Trigonometric based multivariate smooth Picard singular integrals L_p approximation

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Abstract

In this article we reexamine the L_p, $1 \leq p < \infty$ approximation properties of general smooth multivariate singular integral operators over $\mathbb{R}^N$, $N \geq 1$. It is a trigonometric based approach with detailed applications to the corresponding smooth multivariate Picard singular integral operators. The results are quantitative via Jackson type inequalities involving their first L_p modulus of continuity.

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1 Introduction

The rate of convergence of univariate and multivariate singular integral operators has been studied extensively in [1]-[3] and [5], [6] and [8]. All these motivate our current work. In particular we studied the smooth singular integral operators in [1]-[3] and [6], which are not in general positive ones, their L_p approximation properties.

Here we continue the L_p study of the last ones at the multivariate level, at first in general, and then apply our theory to the smooth Picard ones. The main tool, we are based on, is a new trigonometric multivariate Taylor formula from [4]. Our quantitative estimates are related to L_p approximations, using the multivariate first L_p modulus of continuity.

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2 Results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_{+}$, we define

$$\alpha_j := \alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, ..., r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0. \end{cases} \tag{1}$$

and

$$\delta_k := \delta_{k,r}^{[m]} := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, ..., m \in \mathbb{N}. \tag{2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{4}$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\theta_n (f; x_1, ..., x_N) := \theta_{r,n}^{[m]} (f; x_1, ..., x_N) :=$$

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f (x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) \, d\mu_{\xi_n} (s), \tag{5}$$

where $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{Z}$, $m \in \mathbb{Z}_{+}$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers.

The operators $\theta_{r,n}^{[m]}$ preserve constants and are not in general positive operators, see [2], pp. 2-3.

Here we deal with $f \in C^m (\mathbb{R}^N)$, $m \in \mathbb{Z}^+$, with $f_\alpha \in L_p (\mathbb{R}^N)$, $|\alpha| = m \in \mathbb{Z}^+$, $p \geq 1$; where f_α denotes the mixed partial $\frac{\partial^{\widetilde{j}} f(\cdot,...,\cdot)}{\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\alpha| := \sum_{j=1}^{N} \alpha_j = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

We need

Definition 1 *(see also [2], p. 20) We call*

$$\Delta_u^r f (x) := \Delta_{u_1, u_2, ..., u_N}^r f (x_1, ..., x_N) := \tag{6}$$

$$\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f (x_1 + j u_1, x_2 + j u_2, ..., x_N + j u_N) .$$

Let $p \geq 1$, the modulus of smoothness of order r is given by

$$\omega_r(f;h)_p := \sup_{\|u\|_2 \leq h} \|\Delta_u^r(f)\|_p, \tag{7}$$

$h > 0$.

(I) We consider the general case of $m \in \mathbb{N}$, $p > 1$.

We make

Remark 2 *For $\widetilde{j} = 1, ..., m,$ and $\alpha := (\alpha_1, ..., \alpha_N),\ \alpha_i \in \mathbb{Z}^+,\ i = 1, ..., N,$ $|\alpha| := \sum_{i=1}^N \alpha_i = \widetilde{j}$, under the assumption of Theorem 4,*

$$c_{\alpha,n,\widetilde{j}} := c_{\alpha,n} := \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} d\mu_{\xi_n}(s_1, ..., s_N) \in \mathbb{R}, \tag{8}$$

see also [2].

From [2] we get

$$E_{r,n}^{[m]}(x) := \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^m \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n} f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) = \tag{9}$$

$$m \sum_{|\alpha|=\widetilde{j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left(\Delta_{\theta s}^r f_\alpha(x) \right) d\theta \right) d\mu_{\xi_n}(s) \right)$$

$$=: R_{r,n}^{[m]}(x), \quad \forall\, x \in \mathbb{R}^N. \tag{10}$$

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| E_{r,n}^{[m]}(x) \right|^p = \left| R_{r,n}^{[m]}(x) \right|^p$$

$$\left(set\ c_1 := m \sum_{|\alpha|=\widetilde{j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \right)$$

$$= \left| c_1 \int_{\mathbb{R}^N} \left(\prod_{i=1}^N s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left(\Delta_{\theta s}^r f_\alpha(x) \right) d\theta \right) d\mu_{\xi_n}(s) \right|^p \leq$$

$$\left(c_1 \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left| \Delta_{\theta s}^r f_\alpha(x) \right| d\theta \right) d\mu_{\xi_n}(s) \right)^p. \tag{11}$$

