Uniform Approximation by smooth Picard Multivariate Singular Integral Operators revisited

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Abstract

In this article we reexamine the uniform approximation properties of smooth Picard multivariate singular integral operators over $\mathbb{R}^N$, $N \geq 1$. We establish their convergence to the unit operator with rates. The estimates are pointwise and uniform. The established inequalities involve the multivariate first modulus of continuity. Our approach is based on a new multivariate trigonometric Taylor formula. At first we present in detail the general theory of uniform approximation by general smooth multivariate singular integral operators, which then is applied to the Picard operators case.

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1 Introduction

The rate of convergence of univariate and multivariate singular integral operators has been studied extensively in [1]-[3] and [5], [6] and [8]. All these motivate our current work. In particular we studied the smooth singular integral operators in [1]-[3] and [6], which are not in general positive ones.

Here we continue the study of the last ones at the multivariate level, at first in general, and then apply our theory to the smooth Picard ones. The main tool here, we are based on, is a new trigonometric multivariate Taylor

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formula from [4]. Our quantitative estimates are pointwise and uniform, using the multivariate first modulus of continuity.

2 Results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, and define

$$\alpha_j := \alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, ..., r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0. \end{cases} \quad (1)$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \quad (2)$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \quad (3)$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\theta_n (f; x_1, ..., x_N) := \theta_{r,n}^{[m]} (f; x_1, ..., x_N) :=$$

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f (x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) \, d\mu_{\xi_n} (s), \quad (4)$$

where $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers, we take $0 < \xi_n \leq 1$.

Remark 1 *The operators $\theta_{r,n}^{[m]}$ are not in general positive, see [2], p. 2.*

We observe that

Lemma 2 *It holds*

$$\theta_{r,n}^{[m]} (c; x_1, ..., x_n) = c,$$

where c is a constant.

We need

Definition 3 *Let $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$. We define the first modulus of continuity of f as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in\mathbb{R}^N:\\ \|x-y\|_\infty \leq \delta}} |f(x)-f(y)|, \quad \delta > 0, \tag{5}$$

where $\|\cdot\|_\infty$ is the max norm in $\mathbb{R}^N$. The functional $\omega_1(f,\delta)$ is bounded for f being bounded or uniformly continuous, and $\omega_1(f,\delta) \to 0$ as $\delta \to 0$, in the case of f being uniformly continuous.

We present the main general approximation result regarding the operator θ_n.

Theorem 4 *Here $f \in C^2\left(\mathbb{R}^N\right)$ and let all $\alpha_i \in \mathbb{Z}^+$, $i = 1,...,N$, $N \geq 1$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = 2$; $x \in \mathbb{R}^N$, and all the partials f_α of order 2, along with $f \in C_B\left(\mathbb{R}^N\right)$ (continuous and bounded functions); or all f_α of order 2, $f \in C_U\left(\mathbb{R}^N\right)$ (uniformly continuous functions). Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$, for $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.*

Suppose that for all $\alpha := (\alpha_1,...,\alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1,...,N$, $|\alpha| = \sum_{i=1}^{N} \alpha_i = 2$, $j = 0,1,...,r$, we have that both

$$I_{1j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right)\left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s), \tag{6}$$

$$I_{2j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) d\mu_{\xi_n}(s), \tag{7}$$

are uniformly bounded in $\xi_n \in (0,1]$.

Denote ($n \in \mathbb{N}$)

$$\Delta_n(x) := \theta_n(f,x) - f(x) - \left(\sum_{j=0}^{r} \alpha_j j\right) \sin(1) \left[\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} \left(\int_{\mathbb{R}^N} s_i d\mu_{\xi_n}(s)\right)\right]$$

$$-2\left(\sum_{j=0}^{r} \alpha_j j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)\right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \tag{8}$$

$$\left. \sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}} \left(\int_{\mathbb{R}^N} s_i s_{j^*} d\mu_{\xi_n}(s)\right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}} \right\}.$$

