2\xi}\int_{-\infty}^{\infty}R(t)e^{-\frac{|t|}{\xi}}dt.$$

Hence we get

$$|E_1(x)| \leq \frac{1}{2\xi}\int_{-\infty}^{\infty} |R(t)|\, e^{-\frac{|t|}{\xi}} dt \overset{(19)}{\leq}$$

$$\frac{\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|}$$

$$\sum_{j=0}^{r}|\alpha_j|\, j^2\left[\frac{1}{2\xi}\int_{-\infty}^{\infty} t^2 e^{-\frac{|t|}{\xi}} dt + \frac{j}{3\xi}\frac{1}{2\xi}\int_{-\infty}^{\infty}|t|^3 e^{-\frac{|t|}{\xi}} dt\right] = \tag{22}$$

(we use $\int_{-\infty}^{\infty} t^k e^{-\frac{|t|}{\xi}} dt = \left\{\begin{matrix} 0, & k \text{ odd} \\ 2k!\xi^{k+1}, & k \text{ even}\end{matrix}\right\}$)

$$\frac{\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|}$$

$$\sum_{j=0}^{r}|\alpha_j|\, j^2\left[\frac{1}{2\xi}4\xi^3 + \frac{j\xi^4}{3\xi^2}\int_0^{\infty}\left(\frac{t}{\xi}\right)^3 e^{-\frac{t}{\xi}} d\frac{t}{\xi}\right] = \tag{23}$$

$$\frac{\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|}$$

$$\sum_{j=0}^{r}|\alpha_j|\, j^2\left[2\xi^2+\xi^2\frac{j}{3}\int_0^{\infty} z^3 e^{-z} dz\right] =$$

$$\frac{\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|}\sum_{j=0}^{r}|\alpha_j|\, j^2\left[2\xi^2+\xi^2\frac{j}{3}6\right] = \tag{24}$$

$$\frac{2\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)\xi^2}{\left|\beta^2-\alpha^2\right|}\left(\sum_{j=0}^{r}|\alpha_j|\, j^2(j+1)\right).$$

We have proved that

$$|E_1(x)| \leq \frac{2\xi^2\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|}\left(\sum_{j=0}^{r}|\alpha_j|\, j^2(j+1)\right); \tag{25}$$

$\xi > 0$, $\forall\, x \in \mathbb{R}$.

Notice that

$$E_1(x) \to 0, \quad \text{as } \xi \to 0.$$

Next, we simplify $E_1(x)$:

We observe that

$$\int_{-\infty}^{\infty}\sin(\alpha j t)\, e^{-\frac{|t|}{\xi}} dt = \int_{-\infty}^{0}\sin(\alpha j t)\, e^{-\frac{|t|}{\xi}} dt + \int_0^{\infty}\sin(\alpha j t)\, e^{-\frac{|t|}{\xi}} dt. \tag{26}$$

Notice $-\infty \leq t \leq 0 \Rightarrow \infty \geq -t \geq 0$. So that

$$\int_{-\infty}^{0} \sin(\alpha j t)\, e^{-\frac{|t|}{\xi}} dt = -\int_{-\infty}^{0} \sin(\alpha j(-(-t)))\, e^{-\frac{|t|}{\xi}} d(-t) =$$

$$-\int_{-\infty}^{0} (-\sin(\alpha j(-t)))\, e^{-\frac{|t|}{\xi}} d(-t) = \int_{-\infty}^{0} \sin(\alpha j(-t))\, e^{-\frac{|t|}{\xi}} d(-t) = \tag{27}$$

$$\int_{\infty}^{0} \sin(\alpha j t)\, e^{-\frac{|t|}{\xi}} dt = -\int_{0}^{\infty} \sin(\alpha j t)\, e^{-\frac{|t|}{\xi}} dt.$$

So that

$$\int_{-\infty}^{\infty} \sin(\alpha j t)\, e^{-\frac{|t|}{\xi}} dt = 0,$$

and all sine integrals in (21) are zeros.