Hence we have

$$I_1 := \int_{\mathbb{R}^N} \left| E_{r,n}^{[m]}(x) \right|^p dx \leq \tag{12}$$

$$\int_{\mathbb{R}^N} \left(c_1 \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left| \Delta_{\theta s}^r f_\alpha(x) \right| d\theta \right) d\mu_{\xi_n}(s) \right)^p dx =$$

(Call $0 \leq \gamma_\alpha(s,x) := \prod_{i=1}^{N} |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left| \Delta_{\theta s}^r f_\alpha(x) \right| d\theta$*)*

$$\int_{\mathbb{R}^N} \left(c_1 \int_{\mathbb{R}^N} \gamma_\alpha(s,x) \, d\mu_{\xi_n}(s) \right)^p dx =: I_2. \tag{13}$$

Therefore it holds

$$I_2 \leq \int_{\mathbb{R}^N} \left(\binom{m+N-1}{m}^{\frac{p}{q}} c_1 \int_{\mathbb{R}^N} \gamma_\alpha^p(s,x) \, d\mu_{\xi_n}(s) \right) dx =: I_3, \tag{14}$$

as there are $\binom{m+N-1}{m}$ *distinct partials of order* m*.*

But we have

$$\gamma_\alpha(s,x) \leq \prod_{i=1}^{N} |s_i|^{\alpha_i} \left(\int_0^1 \left((1-\theta)^{m-1} \right)^q d\theta \right)^{\frac{1}{q}} \left(\int_0^1 \left| \Delta_{\theta s}^r f_\alpha(x) \right|^p d\theta \right)^{\frac{1}{p}} \tag{15}$$

$$= \prod_{i=1}^{N} |s_i|^{\alpha_i} \frac{1}{(q(m-1)+1)^{\frac{1}{q}}} \left(\int_0^1 \left| \Delta_{\theta s}^r f_\alpha(x) \right|^p d\theta \right)^{\frac{1}{p}}.$$

Hence we get

$$\gamma_\alpha^p(s,x) \leq \prod_{i=1}^{N} |s_i|^{\alpha_i p} \frac{1}{(q(m-1)+1)^{\frac{p}{q}}} \left(\int_0^1 \left| \Delta_{\theta s}^r f_\alpha(x) \right|^p d\theta \right). \tag{16}$$

Thus we obtain

$$I_3 \leq \int_{\mathbb{R}^N} \left(c_1^* \int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i p} \left(\int_0^1 \left| \Delta_{\theta s}^r f_\alpha(x) \right|^p d\theta \right) d\mu_{\xi_n}(s) \right) dx \tag{17}$$

(where $c_1^* := \binom{m+N-1}{m}^{\frac{p}{q}} \frac{1}{(q(m-1)+1)^{\frac{p}{q}}} c_1$*)*

$$= c_1^* \int_{\mathbb{R}^N} \left(\int_0^1 \left(\int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i p} \left| \Delta_{\theta s}^r f_\alpha(x) \right|^p dx \right) d\theta \right) d\mu_{\xi_n}(s)$$

$$= c_1^* \int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i p} \left(\int_0^1 \left(\int_{\mathbb{R}^N} \left| \Delta_{\theta s}^r f_\alpha(x) \right|^p dx \right) d\theta \right) d\mu_{\xi_n}(s) =: I_4. \tag{18}$$

Consequently we derive

$$I_4 \le c_1^* \int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i p} \left(\int_0^1 \left(\omega_r \left(f_\alpha; \theta \|s\|_2 \right)_p \right)^p d\theta \right) d\mu_{\xi_n}(s) \tag{19}$$

$$\le c_1^* \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i p} \right) \omega_r \left(f_\alpha; \|s\|_2 \right)_p^p d\mu_{\xi_n}(s)$$

$$= c_1^* \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i p} \right) \omega_r \left(f_\alpha; \xi_n \frac{\|s\|_2}{\xi_n} \right)_p^p d\mu_{\xi_n}(s)$$

(by $\omega_r (f, \lambda h)_p \le (1+\lambda)^r \omega_r (f, h)_p$, for any $h, \lambda > 0$, $p \ge 1$)

$$\le c_1^* \omega_r \left(f_\alpha; \xi_n \right)_p^p \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i p} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{rp} d\mu_{\xi_n}(s) \right). \tag{20}$$

We have proved that

$$\int_{\mathbb{R}^N} \left| E_{r,n}^{[m]}(x) \right|^p dx \le$$

$$c_1^* \omega_r \left(f_\alpha; \xi_n \right)_p^p \left(\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right). \tag{21}$$

Thus we get ($p > 1$)

$$\left\| E_{r,n}^{[m]}(x) \right\|_p \le$$

$$\left[c_1^* \left[\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right] \omega_r \left(f_\alpha; \xi_n \right)_p^p \right]^{\frac{1}{p}}. \tag{22}$$

We make

Remark 3 *Notice that ($p > 1$)*

$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \le \tag{23}$$

$$\left[\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} < \infty,$$

by assumption of Theorem 4.