Then

(i)

$$\left|\Delta_n(x)\right| \leq \left\|\Delta_n(x)\right\|_\infty \leq \sum_{j=0}^{r}\left|\alpha_j\right|$$

$$\left[\left[j^2 \sum_{\substack{\alpha_i\in\mathbb{Z}^+,\\ \alpha:|\alpha|=2}} \left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\omega_1\left(f_\alpha,\xi_n\right)\int_{\mathbb{R}^N}\left(1+\frac{j\left\|s\right\|_\infty}{3\xi_n}\right)\left(\prod_{i=1}^{N}\left|s_i\right|^{\alpha_i}\right)d\mu_{\xi_n}(s)\right]+\right.$$

$$\left.\frac{1}{2}\omega_1(f,\delta)\int_{\mathbb{R}^N}\left(1+\frac{j\left\|s\right\|_\infty}{3\xi_n}\right)d\mu_{\xi_n}(s)\right] =: \varphi_{\xi_n}. \tag{9}$$

In case of all f_α *of order* 2 *and* $f \in C_U\left(\mathbb{R}^N\right)$ *and* $\xi_n \to 0$, *as* $n \to \infty$, *then* $\Delta_n(x)$, $\left\|\Delta_n(x)\right\|_\infty \to 0$ *with rates.*

(ii) If $\frac{\partial f(x)}{\partial x_i} = 0$, $i = 1, ..., N$, *and* $f_\alpha(x) = 0$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, *with* $|\alpha| = 2$, *then*

$$\left|\theta_n(f,x) - f(x)\right| \leq \varphi_{\xi_n}. \tag{10}$$

And $\theta_n(f,x) \to f(x)$ *in the uniformly continuous case.*

(iii) Additionally assume all partials of order ≤ 2 *are bounded. Hence*

$$\left\|\theta_n(f) - f\right\|_\infty \leq \left(\sum_{j=0}^{r}\left|\alpha_j\right| j\right)(0.8414)\left[\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty\left(\int_{\mathbb{R}^N} s_i d\mu_{\xi_n}(s)\right)\right]+$$

$$\left(\sum_{j=0}^{r}\left|\alpha_j\right| j^2\right)(0.4596)\left\{\sum_{i=1}^{N}\left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_\infty+\right.$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}}\left(\int_{\mathbb{R}^N}\left|s_i\right|\left|s_{j^*}\right| d\mu_{\xi_n}(s)\right)\left\|\frac{\partial^2 f}{\partial x_i \partial x_{j^*}}\right\|_\infty\right\} + \varphi_{\xi_n}. \tag{11}$$

If all $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)$ *and* $\int_{\mathbb{R}^N}\left|s_i\right|\left|s_{j^*}\right| d\mu_{\xi_n}(s)$ *converge to zero, as* $n \to \infty$, *with* $\xi_n \to 0$, *and all* f_α *of order* 2, $f \in C_U\left(\mathbb{R}^N\right)$, *then*

$$\left\|\theta_n(f) - f\right\|_\infty \to 0 \text{ with rates, as } \xi_n \to 0,\ n \to +\infty.$$

Proof. Let $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N)$, $z := (z_1, ..., z_N) := (x_1 + s_1 j, x_2 + s_2 j, ...x_N + s_N j) = x + sj$; $j = 0, 1, ..., r$, and $x := x_0 = (x_{01}, ..., x_{0N}) = (x_1, ..., x_N)$, all in $\mathbb{R}^N$.

Here $f \in C^2\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}$, and clearly all the mixed partials commnute.

Consider

$$g_{x+sj}(t) := f(x + t(sj)), \quad 0 \leq t \leq 1. \tag{12}$$

Notice that $g_{x+sj}(0) = f(x)$, $g_{x+sj}(1) = f(x + sj)$.

