Parametrized and Trigonometric L_p quantitative convergence of smooth Picard singular integral operators

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Abstract

In this article we continue the research on the smooth Picard singular integral operators that started in [2], see there chapters 10-14. This time the foundation of our research is a parametrized trigonometric Taylor's formula. We establish the L_p, $p \geq 1$,, convergence of our operators to the unit operator with rates via Jackson type inequalities engaging the first L_p modulus of smoothness. Of interest here is a parametrized residual appearing term. Note that our operators are not positive.

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1 Introduction

We are motivated by [1], [2] chapters 10-14, and [3], [4]. We use a parametrized trigonometric new Taylor formula from [3], see also [4]. Here we consider some very general operators, the smooth Picard singular integral operators over the real line and we study further their parametrized L_p convergence properties quantitatively. We establish related L_p, $p \geq 1$, inequalities involving the first modulus of smoothness with respect to the L_p, $p \geq 1$, norm. We provide detailed proofs.

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2 Results

By [3], [4], for $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right) \neq 0$, and $f \in C^4(\mathbb{R})$, $a, x \in \mathbb{R}$, we have the following general trigonometric Taylor formula:

$$f(x)-f(a)=f'(a)\left(\frac{\beta^3\sin(\alpha(x-a))-\alpha^3\sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$
$$f''(a)\left(\frac{\cos(\alpha(x-a))-\cos(\beta(x-a))}{\beta^2-\alpha^2}\right)+$$
$$f'''(a)\left(\frac{\beta\sin(\alpha(x-a))-\alpha\sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+ \tag{1}$$
$$\frac{2\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\left(\beta^2\sin^2\left(\frac{\alpha(x-a)}{2}\right)-\alpha^2\sin^2\left(\frac{\beta(x-a)}{2}\right)\right)+$$
$$\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\int_a^x\left[\left(f''''(t)+\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2f(t)\right)\right.$$
$$\left.-\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2f(a)\right)\right]$$
$$[\beta\sin(\alpha(x-t))-\alpha\sin(\beta(x-t))]\,dt.$$

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j}\binom{r}{j}j^{-n}, & j=1,...,r. \\ 1-\sum\limits_{j=1}^{r}(-1)^{r-j}\binom{r}{j}j^{-n}, & n=0, \end{cases} \tag{2}$$

that is

$$\sum_{j=0}^{r}\alpha_j=1. \tag{3}$$

Here we consider all $f, f', f'', f''', f^{(4)} \in L_p(\mathbb{R}) \cap C(\mathbb{R})$, $1 \leq p < \infty$.

For $x \in \mathbb{R}$, $\xi > 0$ we consider the Lebesgue integrals, so called smooth Picard operators

$$P_{r,\xi}(f,x)=\frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+jt)\right)e^{-\frac{|t|}{\xi}}dt, \tag{4}$$

see [1], $P_{r,\xi}$ are not in general positive operators, see [2].

We notice by

$$\frac{1}{2\xi}\int_{-\infty}^{\infty}e^{-\frac{|t|}{\xi}}dt=1, \tag{5}$$

that

$$P_{r,\xi}(c,x)=c, \text{ where } c \text{ is a constant,} \tag{6}$$

and

$$P_{r,\xi}\left(f,x\right)-f\left(x\right)=\frac{1}{2\xi}\left(\sum_{j=0}^{r}\alpha_{j}\int_{-\infty}^{\infty}\left(f\left(x+jt\right)-f\left(x\right)\right)e^{-\frac{|t|}{\xi}}dt\right). \tag{7}$$

Denote by

$$\omega_{1}\left(f,h\right)_{p}:=\sup_{\substack{t\in\mathbb{R}\\|t|\leq h}}\left\|f\left(x+t\right)-f\left(x\right)\right\|_{p,x}, \tag{8}$$

the first L_p modulus of smoothness of f, $1\leq p<\infty$.

By (1) we get that

$$\begin{gathered}f\left(x+jt\right)-f\left(x\right)=f'\left(x\right)\left(\frac{\beta^{3}\sin\left(\alpha jt\right)-\alpha^{3}\sin\left(\beta jt\right)}{\alpha\beta\left(\beta^{2}-\alpha^{2}\right)}\right)+\\ f''\left(x\right)\left(\frac{\cos\left(\alpha jt\right)-\cos\left(\beta jt\right)}{\beta^{2}-\alpha^{2}}\right)+\\ f'''\left(x\right)\left(\frac{\beta\sin\left(\alpha jt\right)-\alpha\sin\left(\beta jt\right)}{\alpha\beta\left(\beta^{2}-\alpha^{2}\right)}\right)+\\ \frac{2\left(f^{(4)}\left(x\right)+\left(\alpha^{2}+\beta^{2}\right)f''\left(x\right)\right)}{\left(\alpha\beta\right)^{2}\left(\beta^{2}-\alpha^{2}\right)}\left(\beta^{2}\sin^{2}\left(\frac{\alpha jt}{2}\right)-\alpha^{2}\sin^{2}\left(\frac{\beta jt}{2}\right)\right)+\\ \frac{1}{\alpha\beta\left(\beta^{2}-\alpha^{2}\right)}\int_{x}^{x+jt}\left[\left(f^{(4)}\left(s\right)+\left(\alpha^{2}+\beta^{2}\right)f''\left(s\right)+\alpha^{2}\beta^{2}f\left(s\right)\right)\right.\\ \left.-\left(f^{(4)}\left(x\right)+\left(\alpha^{2}+\beta^{2}\right)f''\left(x\right)+\alpha^{2}\beta^{2}f\left(x\right)\right)\right]\\ \left[\beta\sin\left(\alpha\left(x+jt-s\right)\right)-\alpha\sin\left(\beta\left(x+jt-s\right)\right)\right]ds,\end{gathered} \tag{9}$$