As in [2] then we get that

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i} d\mu_{\xi_n}(s) < \infty. \tag{24}$$

Hence $c_{\alpha,n,\widetilde{j}} \in \mathbb{R}$.

From the above we have proved

Theorem 4 *Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence. Assume for all $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = m$ that we have*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{25}$$

For $\widetilde{j} = 1, ..., m$, and $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$, call

$$c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} d\mu_{\xi_n}(s). \tag{26}$$

Then

$$\left\| E_{r,n}^{[m]} \right\|_p = \left\| \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) \right\|_{p,x} \leq \tag{27}$$

$$\left[\binom{m+N-1}{m}^{\frac{p}{q}} \left(\frac{m}{(q(m-1)+1)^{\frac{p}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \cdot \right.\right.$$

$$\left.\left. \left[\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right] \omega_r(f_\alpha, \xi_n)_p^p \right) \right]^{\frac{1}{p}}.$$

As $n \to \infty$ and $\xi_n \to 0$, by (27), we obtain that $\left\| E_{r,n}^{[m]} \right\|_p \to 0$ with rates.

One also gets by (27) that

$$\left\| \theta_{r,n}^{[m]}(f; x) - f(x) \right\|_{p,x} \leq$$

$$\sum_{\widetilde{j}=1}^{m} \left| \delta_{\widetilde{j},r}^{[m]} \right| \left(\sum_{|\alpha|=\widetilde{j}} \frac{\left| c_{\alpha,n,\widetilde{j}} \right|}{\prod_{i=1}^{N} \alpha_i!} \|f_\alpha\|_p \right) + R.H.S.(27), \tag{28}$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

Assuming that $c_{\alpha,n,\widetilde{j}} \to 0$, $\xi_n \to 0$, as $n \to \infty$, we get $\left\| \theta_{r,n}^{[m]}(f) - f \right\|_p \to 0$, that is $\theta_{r,n}^{[m]} \to I$ the unit operator, in L_p norm, with rates.

Inequality (27) provides a correction in the constants of the inequality in Theorem 4 in [2], p. 25.

(II) Here we treat the case of $f \in C^2\left(\mathbb{R}^N\right)$ with $f_\alpha \in L_p\left(\mathbb{R}^N\right)$, $|\alpha| = 2$, $p \geq 1$.

We make

Remark 5 *The next are based on a new multivariate trigonometric Taylor formula ([4]).*

Let $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N)$, $z := (z_1, ..., z_N) := (x_1 + s_1 j,\ x_2 + s_2 j, ... x_N + s_N j) = x + sj$; $j = 0, 1, ..., r$, *and* $x := x_0 = (x_{01}, ..., x_{0N}) = (x_1, ..., x_N)$, *all in* $\mathbb{R}^N$; $p > 1$, $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.

Here $f \in C^2\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}$, *and clearly all the mixed partials commnute.*

Consider

$$g_{x+sj}(t) := f(x + t(sj)), \quad 0 \leq t \leq 1. \tag{29}$$

Notice that $g_{x+sj}(0) = f(x)$, $g_{x+sj}(1) = f(x + sj)$.

We have (by [4])

$$f(x+sj) - f(x) = g_{x+sj}(1) - g_{x+sj}(0) =$$

$$g'_{x+sj}(0)\sin(1) + 2g''_{x+sj}(0)\sin^2\left(\frac{1}{2}\right) +$$

$$\int_0^1 \left[\left(g''_{x+sj}(t) + g_{x+sj}(t)\right) - \left(g''_{x+sj}(0) + g_{x+sj}(0)\right)\right]\sin(1-t)\,dt =$$

$$\left(\sum_{i=1}^N (s_i j)\frac{\partial f}{\partial x_i}(x)\right)\sin(1) +$$

$$2\left\{\left[\left(\sum_{i=1}^N (s_i j)\frac{\partial}{\partial x_i}\right)^2 f\right](x)\right\}\sin^2\left(\frac{1}{2}\right) +$$

$$\int_0^1 \left\{\left\{\left[\left(\sum_{i=1}^N (s_i j)\frac{\partial}{\partial x_i}\right)^2 f\right](x + t(sj)) + f(x + t(sj))\right\} - \right. \tag{30}$$

$$\left.\left\{\left[\left(\sum_{i=1}^N (s_i j)\frac{\partial}{\partial x_i}\right)^2 f\right](x) + f(x)\right\}\right\}\sin(1-t)\,dt.$$