We have (by [4])

$$f(x+sj)-f(x)=g_{x+sj}(1)-g_{x+sj}(0)=$$

$$g'_{x+sj}(0)\sin(1)+2g''_{x+sj}(0)\sin^2\left(\frac{1}{2}\right)+$$

$$\int_0^1\left[\left(g''_{x+sj}(t)+g_{x+sj}(t)\right)-\left(g''_{x+sj}(0)+g_{x+sj}(0)\right)\right]\sin(1-t)\,dt=$$

$$\left(\sum_{i=1}^{N}(s_ij)\frac{\partial f}{\partial x_i}(x)\right)\sin(1)+$$

$$2\left\{\left[\left(\sum_{i=1}^{N}(s_ij)\frac{\partial}{\partial x_i}\right)^2 f\right](x)\right\}\sin^2\left(\frac{1}{2}\right)+$$

$$\int_0^1\left\{\left\{\left[\left(\sum_{i=1}^{N}(s_ij)\frac{\partial}{\partial x_i}\right)^2 f\right](x+t(sj))+f(x+t(sj))\right\}-\right.\tag{13}$$

$$\left.\left\{\left[\left(\sum_{i=1}^{N}(s_ij)\frac{\partial}{\partial x_i}\right)^2 f\right](x)+f(x)\right\}\right\}\sin(1-t)\,dt.$$

Denote the remainder $(j=0,1,...,r)$

$$R_j:=\int_0^1\left\{\left\{\left[\left(\sum_{i=1}^{N}(s_ij)\frac{\partial}{\partial x_i}\right)^2 f\right](x+tsj)+f(x+tsj)\right\}\right.$$

$$\left.-\left\{\left[\left(\sum_{i=1}^{N}(s_ij)\frac{\partial}{\partial x_i}\right)^2 f\right](x)+f(x)\right\}\right\}\sin(1-t)\,dt=\tag{14}$$

$$\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(js_i)^{\alpha_i}\right)\left[f_\alpha(x+t(js))-f_\alpha(x)\right]\right.$$

$$\left.+\left(f(x+t(js))-f(x)\right)\right\}\sin(1-t)\,dt.$$

Therefore it holds

$$|R_j|\leq\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\right.$$

$$\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)|f_\alpha(x+tsj)-f_\alpha(x)|+|f(x+tsj)-f(x)|\Bigg\}|\sin(1-t)|\,dt\le \tag{15}$$

$$\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)\omega_1\left(f_\alpha,tj\|s\|_\infty\right)\right. \tag{16}$$

$$\left.+\omega_1\left(f,tj\|s\|_\infty\right)\right\}|\sin(1-t)|\,dt\le$$

$(0<\xi_n\le 1)$

$$\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)\omega_1\left(f_\alpha,\xi_n\right)\left(1+\frac{tj\|s\|_\infty}{\xi_n}\right)\right.$$

$$\left.+\omega_1\left(f,\xi_n\right)\left(1+\frac{tj\|s\|_\infty}{\xi_n}\right)\right\}|\sin(1-t)|\,dt=$$

$$\left\{\left[2\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)\omega_1\left(f_\alpha,\xi_n\right)\right]+\omega_1\left(f,\xi_n\right)\right\}$$

$$\left[\int_0^1\left(1+\frac{tj\|s\|_\infty}{\xi_n}\right)|\sin(1-t)|\,dt\right]\le \tag{17}$$

$$\left\{\left[2\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)\omega_1\left(f_\alpha,\xi_n\right)\right]+\omega_1\left(f,\xi_n\right)\right\}$$

$$\left[\int_0^1\left(1+\frac{tj\|s\|_\infty}{\xi_n}\right)(1-t)\,dt\right].$$

So far we have proved that

$$|R_j| \leq \left\{ \left[2 \sum_{\substack{\alpha_i \in \mathbb{Z}_+, \\ |\alpha| := \sum_{i=1}^{N} \alpha_i = 2}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\prod_{i=1}^{N} (j |s_i|)^{\alpha_i} \right) \omega_1 (f_\alpha, \xi_n) \right] + \omega_1 (f, \xi_n) \right\}$$

$$\left[\int_0^1 \left(1 + \frac{tj \|s\|_\infty}{\xi_n} \right) (1 - t) \, dt \right], \tag{18}$$

$j = 0, 1, ..., r;\ 0 < \xi_n \leq 1$.