or better,

$$\begin{gathered}f\left(x+jt\right)-f\left(x\right)=f'\left(x\right)\left(\frac{\beta^{3}\sin\left(\alpha jt\right)-\alpha^{3}\sin\left(\beta jt\right)}{\alpha\beta\left(\beta^{2}-\alpha^{2}\right)}\right)+\\ f''\left(x\right)\left(\frac{\cos\left(\alpha jt\right)-\cos\left(\beta jt\right)}{\beta^{2}-\alpha^{2}}\right)+\\ f'''\left(x\right)\left(\frac{\beta\sin\left(\alpha jt\right)-\alpha\sin\left(\beta jt\right)}{\alpha\beta\left(\beta^{2}-\alpha^{2}\right)}\right)+\\ \frac{2\left(f^{(4)}\left(x\right)+\left(\alpha^{2}+\beta^{2}\right)f''\left(x\right)\right)}{\left(\alpha\beta\right)^{2}\left(\beta^{2}-\alpha^{2}\right)}\left(\beta^{2}\sin^{2}\left(\frac{\alpha jt}{2}\right)-\alpha^{2}\sin^{2}\left(\frac{\beta jt}{2}\right)\right)+\\ \frac{1}{\alpha\beta\left(\beta^{2}-\alpha^{2}\right)}\int_{0}^{jt}\left[\left(f^{(4)}\left(x+z\right)+\left(\alpha^{2}+\beta^{2}\right)f''\left(x+z\right)+\alpha^{2}\beta^{2}f\left(x+z\right)\right)\right.\end{gathered} \tag{10}$$

$$-\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\right)\Big]$$
$$\left[\beta\sin\left(\alpha\left(jt-z\right)\right)-\alpha\sin\left(\beta\left(jt-z\right)\right)\right]dz.$$

Furthermore it holds

$$\sum_{j=0}^{r}\alpha_j\left[f(x+jt)-f(x)\right]=$$
$$\left(\frac{f'(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta^3\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha^3\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]+$$
$$\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\cos(\alpha jt)-\sum_{j=0}^{r}\alpha_j\cos(\beta jt)\right]+$$
$$\left(\frac{f^{(3)}(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]+$$
$$\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\left(\beta^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\beta jt}{2}\right)\right)+ \tag{11}$$
$$\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\sum_{j=0}^{r}\alpha_j\int_0^{jt}\Big[\left(f^{(4)}(x+z)+\left(\alpha^2+\beta^2\right)f''(x+z)+\alpha^2\beta^2 f(x+z)\right)$$
$$-\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)+\alpha^2\beta^2 f(x)\right)\Big]$$
$$\left[\beta\sin\left(\alpha\left(jt-z\right)\right)-\alpha\sin\left(\beta\left(jt-z\right)\right)\right]dz,$$

or better

$$\sum_{j=0}^{r}\alpha_j\left[f(x+jt)-f(x)\right]=$$
$$\left(\frac{f'(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta^3\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha^3\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]+$$
$$\left(\frac{f''(x)}{\beta^2-\alpha^2}\right)\left[\sum_{j=0}^{r}\alpha_j\cos(\alpha jt)-\sum_{j=0}^{r}\alpha_j\cos(\beta jt)\right]+$$
$$\left(\frac{f^{(3)}(x)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)\left[\beta\sum_{j=0}^{r}\alpha_j\sin(\alpha jt)-\alpha\sum_{j=0}^{r}\alpha_j\sin(\beta jt)\right]+$$
$$\left(\frac{2\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\right)\left(\beta^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\alpha jt}{2}\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{\beta jt}{2}\right)\right)+ \tag{12}$$

$$\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\sum_{j=0}^{r}\alpha_j j\int_0^t\Big[\Big(f^{(4)}\left(x+jw\right)+\left(\alpha^2+\beta^2\right)f''\left(x+jw\right)+\alpha^2\beta^2 f\left(x+jw\right)\Big)$$
$$-\Big(f^{(4)}\left(x\right)+\left(\alpha^2+\beta^2\right)f''\left(x\right)+\alpha^2\beta^2 f\left(x\right)\Big)\Big]$$
$$\left[\beta\sin\left(\alpha j\left(t-w\right)\right)-\alpha\sin\left(\beta j\left(t-w\right)\right)\right]dw.$$

We call

$$R:=R\left(t\right):=\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\sum_{j=0}^{r}\alpha_j j$$
$$\int_0^t\Big[\Big(f^{(4)}\left(x+jw\right)+\left(\alpha^2+\beta^2\right)f''\left(x+jw\right)+\alpha^2\beta^2 f\left(x+jw\right)\Big) \tag{13}$$
$$-\Big(f^{(4)}\left(x\right)+\left(\alpha^2+\beta^2\right)f''\left(x\right)+\alpha^2\beta^2 f\left(x\right)\Big)\Big]$$
$$\left[\beta\sin\left(\alpha j\left(t-w\right)\right)-\alpha\sin\left(\beta j\left(t-w\right)\right)\right]dw,\quad\forall\ t\in\mathbb{R}.$$