Denote the remainder ($j = 0, 1, ..., r$)

$$R_j := \int_0^1 \left\{\left\{\left[\left(\sum_{i=1}^N (s_i j)\frac{\partial}{\partial x_i}\right)^2 f\right](x + tsj) + f(x + tsj)\right\}\right.$$

$$-\left\{\left[\left(\sum_{i=1}^{N}(s_i j)\frac{\partial}{\partial x_i}\right)^2 f\right](x)+f(x)\right\}\Bigg\}\sin(1-t)\,dt= \tag{31}$$

$$\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(js_i)^{\alpha_i}\right)[f_\alpha(x+t(js))-f_\alpha(x)]\right.$$

$$\left.+(f(x+t(js))-f(x))\right\}\sin(1-t)\,dt.$$

Therefore it holds

$$|R_j|\le\int_0^1\left\{\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\right.$$

$$\left.\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)|f_\alpha(x+tsj)-f_\alpha(x)|+|f(x+tsj)-f(x)|\right\}|\sin(1-t)|\,dt. \tag{32}$$

By $|\sin x|\le|x|$, $x\in\mathbb{R}$, *we get*

$$|R_j|\le\int_0^1(1-t)\left\{\left(\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\right.\right. \tag{33}$$

$$\left.\left.\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)|f_\alpha(x+tsj)-f_\alpha(x)|\right)+|f(x+tsj)-f(x)|\right\}dt.$$

Next we can write

$$\sum_{j=0}^{r}\alpha_j[f(x+sj)-f(x)]-\left(\sum_{j=0}^{r}\alpha_j j\right)\left(\sum_{i=1}^{N}s_i\frac{\partial f}{\partial x_i}(x)\right)\sin(1)- \tag{34}$$

$$2\left(\sum_{j=0}^{r}\alpha_j j^2\right)\left\{\left[\left(\sum_{i=1}^{N}s_i\frac{\partial}{\partial x_i}\right)^2 f\right](x)\right\}\sin^2\left(\frac{1}{2}\right)=\sum_{j=0}^{r}\alpha_j R_j.$$

Call

$$R := \sum_{j=0}^{r} \alpha_j R_j. \tag{35}$$

Clearly, it holds

$$\theta_n (f, x) - f (x) = \sum_{j=0}^{r} \alpha_j \int_{\mathbb{R}^N} (f (x + sj) - f (x)) \, d\mu_{\xi_n} (s) \,. \tag{36}$$

Denote $(n \in \mathbb{N})$

$$\Delta_n (x) := \theta_n (f, x) - f (x) - \left(\sum_{j=0}^{r} \alpha_j j \right) \sin (1) \left[\sum_{i=1}^{N} \frac{\partial f (x)}{\partial x_i} \left(\int_{\mathbb{R}^N} s_i d\mu_{\xi} (s) \right) \right]$$

$$-2 \left(\sum_{j=0}^{r} \alpha_j j^2 \right) \sin^2 \left(\frac{1}{2} \right) \left\{ \sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi} (s) \right) \frac{\partial^2 f (x)}{\partial x_i^2} + \right. \tag{37}$$

$$\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, \ldots, N\}}} \left(\int_{\mathbb{R}^N} s_i s_{j^*} d\mu_{\xi} (s) \right) \frac{\partial^2 f (x)}{\partial x_i \partial x_{j^*}} \right\}.$$

We observe that

$$\Delta_n (x) = \sum_{j=0}^{r} \alpha_j \int_{\mathbb{R}^N} R_j d\mu_{\xi_n} (s) = \int_{\mathbb{R}^N} R d\mu_{\xi_n} (s) \,. \tag{38}$$