So, we have found that

$$|R_j| \leq \left[\left[j^2 \sum_{\substack{\alpha_i \in \mathbb{Z}_+, \\ |\alpha| = 2}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \omega_1 (f_\alpha, \xi_n) \right] + \frac{1}{2} \omega_1 (f, \xi_n) \right] \tag{19}$$

$$\left[1 + \frac{j \|s\|_\infty}{3\xi_n} \right], \quad j = 0, 1, ..., r;\ 0 < \xi_n \leq 1.$$

Next we can write

$$\sum_{j=0}^{r} \alpha_j [f(x + sj) - f(x)] - \left(\sum_{j=0}^{r} \alpha_j j \right) \left(\sum_{i=1}^{N} s_i \frac{\partial f}{\partial x_i} (x) \right) \sin(1) - \tag{20}$$

$$2 \left(\sum_{j=0}^{r} \alpha_j j^2 \right) \left\{ \left[\left(\sum_{i=1}^{N} s_i \frac{\partial}{\partial x_i} \right)^2 f \right] (x) \right\} \sin^2 \left(\frac{1}{2} \right) = \sum_{j=0}^{r} \alpha_j R_j.$$

Call

$$R := \sum_{j=0}^{r} \alpha_j R_j. \tag{21}$$

Hence it holds

$$|R| \leq \sum_{j=0}^{r} |\alpha_j| |R_j| \leq \sum_{j=0}^{r} |\alpha_j|$$

$$\left[\left[j^2 \sum_{\left\{ \substack{\alpha := (\alpha_1, ..., \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, ..., N, |\alpha| := \sum_{i=1}^{N} \alpha_i = 2} \right\}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \omega_1 (f_\alpha, \xi_n) \right] + \frac{1}{2} \omega_1 (f, \xi_n) \right] \tag{22}$$

$$\left[1+\frac{j\left\|s\right\|_{\infty}}{3\xi_{n}}\right]=$$

$$\sum_{j=0}^{r}\left|\alpha_{j}\right|\left[\left[j^{2}\sum_{\alpha:|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\omega_{1}\left(f_{\alpha},\xi_{n}\right)\left[1+\frac{j\left\|s\right\|_{\infty}}{3\xi_{n}}\right]\left(\prod_{i=1}^{N}\left|s_{i}\right|^{\alpha_{i}}\right)\right]\right.$$

$$\left.+\frac{1}{2}\omega_{1}\left(f,\xi_{n}\right)\left[1+\frac{j\left\|s\right\|_{\infty}}{3\xi_{n}}\right]\right], \tag{23}$$

$0<\xi_{n}\leq 1$.

See that

$$\sin 1\cong 0.8414$$
$$\left(\sin 0.5\right)^{2}\cong\left(0.4794\right)^{2}\cong 0.2298.$$

Clearly, it holds

$$\theta_{n}\left(f,x\right)-f\left(x\right)=\sum_{j=0}^{r}\alpha_{j}\int_{\mathbb{R}^{N}}\left(f\left(x+sj\right)-f\left(x\right)\right)d\mu_{\xi_{n}}\left(s\right). \tag{24}$$

We observe that

$$\Delta_{n}\left(x\right)=\sum_{j=0}^{r}\alpha_{j}\int_{\mathbb{R}^{N}}R_{j}d\mu_{\xi_{n}}\left(s\right). \tag{25}$$

We have that

$$\sum_{j=0}^{r}\left|\alpha_{j}\right|\int_{\mathbb{R}^{N}}\left|R_{j}\right|d\mu_{\xi_{n}}\left(s\right)\leq$$

$$\sum_{j=0}^{r}\left|\alpha_{j}\right|\left[\left[j^{2}\sum_{\alpha:|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_{i}!}\right)\omega_{1}\left(f_{\alpha},\xi_{n}\right)\int_{\mathbb{R}^{N}}\left(1+\frac{j\left\|s\right\|_{\infty}}{3\xi_{n}}\right)\left(\prod_{i=1}^{N}\left|s_{i}\right|^{\alpha_{i}}\right)d\mu_{\xi_{n}}\left(s\right)\right]\right.$$

$$\left.+\frac{1}{2}\omega_{1}\left(f,\xi_{n}\right)\int_{\mathbb{R}^{N}}\left(1+\frac{j\left\|s\right\|_{\infty}}{3\xi_{n}}\right)d\mu_{\xi_{n}}\left(s\right)\right]=\varphi_{\xi_{n}}. \tag{26}$$

To remind (see also (6), (7))

$$\left\|s\right\|_{\infty}\leq\sum_{i=1}^{N}\left|s_{i}\right|=:\left\|s\right\|_{1}, \tag{27}$$

hence the integrals in (9) ans (26) are uniformly bounded in $\xi_{n}\in(0,1]$.