We set

$$E\left(x\right):=P_{r,\xi}\left(f,x\right)-f\left(x\right)-\left(\frac{f'\left(x\right)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)$$
$$\left[\beta^3\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin\left(\alpha jt\right)e^{-\frac{|t|}{\xi}}dt\right)-\alpha^3\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin\left(\beta jt\right)e^{-\frac{|t|}{\xi}}dt\right)\right]-$$
$$\left(\frac{f''\left(x\right)}{\beta^2-\alpha^2}\right)$$
$$\left[\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\cos\left(\alpha jt\right)e^{-\frac{|t|}{\xi}}dt\right)-\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\cos\left(\beta jt\right)e^{-\frac{|t|}{\xi}}dt\right)\right]-$$
$$\left(\frac{f^{(3)}\left(x\right)}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)$$
$$\left[\beta\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin\left(\alpha jt\right)e^{-\frac{|t|}{\xi}}dt\right)-\alpha\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin\left(\beta jt\right)e^{-\frac{|t|}{\xi}}dt\right)\right]- \tag{14}$$
$$\left(\frac{2\left(f^{(4)}\left(x\right)+\left(\alpha^2+\beta^2\right)f''\left(x\right)\right)}{\left(\alpha\beta\right)^2\left(\beta^2-\alpha^2\right)}\right)$$
$$\left[\beta^2\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\alpha jt}{2}\right)e^{-\frac{|t|}{\xi}}dt\right)-\alpha^2\sum_{j=0}^{r}\alpha_j\frac{1}{2\xi}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{\beta jt}{2}\right)e^{-\frac{|t|}{\xi}}dt\right)\right]$$
$$=\frac{1}{2\xi}\int_{-\infty}^{\infty}R\left(t\right)e^{-\frac{|t|}{\xi}}dt.$$

Next we simplify $E(x)$:

We observe that

$$\int_{-\infty}^{\infty} \sin(\alpha jt)\, e^{-\frac{|t|}{\xi}} dt = \int_{-\infty}^{0} \sin(\alpha jt)\, e^{-\frac{|t|}{\xi}} dt + \int_{0}^{\infty} \sin(\alpha jt)\, e^{-\frac{|t|}{\xi}} dt. \tag{15}$$

Notice $-\infty \leq t \leq 0 \Rightarrow \infty \geq -t \geq 0$. So that

$$\int_{-\infty}^{0} \sin(\alpha jt)\, e^{-\frac{|t|}{\xi}} dt = -\int_{-\infty}^{0} \sin(\alpha j(-(-t)))\, e^{-\frac{|t|}{\xi}} d(-t) =$$

$$-\int_{-\infty}^{0} (-\sin(\alpha j(-t)))\, e^{-\frac{|t|}{\xi}} d(-t) = \int_{-\infty}^{0} \sin(\alpha j(-t))\, e^{-\frac{|t|}{\xi}} d(-t) = \tag{16}$$

$$\int_{\infty}^{0} \sin(\alpha jt)\, e^{-\frac{|t|}{\xi}} dt = -\int_{0}^{\infty} \sin(\alpha jt)\, e^{-\frac{|t|}{\xi}} dt.$$

So that

$$\int_{-\infty}^{\infty} \sin(\alpha jt)\, e^{-\frac{|t|}{\xi}} dt = 0,$$

and all sine integrals in (14) are zeros.

Furthermore we have that

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha jt}{2}\right) e^{-\frac{|t|}{\xi}} dt = 2\int_{0}^{\infty} \sin^2\left(\frac{\alpha jt}{2}\right) e^{-\frac{|t|}{\xi}} dt, \quad j = 0, 1, ..., r. \tag{17}$$

The last follows by

$$\int_{-\infty}^{0} \sin^2\left(\frac{\alpha jt}{2}\right) e^{-\frac{|t|}{\xi}} dt = -\int_{-\infty}^{0} \left(-\sin\left(\frac{-\alpha jt}{2}\right)\right)^2 e^{-\frac{|-t|}{\xi}} d(-t) \overset{(z=-t)}{=} \tag{18}$$

$$-\int_{\infty}^{0} \sin^2\left(\frac{\alpha jz}{2}\right) e^{-\frac{|z|}{\xi}} dz = \int_{0}^{\infty} \sin^2\left(\frac{\alpha jz}{2}\right) e^{-\frac{z}{\xi}} dz.$$

Next, we calculate

$$\int_{0}^{\infty} \sin^2\left(\frac{\alpha j}{2} z\right) e^{-\frac{|z|}{\xi}} dz = \xi \int_{0}^{\infty} \sin^2\left(\left(\frac{\alpha j\xi}{2}\right)\frac{z}{\xi}\right) e^{-\frac{z}{\xi}} d\frac{z}{\xi} = \tag{19}$$

(call $\frac{t}{\xi} =: x$ and $\frac{\alpha j\xi}{2} =: a_1$)

$$\xi \int_{0}^{\infty} \sin^2(a_1 x)\, e^{-x} dx =$$

(by Wolfram Alpha Computational Inteligence)

$$\xi\left(\frac{2a_1^2}{4a_1^2+1}\right) = \frac{\alpha^2 j^2 \xi^3}{2\left(\alpha^2 j^2 \xi^2 + 1\right)}.$$

Thus, it is

$$\int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha j t}{2}\right) e^{-\frac{|t|}{\xi}} dt = \frac{\alpha^2 j^2 \xi^3}{\alpha^2 j^2 \xi^2 + 1}, \quad j = 0, 1, ..., r, \tag{20}$$

and

$$\frac{1}{2\xi} \int_{-\infty}^{\infty} \cos(\alpha j t) e^{-\frac{|t|}{\xi}} dt = \frac{1}{\alpha^2 j^2 \xi^2 + 1}, \tag{21}$$

$j = 0, 1, ..., r$.