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$|\Delta_n (x)|^p = \left| \int_{\mathbb{R}^N} R d\mu_{\xi_n} (s) \right|^p \leq \left(\int_{\mathbb{R}^N} |R| \, d\mu_{\xi_n} (s) \right)^p \leq$$

$$\left(\int_{\mathbb{R}^N} \left(\sum_{j=0}^{r} |\alpha_j| \, |R_j| \right) d\mu_{\xi_n} (s) \right)^p = \left(\sum_{j=0}^{r} |\alpha_j| \left(\int_{\mathbb{R}^N} |R_j| \, d\mu_{\xi_n} (s) \right) \right)^p \leq \tag{39}$$

(by Hölder's inequality)

$$\left(\sum_{j=0}^{r} |\alpha_j|^q \right)^{\frac{p}{q}} \left(\sum_{j=0}^{r} \left(\int_{\mathbb{R}^N} |R_j| \, d\mu_{\xi_n} (s) \right)^p \right) \leq$$

(again by Hölder's inequality)

$$\left(\sum_{j=0}^{r} |\alpha_j|^q \right)^{\frac{p}{q}} \left(\sum_{j=0}^{r} \left(\int_{\mathbb{R}^N} |R_j|^p \, d\mu_{\xi_n} (s) \right) \right).$$

Thus, we have

$$\left|\Delta_n(x)\right|^p \leq \left(\sum_{j=0}^{r}\left|\alpha_j\right|^q\right)^{\frac{p}{q}}\left(\sum_{j=0}^{r}\left(\int_{\mathbb{R}^N}\left|R_j\right|^p d\mu_{\xi_n}(s)\right)\right). \tag{40}$$

By (33), we get that

$$\left|R_j\right| \leq 2\left(\int_0^1 (1-t)^q\,dt\right)^{\frac{1}{q}}\left\{\int_0^1\left[\left(\sum_{|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(j\left|s_i\right|\right)^{\alpha_i}\right)\right.\right.\right.$$

$$\left.\left.\left.\left|f_\alpha(x+tsj)-f_\alpha(x)\right|\right)+\left|f(x+tsj)-f(x)\right|\right]^p dt\right\}^{\frac{1}{p}} \leq$$

$$\frac{2}{(q+1)^{\frac{1}{q}}}\left\{\left\{\sum_{|\alpha|=2}\left(\left(\left(\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(j\left|s_i\right|\right)^{\alpha_i}\right)\right)^p\right)^{\frac{1}{p}}\right.\right.\right.$$

$$\left.\left.\left.\left(\int_0^1\left|f_\alpha(x+tsj)-f_\alpha(x)\right|^p dt\right)^{\frac{1}{p}}\right\}+\left(\int_0^1\left|f(x+tsj)-f(x)\right|^p dt\right)^{\frac{1}{p}}\right\} \leq \tag{41}$$

$$\frac{2}{(q+1)^{\frac{1}{q}}}\left(\frac{N(N+1)+2}{2}\right)^{\frac{1}{q}}\left\{\sum_{|\alpha|=2}\left(\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(j\left|s_i\right|\right)^{\alpha_i}\right)\right)^p\right.$$

$$\left.\left(\int_0^1\left|f_\alpha(x+tsj)-f_\alpha(x)\right|^p dt\right)+\left(\int_0^1\left|f(x+tsj)-f(x)\right|^p dt\right)\right\}^{\frac{1}{p}}.$$

Hence

$$\left|R_j\right|^p \leq \frac{2^p}{(q+1)^{\frac{p}{q}}}\left(\frac{N(N+1)+2}{2}\right)^{\frac{p}{q}} \tag{42}$$

$$\left\{\sum_{|\alpha|=2}\left(\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(j\left|s_i\right|\right)^{\alpha_i}\right)\right)^p\right.$$

$$\left.\left(\int_0^1\left|f_\alpha(x+tsj)-f_\alpha(x)\right|^p dt\right)+\left(\int_0^1\left|f(x+tsj)-f(x)\right|^p dt\right)\right\}.$$

By (40) and (42), we continue as follows

$$\left|\Delta_n(x)\right|^p \leq \left(\sum_{j=0}^{r}\left|\alpha_j\right|^q\right)^{\frac{p}{q}}\left\{\frac{2^p}{(q+1)^{\frac{p}{q}}}\left(\frac{N(N+1)+2}{2}\right)^{\frac{p}{q}}\right\}$$

$$\left\{\sum_{j=0}^{r}\left[\sum_{|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)^p\right.\right. \tag{43}$$

$$\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)^p\left(\int_0^1|f_\alpha(x+tsj)-f_\alpha(x)|^p\,dt\right)d\mu_{\xi_n}(s)\right)$$

$$\left.\left.+\left(\int_{\mathbb{R}^N}\left(\int_0^1|f(x+tsj)-f(x)|^p\,dt\right)d\mu_{\xi_n}(s)\right)\right]\right\}.$$