Notice also that $(j = 0, 1, ..., r)$

$$\begin{aligned} \textstyle\int_{\mathbb{R}^N} |s_i| \, |s_{j^*}| \, d\mu_{\xi_n}(s) \leq I_{1j}(\alpha) < \infty, \\ \textstyle\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \leq I_{1j}(\alpha) < \infty, \end{aligned} \tag{28}$$

and

$$\int_{\mathbb{R}^N} |s_i| \, d\mu_{\xi_n}(s) \leq \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right)^{\frac{1}{2}} < \infty, \tag{29}$$

by Hölder's inequality, and all of them are uniformly bounded in $\xi_n \in (0, 1]$.

Thus, in the uniformly continuous case of f_α, $|\alpha| = 2$, and f we get $\varphi_{\xi_n} \to 0$, as $\xi_n \to 0$.

That is $\|\Delta_n(x)\|_\infty \to 0$, as $\xi_n \to 0$.

The proof of the theorem is now completed. ■

We make

Remark 5 *Next we will apply Theorem 4 to specific multivariate smooth Picard singular integral operators*

$$P_n(f; x_1, ..., x_N) := P_{r,n}^{[m]}(f; x_1, ..., x_N) :=$$

$$\frac{1}{(2\xi_n)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) \, e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N, \tag{30}$$

$r, n \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $0 < \xi_n \leq 1$.

Clearly here it is

$$d\mu_{\xi_n}(s) = \frac{1}{(2\xi_n)^N} e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N, \quad s \in \mathbb{R}^N. \tag{31}$$

We observe that

$$\frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N = 1, \tag{32}$$

see [3], Chap. 22, p. 356.

Here $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\alpha| = \sum_{i=1}^{N} \alpha_i = 2$. We notice that

$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N \leq 4^N \xi_n^{N+2} \leq 4^N, \tag{33}$$

by [3], p. 364.

So (6), (7) are confirmed for $j = 0$ when $d\mu_{\xi_n}$ is as in (31).

We need

Theorem 6 *Let $N \in \mathbb{N}$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\alpha| = \sum_{i=1}^{N} \alpha_i = 2$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$; $j = 1, 2, ..., r$. Then*

$$I_{1j}^{*}(\alpha) := \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1...ds_N \leq \tag{34}$$

$$\xi_n^2 \left[\left(1 + \frac{j}{3}N\right) + \left(1 + \frac{j}{3}\right) \prod_{i=1}^{N} \left(\frac{\lfloor e(\alpha_i + 1)! \rfloor}{e}\right)\right] \leq$$

$$\left[\left(1 + \frac{j}{3}N\right) + \left(1 + \frac{j}{3}\right)\left(\frac{16}{e}\right)^N\right] < +\infty,$$

are uniformly bunded in $\xi_n \in (0,1]$, fulfilling (6). Above $\lfloor \cdot \rfloor$ is the integral part of the number symbol.

Proof. Let $j = 1, ..., r$, then

$$I_{1j}^{*}(\alpha) = \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1...ds_N =$$

$$\frac{1}{\xi_n^N} \int_{\mathbb{R}_+^N} \left(1 + \frac{j\left(\sum_{i=1}^{N} s_i\right)}{3\xi_n}\right) \left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N} s_i}{\xi_n}} ds_1...ds_N =$$

$$\xi_n^2 \int_{\mathbb{R}_+^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} \frac{s_i}{\xi_n}\right)\right) \left(\prod_{i=1}^{N} \left(\frac{s_i}{\xi_n}\right)^{\alpha_i}\right) e^{-\sum_{i=1}^{N} \frac{s_i}{\xi_n}} d\left(\frac{s_1}{\xi_n}\right) ... d\left(\frac{s_N}{\xi_n}\right) = \tag{35}$$