So, based on the above we write down the simplified version of $E(x)$ as follows:

$$E(x) = P_{r,\xi}(f, x) - f(x) -$$

$$\left(\frac{f''(x)}{\beta^2 - \alpha^2}\right) \left\{ \sum_{j=0}^{r} \alpha_j \left[\left(\frac{1}{\alpha^2 j^2 \xi^2 + 1}\right) - \left(\frac{1}{\beta^2 j^2 \xi^2 + 1}\right) \right] \right\} -$$

$$\left(\frac{2\left(f^{(4)}(x) + \left(\alpha^2 + \beta^2\right) f''(x)\right)}{(\alpha\beta)^2 \left(\beta^2 - \alpha^2\right)}\right)$$

$$\left[\beta^2 \sum_{j=0}^{r} \alpha_j \left(\frac{\alpha^2 j^2 \xi^2}{2\left(\alpha^2 j^2 \xi^2 + 1\right)}\right) - \alpha^2 \sum_{j=0}^{r} \alpha_j \left(\frac{\beta^2 j^2 \xi^2}{2\left(\beta^2 j^2 \xi^2 + 1\right)}\right) \right] = \tag{22}$$

$$P_{r,\xi}(f, x) - f(x) - f''(x) \left\{ \sum_{j=0}^{r} \alpha_j \left[\frac{j^2 \xi^2}{\left(\alpha^2 j^2 \xi^2 + 1\right)\left(\beta^2 j^2 \xi^2 + 1\right)} \right] \right\} -$$

$$\left(\frac{2\left(f^{(4)}(x) + \left(\alpha^2 + \beta^2\right) f''(x)\right)}{(\alpha\beta)^2 \left(\beta^2 - \alpha^2\right)}\right) \tag{23}$$

$$\left[\beta^2 \sum_{j=0}^{r} \alpha_j \left(\frac{\alpha^2 j^2 \xi^2}{2\left(\alpha^2 j^2 \xi^2 + 1\right)}\right) - \alpha^2 \sum_{j=0}^{r} \alpha_j \left(\frac{\beta^2 j^2 \xi^2}{2\left(\beta^2 j^2 \xi^2 + 1\right)}\right) \right].$$

Some more simplifications lead to

$$E(x) = P_{r,\xi}(f, x) - f(x) - f''(x) \left\{ \sum_{j=0}^{r} \alpha_j j^2 \left[\frac{1}{\left(\alpha^2 j^2 \xi^2 + 1\right)\left(\beta^2 j^2 \xi^2 + 1\right)} \right] \right\} \xi^2 -$$

$$\left(f^{(4)}(x) + \left(\alpha^2 + \beta^2\right) f''(x)\right) \left[\sum_{j=0}^{r} \alpha_j j^4 \left(\frac{1}{\left(\alpha^2 j^2 \xi^2 + 1\right)\left(\beta^2 j^2 \xi^2 + 1\right)} \right) \right] \xi^4; \tag{24}$$

$\xi > 0$, $x \in \mathbb{R}$.

We present our first result, about L_p approximation, $p > 1$.

Theorem 1 *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right) \neq 0$, and let all $f, f', f'', f^{(3)}, f^{(4)} \in L_p(\mathbb{R}) \cap C(\mathbb{R})$, $1<p<\infty$. Here $q>1 : \frac{1}{p}+\frac{1}{q}=1$; $\xi>0$, $x \in \mathbb{R}$. Then*

$$\|E(x)\|_p = \Big\| P_{r,\xi}(f,x) - f(x) -$$

$$f''(x)\left\{\sum_{j=0}^{r}\alpha_j j^2\left(\frac{1}{\left(\alpha^2 j^2\xi^2+1\right)\left(\beta^2 j^2\xi^2+1\right)}\right)\right\}\xi^2 -$$

$$\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)$$

$$\left(\sum_{j=0}^{r}\alpha_j j^4\left(\frac{1}{\left(\alpha^2 j^2\xi^2+1\right)\left(\beta^2 j^2\xi^2+1\right)}\right)\right)\xi^4\Big\|_p \leq \tag{25}$$

$$\frac{2^{1+\frac{2}{q}}}{q^{\frac{1}{q}}\left|\beta^2-\alpha^2\right|}\left(\frac{r+1}{q+1}\right)^{\frac{1}{q}}\omega_1\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f,\xi\right)_p\xi^2$$

$$\left[\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}\left\{\left(\frac{p}{2}\right)^{-qp+q-p-1}\right.\right.$$

$$\left.\left.\left(\Gamma(qp-q+p+1)+\frac{j^p}{(p+1)}\left(\frac{p}{2}\right)^{-p}\Gamma(qp-q+2p+1)\right)\right\}\right]^{\frac{1}{p}} \to 0,$$

as $\xi \to 0$.

Above Γ is the gamma function.