Furthermore it holds

$$\int_{\mathbb{R}^N}|\Delta_n(x)|^p\,dx \le \left(\sum_{j=0}^{r}|\alpha_j|^q\right)^{\frac{p}{q}}\left\{\frac{2^p}{(q+1)^{\frac{p}{q}}}\left(\frac{N(N+1)+2}{2}\right)^{\frac{p}{q}}\right\}$$

$$\left\{\sum_{j=0}^{r}\left[\sum_{|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)^p\right.\right.$$

$$\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)^p\left(\int_0^1\left(\int_{\mathbb{R}^N}|f_\alpha(x+tsj)-f_\alpha(x)|^p\,dx\right)dt\right)d\mu_{\xi_n}(s)\right)$$

$$\left.\left.+\left(\int_{\mathbb{R}^N}\left(\int_0^1\left(\int_{\mathbb{R}^N}|f(x+tsj)-f(x)|^p\,dx\right)dt\right)d\mu_{\xi_n}(s)\right)\right]\right\} \le \tag{44}$$

$$\left(\sum_{j=0}^{r}|\alpha_j|^q\right)^{\frac{p}{q}}\left\{\frac{2^p}{(q+1)^{\frac{p}{q}}}\left(\frac{N(N+1)+2}{2}\right)^{\frac{p}{q}}\right\}$$

$$\left\{\sum_{j=0}^{r}\left[\sum_{|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)^p\right.\right.$$

$$\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)^p\left(\int_0^1\omega_1\left(f_\alpha, jt\|s\|_2\right)_p^p\,dt\right)d\mu_{\xi_n}(s)\right)$$

$$\left.\left.+\left(\int_{\mathbb{R}^N}\left(\int_0^1\omega_1\left(f_\alpha, jt\|s\|_2\right)_p^p\,dt\right)d\mu_{\xi_n}(s)\right)\right]\right\} \le$$

$$\left(\sum_{j=0}^{r}|\alpha_j|^q\right)^{\frac{p}{q}}\left\{\frac{2^p}{(q+1)^{\frac{p}{q}}}\left(\frac{N(N+1)+2}{2}\right)^{\frac{p}{q}}\right\}$$

$$\left\{\sum_{j=0}^{r}\left[\sum_{|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)^{p}\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}\left(j\left|s_i\right|\right)^{\alpha_i}\right)^{p}\omega_1\left(f_\alpha,\frac{\xi_n j\left\|s\right\|_2}{\xi_n}\right)_p^{p}d\mu_{\xi_n}(s)\right)\right.\right.$$

$$\left.\left.+\left(\int_{\mathbb{R}^N}\omega_1\left(f,\frac{\xi_n j\left\|s\right\|_2}{\xi_n}{}_2\right)_p^{p}d\mu_{\xi_n}(s)\right)\right]\right\}\leq \tag{45}$$

$$\left(\sum_{j=0}^{r}\left|\alpha_j\right|^q\right)^{\frac{p}{q}}\left\{\frac{2^p}{(q+1)^{\frac{p}{q}}}\left(\frac{N(N+1)+2}{2}\right)^{\frac{p}{q}}\right\}$$

$$\left\{\sum_{j=0}^{r}\left[\sum_{|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)^{p}\omega_1\left(f_\alpha,\xi_n\right)_p^p\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}\left(j\left|s_i\right|\right)^{\alpha_i}\right)^{p}\left(1+\frac{j\left\|s\right\|_2}{\xi_n}\right)^{p}d\mu_{\xi_n}(s)\right)\right.\right.$$

$$\left.\left.+\omega_1\left(f,\xi_n\right)_p^p\left(\int_{\mathbb{R}^N}\left(1+\frac{j\left\|s\right\|_2}{\xi_n}\right)^{p}d\mu_{\xi_n}(s)\right)\right]\right\}\leq \tag{46}$$

$$\left(\sum_{j=0}^{r}\left|\alpha_j\right|^q\right)^{\frac{p}{q}}\left\{\frac{2^p}{(q+1)^{\frac{p}{q}}}\left(\frac{N(N+1)+2}{2}\right)^{\frac{p}{q}}\right\}$$