$$\xi_n^2 \int_{\mathbb{R}_+^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right)\right) \left(\prod_{i=1}^{N} z_i^{\alpha_i}\right) e^{-\sum_{i=1}^{N} z_i} dz_1...dz_N =$$

$$\xi_n^2 \left[\int_{[0,1]^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right)\right) \left(\prod_{i=1}^{N} z_i^{\alpha_i}\right) e^{-\sum_{i=1}^{N} z_i} dz_1...dz_N + \right.$$

$$\left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right)\right) \left(\prod_{i=1}^{N} z_i^{\alpha_i}\right) e^{-\sum_{i=1}^{N} z_i} dz_1...dz_N\right] \leq$$

$$\xi_n^2 \left[\left(1 + \frac{j}{3}N\right) + \int_{(\mathbb{R}_+ - [0,1])^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right)\right) \left(\prod_{i=1}^{N} z_i^{\alpha_i}\right) e^{-\sum_{i=1}^{N} z_i} dz_1...dz_N\right] \leq \tag{36}$$

$$\xi_n^2\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\int_{(\mathbb{R}_+-[0,1])^N}\left(\sum_{i=1}^{N} z_i\right)\left(\prod_{i=1}^{N} z_i^{\alpha_i}\right)e^{-\sum\limits_{i=1}^{N} z_i}dz_1...dz_N\right]\leq$$

$$\xi_n^2\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\int_{(\mathbb{R}_+-[0,1])^N}\left(\prod_{i=1}^{N} z_i\right)\left(\prod_{i=1}^{N} z_i^{\alpha_i}\right)\left(\prod_{i=1}^{N} e^{-z_i}\right)\left(\prod_{i=1}^{N} dz_i\right)\right]=$$

$$\xi_n^2\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\prod_{i=1}^{N}\int_1^{\infty} z_i^{\alpha_i+1}e^{-z_i}dz_i\right]=$$

$$\xi_n^2\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\prod_{i=1}^{N}\int_1^{\infty} z_i^{(\alpha_i+2)-1}e^{-z_i}dz_i\right]$$

(by [7], p. 348)

$$\xi_n^2\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\prod_{i=1}^{N}\Gamma\left((\alpha_i+2),1\right)\right], \tag{37}$$

where $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function.

We have proved that, $j=1,...,r$, that

$$I_{1j}^*(\alpha)\leq\xi_n^2\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\prod_{i=1}^{N}\Gamma\left((\alpha_i+2),1\right)\right]\leq \tag{38}$$

$$\xi_n^2\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\prod_{i=1}^{N}\left(\frac{\lfloor e(\alpha_i+1)!\rfloor}{e}\right)\right]\leq$$

$$\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\left(\frac{16}{e}\right)^N\right]<+\infty,$$

therefore $I_{1j}^*(\alpha)$ are uniformly bounded.

Above we used the formula

$$\Gamma(s+1,1)=\frac{\lfloor es!\rfloor}{e},\quad s\in\mathbb{N}. \tag{39}$$

Here $\alpha_i+2\in\mathbb{N}$, hence

$$\Gamma\left((\alpha_i+2),1\right)=\frac{\lfloor e(\alpha_i+1)!\rfloor}{e}\leq\frac{\lfloor e3!\rfloor}{e}=\frac{\lfloor 6e\rfloor}{e}=\frac{\lfloor 16.30968\rfloor}{e}=\frac{16}{e}. \tag{40}$$

The claim is proved. ■

It follows

Theorem 7 *Let $N \in \mathbb{N}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$; $j = 1,2,...,r$. Then*

$$I_{2j}^{*}(\alpha) := \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\left(\sum\limits_{i=1}^{N} |s_i|\right)}{3\xi_n} \right) e^{-\frac{\sum\limits_{i=1}^{N}|s_i|}{\xi_n}} ds_1...ds_N \leq \tag{41}$$

$$\left[\left(1 + \frac{j}{3}N\right) + \left(1 + \frac{j}{3}\right)\left(\frac{2}{e}\right)^N \right] < +\infty,$$

are uniformly bounded in $\xi_n \in (0,1]$, fulfilling (7).