Proof. Call the function

$$F := f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f. \tag{26}$$

Then, we get

$$R = R(t) = \frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\sum_{j=0}^{r}\alpha_j j\int_0^t\left[F(x+jw)-F(x)\right]$$

$$\left[\beta\sin(\alpha j(t-w))-\alpha\sin(\beta j(t-w))\right]dw, \quad \forall\, t \in \mathbb{R}. \tag{27}$$

We isolate and study

$$I := \int_0^t\left[F(x+jw)-F(x)\right]$$

$$\left[\beta\sin(\alpha j(t-w))-\alpha\sin(\beta j(t-w))\right]dw, \quad \forall\, t \in \mathbb{R}. \tag{28}$$

For $t<0$, we have that

$$|I| = \left|\int_t^0\left[F(x+jw)-F(x)\right]\left[\beta\sin(\alpha j(t-w))-\alpha\sin(\beta j(t-w))\right]dw\right| \leq$$

$$\int_t^0 |F(x+jw)-F(x)|\,|\beta\sin(\alpha j(t-w))-\alpha\sin(\beta j(t-w))|\,dw \leq \tag{29}$$

$$2|\alpha||\beta| j\int_t^0 |F(x+jw)-F(x)|(w-t)\,dw =$$

$$-2|\alpha||\beta| j\int_t^0 |F(x-j(-w))-F(x)|(-t-(-w))\,d(-w) =$$

$(t \leq w \leq 0 \Rightarrow -t \geq -w =: \theta \geq 0)$

$$-2|\alpha||\beta| j\int_{-t}^0 |F(x-j\theta)-F(x)|(-t-\theta)\,d\theta =$$

$$2|\alpha||\beta| j\int_0^{-t} |F(x-j\theta)-F(x)|(-t-\theta)\,d\theta =$$

$$2|\alpha\beta| j\int_0^{|t|} |F(x+sign(t)j\theta)-F(x)|(|t|-\theta)\,d\theta.$$

So, we have proved that

$$|I| \leq 2|\alpha\beta| j\int_0^{|t|} |F(x+sign(t)j\theta)-F(x)|(|t|-\theta)\,d\theta, \quad \forall\ t\in\mathbb{R}, \tag{30}$$

and, by (27),

$$|R(t)| \leq \frac{2}{|\beta^2-\alpha^2|}\sum_{j=0}^{r}|\alpha_j| j^2\int_0^{|t|} |F(x+jsign(t)\theta)-F(x)|(|t|-\theta)\,d\theta, \tag{31}$$

$\forall\ t\in\mathbb{R}$.

By (14), we have

$$E(x)=\frac{1}{2\xi}\int_{-\infty}^{\infty} R(t)\,e^{-\frac{|t|}{\xi}}\,dt. \tag{32}$$

Hence it holds $(p,q>1: \frac{1}{p}+\frac{1}{q}=1)$,

$$\int_{-\infty}^{\infty}|E(x)|^p\,dx = \frac{1}{(2\xi)^p}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}R(t)\,e^{-\frac{|t|}{\xi}}\,dt\right|^p dx \leq$$

$$\frac{1}{(2\xi)^p}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|\,e^{-\frac{|t|}{\xi}}\,dt\right)^p dx = \tag{33}$$

$$\frac{1}{(2\xi)^p}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|\,e^{-\frac{|t|}{2\xi}}e^{-\frac{|t|}{2\xi}}\,dt\right)^p dx \leq$$

$$\frac{1}{(2\xi)^p}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|^p\,e^{-\frac{|pt|}{2\xi}}\,dt\right)\left(\int_{-\infty}^{\infty}e^{-\frac{|qt|}{2\xi}}\,dt\right)^{\frac{p}{q}} dx =$$

$$\frac{1}{(2\xi)^p}\left(\frac{4\xi}{q}\right)^{\frac{p}{q}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|^p e^{-\frac{|pt|}{2\xi}}dt\right)dx = \tag{34}$$

$$\frac{2^{p-2}\xi^{-1}}{q^{p-1}}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|^p e^{-\frac{|pt|}{2\xi}}dt\right)dx\right) =: (*) .$$

But, we need to treat

$$|R(t)| \leq \frac{2}{\left|\beta^2-\alpha^2\right|}\sum_{j=0}^{r}|\alpha_j|\, j^2$$

$$\left(\int_0^{|t|}|F(x+jsign(t)\theta)-F(x)|^p d\theta\right)^{\frac{1}{p}}\left(\int_0^{|t|}(|t|-\theta)^q d\theta\right)^{\frac{1}{q}} = \tag{35}$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\sum_{j=0}^{r}|\alpha_j|\, j^2\left(\int_0^{|t|}|F(x+jsign(t)\theta)-F(x)|^p d\theta\right)^{\frac{1}{p}}\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} =$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\left\{\sum_{j=0}^{r}\left(\left(|\alpha_j|\, j^2\right)^p\right)^{\frac{1}{p}}\left(\int_0^{|t|}|F(x+jsign(t)\theta)-F(x)|^p d\theta\right)^{\frac{1}{p}}\right\}$$

$$\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \leq$$

$(0<\frac{1}{p}<1)$

$$\frac{2(r+1)^{\frac{1}{q}}}{\left|\beta^2-\alpha^2\right|}\left(\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\int_0^{|t|}|F(x+jsign(t)\theta)-F(x)|^p d\theta\right)^{\frac{1}{p}}\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}.$$

Hence, we find that

$$|R(t)|^p \leq \frac{2^p}{\left|\beta^2-\alpha^2\right|^p}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}$$

$$\left[\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\int_0^{|t|}|F(x+jsign(t)\theta)-F(x)|^p d\theta\right]|t|^{\frac{(q+1)p}{q}} . \tag{36}$$

Therefore, we get

$$(*) \leq \left(\frac{2^{p-2}\xi^{-1}}{q^{p-1}}\frac{2^p}{\left|\beta^2-\alpha^2\right|^p}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\right)\left\{\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left[\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\right.\right.\right.$$

$$\left.\left.\left.\int_0^{|t|}|F(x+jsign(t)\theta)-F(x)|^p d\theta\right]|t|^{\frac{(q+1)p}{q}} e^{-\frac{|pt|}{2\xi}}dt\right)dx\right\} =$$