$$\left\{\sum_{j=0}^{r}\left[\sum_{|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)^{p}\omega_1\left(f_\alpha,\xi_n\right)_p^p\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}\left(j\left|s_i\right|\right)^{\alpha_i}\right)^{p}\left(1+\frac{j^p\left\|s\right\|_2^p}{\xi_n^p}\right)d\mu_{\xi_n}(s)\right)\right.\right.$$

$$\left.\left.+\omega_1\left(f,\xi_n\right)_p^p\left(\int_{\mathbb{R}^N}\left(1+\frac{j^p\left\|s\right\|_2^p}{\xi_n^p}\right)d\mu_{\xi_n}(s)\right)\right]\right\}. \tag{47}$$

Then it holds

$$\left\|\Delta_n(x)\right\|_p\leq\left(\sum_{j=0}^{r}\left|\alpha_j\right|^q\right)^{\frac{1}{q}}\left\{\frac{2}{(q+1)^{\frac{1}{q}}}\left(\frac{N(N+1)+2}{2}\right)^{\frac{1}{q}}\right\}$$

$$\left\{\sum_{j=0}^{r}\left[j^{2p}\sum_{|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)^{p}\omega_1\left(f_\alpha,\xi_n\right)_p^p\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}\left|s_i\right|^{p\alpha_i}\right)\left(1+\frac{j^p\left\|s\right\|_2^p}{\xi_n^p}\right)d\mu_{\xi_n}(s)\right)\right.\right.$$

$$\left.\left.+\omega_1\left(f,\xi_n\right)_p^p\left(\int_{\mathbb{R}^N}\left(1+\frac{j^p\left\|s\right\|_2^p}{\xi_n^p}\right)d\mu_{\xi_n}(s)\right)\right]\right\}^{\frac{1}{p}}. \tag{48}$$

We have proved the following trigonometric induced alternative L_p approximation result for θ_n operators.

Theorem 6 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $0<\xi_n\leq 1$, $n\in\mathbb{N}$. Here we deal with $f\in C^2\left(\mathbb{R}^N\right)$, $N\geq 1$, with $f_\alpha\in L_p\left(\mathbb{R}^N\right)$, $|\alpha|=2$, where $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, and $|\alpha|=\sum_{i=1}^{N}\alpha_i$; $x\in\mathbb{R}^N$. Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$. Suppose that for all $\alpha:|\alpha|=2$, $j=0,1,...,r$, we have that both*

$$I_{1j}\left(\alpha\right):=\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}|s_i|^{p\alpha_i}\right)\left(1+\frac{j^p\left\|s\right\|_2^p}{\xi_n^p}\right)d\mu_{\xi_n}\left(s\right), \tag{49}$$

$$I_2:=\int_{\mathbb{R}^N}\left(\frac{\left\|s\right\|_2}{\xi_n}\right)^p d\mu_{\xi_n}\left(s\right), \tag{50}$$

are uniformly bounded in $\xi_n\in(0,1]$.

Denote ($n\in\mathbb{N}$)

$$\Delta_n\left(x\right):=\theta_n\left(f,x\right)-f\left(x\right)-\left(\sum_{j=0}^{r}\alpha_j j\right)\sin\left(1\right)\left[\sum_{i=1}^{N}\frac{\partial f\left(x\right)}{\partial x_i}\left(\int_{\mathbb{R}^N}s_i d\mu_{\xi_n}\left(s\right)\right)\right]$$

$$-2\left(\sum_{j=0}^{r}\alpha_j j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{N}\left(\int_{\mathbb{R}^N}s_i^2 d\mu_{\xi_n}\left(s\right)\right)\frac{\partial^2 f\left(x\right)}{\partial x_i^2}+\right. \tag{51}$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}}\left(\int_{\mathbb{R}^N}s_i s_{j^*}d\mu_{\xi_n}\left(s\right)\right)\frac{\partial^2 f\left(x\right)}{\partial x_i\partial x_{j^*}}\right\}.$$