Furthermore we have that

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha j t}{2}\right) e^{-\frac{|t|}{\xi}} dt = 2\int_{0}^{\infty} \sin^2\left(\frac{\alpha j t}{2}\right) e^{-\frac{|t|}{\xi}} dt, \quad j = 0, 1, ..., r. \tag{28}$$

The last follows by

$$\int_{-\infty}^{0} \sin^2\left(\frac{\alpha j t}{2}\right) e^{-\frac{|t|}{\xi}} dt = -\int_{-\infty}^{0} \left(-\sin\left(\frac{-\alpha j t}{2}\right)\right)^2 e^{-\frac{|-t|}{\xi}} d(-t) \overset{(z=-t)}{=} \tag{29}$$

$$-\int_{\infty}^{0} \sin^2\left(\frac{\alpha j z}{2}\right) e^{-\frac{|z|}{\xi}} dz = \int_{0}^{\infty} \sin^2\left(\frac{\alpha j z}{2}\right) e^{-\frac{z}{\xi}} dz.$$

Next, we calculate

$$\int_{0}^{\infty} \sin^2\left(\frac{\alpha j}{2} z\right) e^{-\frac{|z|}{\xi}} dz = \xi \int_{0}^{\infty} \sin^2\left(\left(\frac{\alpha j \xi}{2}\right) \frac{z}{\xi}\right) e^{-\frac{z}{\xi}} d\frac{z}{\xi} = \tag{30}$$

(call $\frac{t}{\xi} =: x$ and $\frac{\alpha j \xi}{2} =: a_1$)

$$\xi \int_{0}^{\infty} \sin^2(a_1 x)\, e^{-x} dx =$$

(by Wolfram Alpha Computational Inteligence)

$$\xi\left(\frac{2a_1^2}{4a_1^2+1}\right) = \frac{\alpha^2 j^2 \xi^3}{2\left(\alpha^2 j^2 \xi^2 + 1\right)}.$$

Thus, it is

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha j t}{2}\right) e^{-\frac{|t|}{\xi}} dt = \frac{\alpha^2 j^2 \xi^3}{\alpha^2 j^2 \xi^2 + 1}, \quad j = 0, 1, ..., r. \tag{31}$$

Consequently, it holds

$$\frac{1}{2\xi}\int_{-\infty}^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)e^{-\frac{|t|}{\xi}}dt=\frac{\alpha^2j^2\xi^2}{2\left(\alpha^2j^2\xi^2+1\right)}\to 0,\text{ as }\xi\to 0,\quad j=0,1,...,r. \tag{32}$$

We notice that

$$\int_{-\infty}^{\infty}\cos\left(\alpha jt\right)e^{-\frac{|t|}{\xi}}dt=2\int_{0}^{\infty}\cos\left(\alpha jt\right)e^{-\frac{t}{\xi}}dt=$$

$$2\xi\int_{0}^{\infty}\cos\left(\left(\alpha j\xi\right)\frac{t}{\xi}\right)e^{-\frac{t}{\xi}}d\frac{t}{\xi}= \tag{33}$$

(call $\alpha j\xi =: \alpha_2$, $\frac{t}{\xi} =: x$)

$$2\xi\int_{0}^{\infty}\cos\left(\alpha_2 x\right)e^{-x}dx=2\xi\left(\frac{1}{\alpha_2^2+1}\right)=2\xi\left(\frac{1}{\alpha^2j^2\xi^2+1}\right),$$

and

$$\frac{1}{2\xi}\int_{-\infty}^{\infty}\cos\left(\alpha jt\right)e^{-\frac{|t|}{\xi}}dt=\frac{1}{\alpha^2j^2\xi^2+1}, \tag{34}$$

$j=0,1,...,r$.

So, based on the above we write down the simplified version of $E_1(x)$ as follows:

$$E_1(x)=P_{r,\xi}(f,x)-f(x)-$$

$$\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left\{\sum_{j=0}^{r}\alpha_j\left[\left(\frac{1}{\alpha^2j^2\xi^2+1}\right)-\left(\frac{1}{\beta^2j^2\xi^2+1}\right)\right]\right\}-$$

$$\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{\left(\alpha\beta\right)^2\left(\beta^2-\alpha^2\right)}\right)$$

$$\left[\beta^2\sum_{j=0}^{r}\alpha_j\left(\frac{\alpha^2j^2\xi^2}{2\left(\alpha^2j^2\xi^2+1\right)}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\left(\frac{\beta^2j^2\xi^2}{2\left(\beta^2j^2\xi^2+1\right)}\right)\right]= \tag{35}$$