Proof. We have

$$I_{2j}^{*}(\alpha) = \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\left(\sum\limits_{i=1}^{N} |s_i|\right)}{3\xi_n} \right) e^{-\frac{\sum\limits_{i=1}^{N}|s_i|}{\xi_n}} ds_1...ds_N =$$

$$\frac{1}{\xi_n^N} \int_{\mathbb{R}_+^N} \left(1 + \frac{j\left(\sum\limits_{i=1}^{N} s_i\right)}{3\xi_n} \right) e^{-\frac{\sum\limits_{i=1}^{N} s_i}{\xi_n}} ds_1...ds_N = \tag{42}$$

$$\int_{\mathbb{R}_+^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right) \right) e^{-\sum\limits_{i=1}^{N} z_i} dz_1...dz_N =$$

$$\int_{[0,1]^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right) \right) e^{-\sum\limits_{i=1}^{N} z_i} dz_1...dz_N +$$

$$\int_{(\mathbb{R}_+ - [0,1])^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right) \right) e^{-\sum\limits_{i=1}^{N} z_i} dz_1...dz_N \leq$$

$$\left(1 + \frac{j}{3}N\right) + \int_{(\mathbb{R}_+ - [0,1])^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right) \right) e^{-\sum\limits_{i=1}^{N} z_i} dz_1...dz_N \leq$$

$$\left(1 + \frac{j}{3}N\right) + \left(1 + \frac{j}{3}\right) \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^{N} z_i\right) e^{-\sum\limits_{i=1}^{N} z_i} dz_1...dz_N \leq$$

$$\left(1 + \frac{j}{3}N\right) + \left(1 + \frac{j}{3}\right) \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^{N} z_i\right) \left(\prod_{i=1}^{N} e^{-z_i}\right) \left(\prod_{i=1}^{N} dz_i\right) =$$

$$\left(1 + \frac{j}{3}N\right) + \left(1 + \frac{j}{3}\right) \prod_{i=1}^{N} \int_1^{\infty} z_i e^{-z_i} dz_i = \tag{43}$$

$$\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\prod_{i=1}^{N}\int_{1}^{\infty} z_i^{2-1}e^{-z_i}dz_i =$$

$$\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\prod_{i=1}^{N}\Gamma\left(2,1\right) =$$

$$\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\left(\frac{\lfloor e\rfloor}{e}\right)^{N} =$$

$$\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\left(\frac{2}{e}\right)^{N} < +\infty.$$

■

We make

Remark 8 *By (28), (29), Remark 5, and Theorem 6, we observe that* ($j = 0, 1, ..., r$)

$$\frac{1}{(2\xi_n)^N}\int_{\mathbb{R}^N} |s_i|\,|s_{j^*}|\, e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1...ds_N,$$

$$\frac{1}{(2\xi_n)^N}\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1...ds_N, \tag{44}$$

$$\frac{1}{(2\xi_n)^N}\int_{\mathbb{R}^N} |s_i|\, e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1...ds_N,$$

are uniformly bounded in $\xi_n \in (0,1]$ *and they converge to zero as* $\xi_n \to 0$.

Based on Theorem 4, Remark 5, Theorem 6, Theorem 7 and Remark 8, we present our major result about approximation properties of smooth Picard singular integral operators P_n, see (30).

Theorem 9 *Here* $f \in C^2\left(\mathbb{R}^N\right)$ *and let* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $N \geq 1$, $|\alpha| := \sum_{i=1}^{N}\alpha_i = 2$; $x \in \mathbb{R}^N$, *and* f_α *of order* 2, $f \in C_B\left(\mathbb{R}^N\right) \cup C_U\left(\mathbb{R}^N\right)$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.