(set $c_1 = \frac{2^{2(p-1)}\xi^{-1}}{q^{p-1}|\beta^2-\alpha^2|^p}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}$)

$$c_1 \left\{ \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p \, j^{2p} \right.\right.\right.$$

$$\left.\left.\left. \int_0^{|t|} |F(x + jsign(t)\,\theta) - F(x)|^p \, d\theta \right] dx \right) |t|^{\frac{(q+1)p}{q}} e^{-\frac{|pt|}{2\xi}} dt \right\} = \tag{37}$$

$$= c_1 \left\{ \int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p \, j^{2p} \int_0^{|t|} \left(\int_{-\infty}^{\infty} |F(x + jsign(t)\,\theta) - F(x)|^p \, dx \right) d\theta \right] \right.$$

$$\left. |t|^{\frac{(q+1)p}{q}} e^{-\frac{|pt|}{2\xi}} dt \right\} \leq$$

($\xi > 0$)

$$c_1 \left\{ \int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p \, j^{2p} \left(\int_0^{|t|} \omega_1 \left(F, \xi \frac{j\theta}{\xi} \right)_p^p d\theta \right) \right] |t|^{\frac{(q+1)p}{q}} e^{-\frac{|pt|}{2\xi}} dt \right\} \leq \tag{38}$$

$$c_1 \omega_1 (F, \xi)_p^p \left\{ \int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p \, j^{2p} \left(\int_0^{|t|} \left(1 + \frac{j}{\xi}\theta \right)^p d\theta \right) \right] \right.$$

$$\left. |t|^{(q+1)(p-1)} e^{-\frac{|pt|}{2\xi}} dt \right\} \leq$$

$$c_1 \omega_1 (F, \xi)_p^p \left\{ \int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p \, j^{2p} 2^{p-1} \left(\int_0^{|t|} \left(1 + \frac{j^p}{\xi^p}\theta^p \right) d\theta \right) \right] \right.$$

$$\left. |t|^{(q+1)(p-1)} e^{-\frac{|pt|}{2\xi}} dt \right\} =$$

$$2^{p-1} c_1 \omega_1 (F, \xi)_p^p$$

$$\left\{ \int_{-\infty}^{\infty} \left[\sum_{j=0}^{r} |\alpha_j|^p \, j^{2p} \left(|t| + \frac{j^p}{\xi^p} \frac{|t|^{p+1}}{p+1} \right) \right] |t|^{(q+1)(p-1)} e^{-\frac{|pt|}{2\xi}} dt \right\} =$$

$$2^{p-1} c_1 \omega_1 (F, \xi)_p^p$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p \, j^{2p} \int_{-\infty}^{\infty} \left(|t| + \frac{j^p}{\xi^p} \frac{|t|^{p+1}}{p+1} \right) |t|^{(q+1)(p-1)} e^{-\frac{|pt|}{2\xi}} dt \right\} = \tag{39}$$

$$2^p c_1 \omega_1 (F, \xi)_p^p$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p \, j^{2p} \int_0^{\infty} \left(t + \frac{j^p}{\xi^p (p+1)} t^{p+1} \right) t^{(q+1)(p-1)} e^{-\frac{pt}{2\xi}} dt \right\} =$$

$$2^p c_1 \omega_1 (F,\xi)_p^p$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\int_0^\infty t^{(q+1)(p-1)+1} e^{-\frac{pt}{2\xi}} dt + \frac{j^p}{\xi^p (p+1)} \int_0^\infty t^{(q+1)(p-1)+p+1} e^{-\frac{pt}{2\xi}} dt \right] \right\} = \tag{40}$$

$$2^p c_1 \omega_1 (F,\xi)_p^p \left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\xi^{qp-q+p+1} \int_0^\infty \left(\frac{t}{\xi}\right)^{(q+1)(p-1)+1} e^{-\left(\frac{p}{2}\right)\frac{t}{\xi}} d\frac{t}{\xi} \right.\right.$$

$$\left.\left. + \frac{j^p}{\xi^p (p+1)} \xi^{qp-q+2p+1} \int_0^\infty \left(\frac{t}{\xi}\right)^{qp-q+2p} e^{-\left(\frac{p}{2}\right)\frac{t}{\xi}} d\frac{t}{\xi} \right] \right\} =$$

(above it is $(q+1)(p-1)+1 = qp-q+p > 0$, $(q+1)(p-1)+p+1 = qp-q+2p > 0$)

$$2^p c_1 \omega_1 (F,\xi)_p^p \xi^{qp-q+p+1} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left\{ \int_0^\infty x^{qp-q+p} e^{-\left(\frac{p}{2}\right)x} dx \right.\right. \tag{41}$$

$$\left.\left. + \frac{j^p}{(p+1)} \int_0^\infty x^{qp-q+2p} e^{-\left(\frac{p}{2}\right)x} dx \right] \right\} =$$

(next we use, for $a,b > 0$ that it holds $\int_0^\infty x^a e^{-bx} dx = b^{-a-1}\Gamma(a+1)$, by Wolfram Alpha computational intelligence)

$$2^p c_1 \omega_1 (F,\xi)_p^p \xi^{qp-q+p+1} \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left\{ \left(\frac{p}{2}\right)^{-qp+q-p-1} \Gamma(qp-q+p+1) \right.\right.$$

$$\left.\left. + \frac{j^p}{(p+1)} \left(\frac{p}{2}\right)^{-qp+q-2p-1} \Gamma(qp-q+2p+1) \right\} \right] =$$

$$\frac{2^{3p-2}}{q^{p-1} \left|\beta^2 - \alpha^2\right|^p} \left(\frac{r+1}{q+1}\right)^{\frac{p}{q}} \omega_1 (F,\xi)_p^p \xi^{qp-q+p} \tag{42}$$

$$\left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left\{ \left(\frac{p}{2}\right)^{-qp+q-p-1} \right.\right.$$

$$\left.\left. \left(\Gamma(qp-q+p+1) + \frac{j^p}{(p+1)} \left(\frac{p}{2}\right)^{-p} \Gamma(qp-q+2p+1) \right) \right\} \right].$$