$$P_{r,\xi}(f,x)-f(x)-f''(x)\left\{\sum_{j=0}^{r}\alpha_j\left[\frac{j^2\xi^2}{\left(\alpha^2j^2\xi^2+1\right)\left(\beta^2j^2\xi^2+1\right)}\right]\right\}- \tag{36}$$

$$\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{\left(\alpha\beta\right)^2\left(\beta^2-\alpha^2\right)}\right)$$

$$\left[\beta^2\sum_{j=0}^{r}\alpha_j\left(\frac{\alpha^2j^2\xi^2}{2\left(\alpha^2j^2\xi^2+1\right)}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\left(\frac{\beta^2j^2\xi^2}{2\left(\beta^2j^2\xi^2+1\right)}\right)\right]=$$

$$P_{r,\xi}(f,x) - f(x) - f''(x)\left\{\sum_{j=0}^{r}\alpha_j j^2\left[\frac{1}{(\alpha^2 j^2\xi^2+1)(\beta^2 j^2\xi^2+1)}\right]\right\}\xi^2 -$$

$$\left(f^{(4)}(x) + \left(\alpha^2+\beta^2\right)f''(x)\right)\left[\sum_{j=0}^{r}\alpha_j j^4\left(\frac{1}{(\alpha^2 j^2\xi^2+1)(\beta^2 j^2\xi^2+1)}\right)\right]\xi^4 = E(x). \tag{37}$$

The proof of the theorem is now complete. ■

In the special case of $\alpha = 2$, $\beta = 1$, we set

$$\overline{E(x)} := P_{r,\xi}(f,x) - f(x) -$$

$$f''(x)\left\{\sum_{j=0}^{r}\alpha_j\left[\frac{j^2}{(4j^2\xi^2+1)(j^2\xi^2+1)}\right]\right\}\xi^2 - \tag{38}$$

$$\left(f^{(4)}(x) + 5f''(x)\right)\left[\sum_{j=0}^{r}\alpha_j\frac{j^4}{(4j^2\xi^2+1)(j^2\xi^2+1)}\right]\xi^4;$$

$\xi > 0$, $x \in \mathbb{R}$.

That is $\overline{E(x)}$ is $E(x)$ for $\alpha = 2$, $\beta = 1$.

We mention

Corollary 2 *It holds*

$$\left\|\overline{E(x)}\right\|_\infty \le \frac{2\xi^2\omega_1\left(\left(f^{(4)}+5f''+4f\right),\xi\right)}{3}\left(\sum_{j=0}^{r}|\alpha_j|\,j^2(j+1)\right); \tag{39}$$

$\xi > 0$, $x \in \mathbb{R}$.

And $\left\|\overline{E(x)}\right\| \to 0$, *as* $\xi \to 0$.

We finish with

Corollary 3 *(to Theorem 1) Additionally let $f'', f^{(4)} \in C_B(\mathbb{R})$. It holds*

$$|P_{r,\xi}(f,x) - f(x)| \le \|P_{r,\xi}(f,x) - f(x)\|_\infty \le$$

$$\|f''\|_\infty\left\{\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\frac{1}{(\alpha^2 j^2\xi^2+1)(\beta^2 j^2\xi^2+1)}\right]\right\}\xi^2 +$$

$$\left(\left\|f^{(4)}\right\|_\infty + \left(\alpha^2+\beta^2\right)\|f''\|_\infty\right)\left[\sum_{j=0}^{r}|\alpha_j|\,j^4\left(\frac{1}{(\alpha^2 j^2\xi^2+1)(\beta^2 j^2\xi^2+1)}\right)\right]\xi^4 + \tag{40}$$

$$\frac{2\xi^2\omega_1\left(\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right),\xi\right)}{\left|\beta^2-\alpha^2\right|}$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\,(j+1)\right)\to 0, \quad \text{as } \xi\to 0.$$

References

[1] G.A. Anastassiou, *Basic convergence with rates of smooth Picard singular integral operators*, J. Computational Analysis and Appl., 8(4) (2006), 313-334.

[2] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, New York, Chapter 10, 2011.

[3] G.A. Anastassiou, *Opial and Ostrowski type inequalities based on trigonometric and hyperbolic type Taylor formulae*, submitted, 2023.

[4] Ali Hasan Ali, Zsolt Pales, *Taylor-type expansions in terms of exponential polynomials*, Mathematical Inequalities and Applications, 25(4) (2022), 1123-1141.