Denote ($n \in \mathbb{N}$)

$$\overline{\Delta}_n(x) := P_n(f,x) - f(x) -$$

$$\left(\sum_{j=0}^{r}\alpha_j j\right)\sin(1)\left[\sum_{i=1}^{N}\frac{\partial f(x)}{\partial x_i}\left(\frac{1}{(2\xi_n)^N}\int_{\mathbb{R}^N} s_i e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1...ds_N\right)\right] \tag{45}$$

$$-2\left(\sum_{j=0}^{r}\alpha_j j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{N}\left(\frac{1}{(2\xi_n)^N}\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1...ds_N\right)\frac{\partial^2 f(x)}{\partial x_i^2}\right.$$

$$+\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}}\left(\frac{1}{(2\xi_n)^N}\int_{\mathbb{R}^N} s_i s_{j^*} e^{-\frac{\sum_{i=1}^N |s_i|}{\xi_n}} ds_1...ds_N\right)\frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}}\right\}.$$

Then

(i)

$$\left|\overline{\Delta}_n(x)\right| \leq \left\|\overline{\Delta}_n(x)\right\|_\infty \leq$$

$$\sum_{j=0}^r |\alpha_j| \left[\left[j^2 \sum_{\substack{\alpha_i\in\mathbb{Z}^+,\\ \alpha:|\alpha|=2}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right) \omega_1(f_\alpha, \xi_n) \frac{1}{(2\xi_n)^N}\right.\right.$$

$$\left.\int_{\mathbb{R}^N}\left(1+\frac{j\|s\|_\infty}{3\xi_n}\right)\left(\prod_{i=1}^N |s_i|^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^N |s_i|}{\xi_n}} ds_1...ds_N\right] + \tag{46}$$

$$\left.\frac{1}{2}\omega_1(f,\delta)\frac{1}{(2\xi_n)^N}\int_{\mathbb{R}^N}\left(1+\frac{j\|s\|_\infty}{3\xi_n}\right) e^{-\frac{\sum_{i=1}^N |s_i|}{\xi_n}} ds_1...ds_N\right] =: \overline{\varphi}_{\xi_n}.$$

In case of all f_α of order 2 and $f \in C_U(\mathbb{R}^N)$ and $\xi_n \to 0$, as $n \to \infty$, then $\overline{\Delta}_n(x)$, $\left\|\overline{\Delta}_n(x)\right\|_\infty \to 0$ with rates.

(ii) If $\frac{\partial f(x)}{\partial x_i} = 0$, $i = 1,...,N$, and $f_\alpha(x) = 0$, $\alpha_i \in \mathbb{Z}^+$, $i = 1,...,N$, with $|\alpha| = 2$, then

$$|P_n(f,x) - f(x)| \leq \overline{\varphi}_{\xi_n}. \tag{47}$$

And $P_n(f,x) \to f(x)$ in the uniformly continuous case.

(iii) Additionally assume that all partials of order ≤ 2 are bounded. Hence

$$\|P_n(f) - f\|_\infty \leq \left(\sum_{j=0}^r |\alpha_j| j\right)(0.8414)$$

$$\left[\sum_{i=1}^N \left\|\frac{\partial f}{\partial x_i}\right\|_\infty \frac{1}{(2\xi_n)^N}\left(\int_{\mathbb{R}^N} |s_i| e^{-\frac{\sum_{i=1}^N |s_i|}{\xi_n}} ds_1...ds_N\right)\right]$$

$$+\left(\sum_{j=0}^r |\alpha_j| j^2\right)(0.4596)$$

$$\left\{\sum_{i=1}^N \frac{1}{(2\xi_n)^N}\left(\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^N |s_i|}{\xi_n}} ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_\infty +\right.$$

$$\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}} \frac{1}{(2\xi_n)^N} \left(\int_{\mathbb{R}^N} |s_i|\,|s_{j^*}|\, e^{-\frac{\sum_{i=1}^N |s_i|}{\xi_n}} ds_1...ds_N \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_\infty \Bigg\} + \overline{\varphi}_{\xi_n}. \tag{48}$$

If all f_α *of order* 2*,* $f \in C_U\left(\mathbb{R}^N\right)$*, then*

$$\|P_n(f) - f\|_\infty \to 0 \text{ with rates, as } \xi_n \to 0,\ n \to +\infty.$$

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