So for $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\xi > 0$, we have proved:

$$\int_{-\infty}^{\infty} |E(x)|^p dx \leq \frac{2^{3p-2}}{q^{p-1} \left|\beta^2 - \alpha^2\right|^p} \left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}$$

$$\omega_1 \left(f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2 \beta^2 f, \xi \right)_p^p \xi^{qp-q+p} \tag{43}$$

$$\left[\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left\{\left(\frac{p}{2}\right)^{-qp+q-p-1}\right.\right.$$

$$\left.\left.\left(\Gamma(qp-q+p+1)+\frac{j^p}{(p+1)}\left(\frac{p}{2}\right)^{-p}\Gamma(qp-q+2p+1)\right)\right\}\right].$$

The proof of the theorem is now completed. ■

Next comes the L_1 related approximation.

Theorem 2 *Let $\alpha,\beta\in\mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$, and let all $f,f',f'',f^{(3)},f^{(4)}\in L_1(\mathbb{R})\cap C(\mathbb{R})$, $\xi>0$, $x\in\mathbb{R}$. Then*

$$\|E(x)\|_1=\|P_{r,\xi}(f,x)-f(x)-$$

$$f''(x)\left\{\sum_{j=0}^{r}\alpha_j j^2\left(\frac{1}{\left(\alpha^2j^2\xi^2+1\right)\left(\beta^2j^2\xi^2+1\right)}\right)\right\}\xi^2-$$

$$\left(f^{(4)}(x)+\left(\alpha^2+\beta^2\right)f''(x)\right)$$

$$\left(\sum_{j=0}^{r}\alpha_j j^4\left(\frac{1}{\left(\alpha^2j^2\xi^2+1\right)\left(\beta^2j^2\xi^2+1\right)}\right)\right)\xi^4\Bigg\|_1\leq$$

$$\frac{2\xi^2\omega_1\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2f,\xi\right)_1}{\left|\beta^2-\alpha^2\right|}\left(\sum_{j=0}^{r}|\alpha_j|\,j^2(2+3j)\right)\to 0, \tag{44}$$

as $\xi\to 0$.

Proof. We have that (by (14))

$$|E(x)|=\frac{1}{2\xi}\left|\int_{-\infty}^{\infty}R(t)\,e^{-\frac{|t|}{\xi}}dt\right|\leq\frac{1}{2\xi}\int_{-\infty}^{\infty}|R(t)|\,e^{-\frac{|t|}{\xi}}dt\overset{(31)}{\leq}$$

$$\frac{1}{\left|\beta^2-\alpha^2\right|\xi}\sum_{j=0}^{r}|\alpha_j|\,j^2 \tag{45}$$

$$\int_{-\infty}^{\infty}\left(\int_0^{|t|}|F(x+jsign(t)\theta)-F(x)|\,(|t|-\theta)\,d\theta\right)e^{-\frac{|t|}{\xi}}dt.$$

Thus, we get

$$\|E(x)\|_1=\int_{-\infty}^{\infty}|E(x)|\,dx\leq$$

$$\frac{1}{\left|\beta^2-\alpha^2\right|\xi}\sum_{j=0}^{r}|\alpha_j|\,j^2$$

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\left|F\left(x+jsign\left(t\right)\theta\right)-F\left(x\right)\right|\left(|t|-\theta\right)d\theta\right)e^{-\frac{|t|}{\xi}}dt\right)dx\right)\leq \tag{46}$$

$$\frac{1}{\left|\beta^{2}-\alpha^{2}\right|\xi}\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}$$

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(|t|\int_{0}^{|t|}\left|F\left(x+jsign\left(t\right)\theta\right)-F\left(x\right)\right|d\theta\right)e^{-\frac{|t|}{\xi}}dt\right)dx\right)=$$

$$\frac{1}{\left|\beta^{2}-\alpha^{2}\right|\xi}\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}$$

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(|t|\int_{0}^{|t|}\left|F\left(x+jsign\left(t\right)\theta\right)-F\left(x\right)\right|d\theta\right)e^{-\frac{|t|}{\xi}}dx\right)dt\right)=$$

$$\frac{1}{\left|\beta^{2}-\alpha^{2}\right|\xi}\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}$$

$$\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\left|F\left(x+jsign\left(t\right)\theta\right)-F\left(x\right)\right|d\theta\right)dx\right)|t|\,e^{-\frac{|t|}{\xi}}dt\right)=$$

$$\frac{1}{\left|\beta^{2}-\alpha^{2}\right|\xi}\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}$$

$$\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\left(\int_{-\infty}^{\infty}\left|F\left(x+jsign\left(t\right)\theta\right)-F\left(x\right)\right|dx\right)d\theta\right)|t|\,e^{-\frac{|t|}{\xi}}dt\right)\leq \tag{47}$$

$$\frac{1}{\left|\beta^{2}-\alpha^{2}\right|\xi}\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\omega_{1}\left(F,j\theta\right)_{1}d\theta\right)|t|\,e^{-\frac{|t|}{\xi}}dt\right)\leq$$

$$\frac{\omega_{1}\left(F,\xi\right)_{1}}{\left|\beta^{2}-\alpha^{2}\right|\xi}\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\left(1+\frac{j}{\xi}\theta\right)d\theta\right)|t|\,e^{-\frac{|t|}{\xi}}dt\right)=$$

$$\frac{\omega_{1}\left(F,\xi\right)_{1}}{\left|\beta^{2}-\alpha^{2}\right|\xi}\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{-\infty}^{\infty}\left[|t|+\frac{j}{\xi}\frac{|t|^{2}}{2}\right]|t|\,e^{-\frac{|t|}{\xi}}dt\right)=$$

$$\frac{2\xi^{2}\omega_{1}\left(F,\xi\right)_{1}}{\left|\beta^{2}-\alpha^{2}\right|}\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{0}^{\infty}\left[\frac{t}{\xi}+\frac{j}{2}\left(\frac{t}{\xi}\right)^{2}\right]\frac{t}{\xi}e^{-\frac{t}{\xi}}d\frac{t}{\xi}\right)= \tag{48}$$

$$\frac{2\xi^{2}\omega_{1}\left(F,\xi\right)_{1}}{\left|\beta^{2}-\alpha^{2}\right|}\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{0}^{\infty}\left(t+\frac{j}{2}t^{2}\right)te^{-t}dt\right)=$$

$$\frac{2\xi^{2}\omega_{1}\left(F,\xi\right)_{1}}{\left|\beta^{2}-\alpha^{2}\right|}\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left[\int_{0}^{\infty}t^{2}e^{-t}dt+\frac{j}{2}\int_{0}^{\infty}t^{3}e^{-t}dt\right]=$$

$$\frac{2\xi^2\omega_1(F,\xi)_1}{\left|\beta^2-\alpha^2\right|}\sum_{j=0}^{r}|\alpha_j|\,j^2\left[2+\frac{j}{2}\cdot 6\right]=$$

$$\frac{2\xi^2\omega_1(F,\xi)_1}{\left|\beta^2-\alpha^2\right|}\sum_{j=0}^{r}|\alpha_j|\,j^2\,(2+3j)\,. \tag{49}$$

Above we used that

$$\int_0^\infty t^2e^{-t}dt=2 \text{ and } \int_0^\infty t^3e^{-t}dt=6.$$

The theorem is proved. ■

We finish with the following:

Corollary 3 *(to Theorem 1, $p>1$) It holds*

$$\|P_{r,\xi}(f)-f\|_p\leq$$

$$\|f''\|_p\left\{\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\frac{1}{(\alpha^2j^2\xi^2+1)(\beta^2j^2\xi^2+1)}\right)\right\}\xi^2+$$

$$\left(\left\|f^{(4)}\right\|_p+\left(\alpha^2+\beta^2\right)\|f''\|_p\right)$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\,j^4\left(\frac{1}{(\alpha^2j^2\xi^2+1)(\beta^2j^2\xi^2+1)}\right)\right)\xi^4+$$

$$\frac{2^{1+\frac{2}{q}}}{q^{\frac{1}{q}}\left|\beta^2-\alpha^2\right|}\left(\frac{r+1}{q+1}\right)^{\frac{1}{q}}\omega_1\left(f^{(4)}+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2f,\xi\right)_p\xi^2$$

$$\left[\sum_{j=0}^{r}|\alpha_j|^p\,j^{2p}\left\{\left(\frac{p}{2}\right)^{-qp+q-p-1}\right.\right. \tag{50}$$

$$\left.\left.\left(\Gamma(qp-q+p+1)+\frac{j^p}{(p+1)}\left(\frac{p}{2}\right)^{-p}\Gamma(qp-q+2p+1)\right)\right\}\right]^{\frac{1}{p}}\to 0,$$

as $\xi\to 0$.

Corollary 4 *(to Theorem 2, $p=1$) It holds*

$$\|P_{r,\xi}(f)-f\|_1\leq$$

$$\|f''\|_1\left\{\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\frac{1}{(\alpha^2j^2\xi^2+1)(\beta^2j^2\xi^2+1)}\right)\right\}\xi^2+$$

$$\left(\left\|f^{(4)}\right\|_1+\left(\alpha^2+\beta^2\right)\|f''\|_1\right)$$

$$\left(\sum_{j=0}^{r} |\alpha_j| \, j^4 \left(\frac{1}{\left(\alpha^2 j^2 \xi^2 + 1\right)\left(\beta^2 j^2 \xi^2 + 1\right)} \right) \right) \xi^4 +$$

$$\frac{2\xi^2 \omega_1 \left(f^{(4)} + \left(\alpha^2 + \beta^2\right) f'' + \alpha^2 \beta^2 f, \xi \right)_1}{\left| \beta^2 - \alpha^2 \right|} \left(\sum_{j=0}^{r} |\alpha_j| \, j^2 \, (2 + 3j) \right) \to 0, \quad (51)$$

as $\xi \to 0$.

References

[1] G.A. Anastassiou, *Basic convergence with rates of smooth Picard singular integral operators*, J. Computational Analysis and Appl., 8(4) (2006), 313-334.

[2] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, New York, Chapter 12, 2011.

[3] G.A. Anastassiou, *Opial and Ostrowski type inequalities based on trigonometric and hyperbolic type Taylor formulae*, submitted, 2023.

[4] Ali Hasan Ali, Zsolt Pales, *Taylor-type expansions in terms of exponential polynomials*, Mathematical Inequalities and Applications, 25(4) (2022), 1123-1141.