Then, it holds

$$\left\|\Delta_n\left(x\right)\right\|_p\leq\left(\sum_{j=0}^{r}|\alpha_j|^q\right)^{\frac{1}{q}}\left\{\frac{2}{(q+1)^{\frac{1}{q}}}\left(\frac{N\left(N+1\right)+2}{2}\right)^{\frac{1}{q}}\right\}$$

$$\left\{\sum_{j=0}^{r}\left[j^{2p}\sum_{|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)^p\omega_1\left(f_\alpha,\xi_n\right)_p^p\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}|s_i|^{p\alpha_i}\right)\left(1+\frac{j^p\left\|s\right\|_2^p}{\xi_n^p}\right)d\mu_{\xi_n}\left(s\right)\right)\right.\right. \tag{52}$$

$$\left.\left.+\omega_1\left(f,\xi_n\right)_p^p\left(\int_{\mathbb{R}^N}\left(1+\frac{j^p\left\|s\right\|_2^p}{\xi_n^p}\right)d\mu_{\xi_n}\left(s\right)\right)\right]\right\}^{\frac{1}{p}}.$$

As $n\to\infty$ and $\xi_n\to 0$, by (52), we obtain that $\left\|\Delta_n\right\|_p\to 0$ with rates. One also gets by (52) that

$$\left\|\theta_n\left(f,x\right)-f\left(x\right)\right\|_{p,x}\leq$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\, j\right)\sin(1)\left[\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_p\left(\int_{\mathbb{R}^N}|s_i|\, d\mu_{\xi_n}(s)\right)\right]$$

$$+2\left(\sum_{j=0}^{r}|\alpha_j|\, j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{N}\left(\int_{\mathbb{R}^N}s_i^2 d\mu_{\xi_n}(s)\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_p+\right.$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,\ldots,N\}}}\left(\int_{\mathbb{R}^N}|s_i s_{j^*}|\, d\mu_{\xi_n}(s)\right)\left\|\frac{\partial^2 f}{\partial x_i\partial x_{j^*}}\right\|_p\right\}+R.H.S.\,(52)\,, \qquad (53)$$

given that $\|f_\alpha\|_p<\infty$, $|\alpha|=\widetilde{j}$, $\widetilde{j}=1,2$. *Assuming that* $\int_{\mathbb{R}^N}s_i^2 d\mu_{\xi_n}(s)$, $\int_{\mathbb{R}^N}|s_i s_{j^*}|\, d\mu_{\xi_n}(s)$, $i,j^*\in\{1,...,N\}$, $i\neq j^*$, *converge to zero as* $\xi_n\to 0$, *we get* $\|\theta_n(f,x)-f(x)\|_p\to 0$, *that is* $\theta_n\to I$ *the unit operator, in* L_p *norm, with rates.*

We make

Remark 7 *It follows the* L_1 *approximation by* θ_n *operators, i.e.* $p=1$ *case. They are all the same, as in Remark 5, till including (38):*

$$\Delta_n(x)=\sum_{j=0}^{r}\alpha_j\int_{\mathbb{R}^N}R_j d\mu_{\xi_n}(s)=\int_{\mathbb{R}^N}R d\mu_{\xi_n}(s)\,.$$

Hence it holds

$$|\Delta_n(x)|\leq\left(\sum_{j=0}^{r}|\alpha_j|\left(\int_{\mathbb{R}^N}|R_j|\, d\mu_{\xi_n}(s)\right)\right)\overset{(by\ (33))}{\leq} \qquad (54)$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|\int_{\mathbb{R}^N}\left(\int_0^1\left\{\sum_{|\alpha|=2}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)\right.\right.\right.$$

$$\left.\left.\left.|f_\alpha(x+tsj)-f_\alpha(x)|\right)+|f(x+tsj)-f(x)|\right\}dt\right)d\mu_{\xi_n}(s)\right\}.$$

Therefore we have

$$\int_{\mathbb{R}^N}|\Delta_n(x)|\, dx\leq$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|\left[\left(\int_{\mathbb{R}^N}\left(\int_0^1\left\{\sum_{|\alpha|=2}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)\right.\right.\right.\right.\right.$$

$$\left.\left.\left(\int_{\mathbb{R}^N}|f_\alpha(x+tsj)-f_\alpha(x)|\, dx\right)dt\right\}d\mu_{\xi_n}(s)\right)+$$

$$\left(\int_{\mathbb{R}^N}\left(\int_0^1\left(\int_{\mathbb{R}^N}\left|f(x+tsj)-f(x)\right|dx\right)dt\right)d\mu_{\xi_n}(s)\right)\Bigg]\Bigg\}\leq$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|\left[\left(\int_{\mathbb{R}^N}\left(\int_0^1\left\{\sum_{|\alpha|=2}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)\right.\right.\right.\right.\right.$$

$$\left.\omega_1\left(f_\alpha,jt\|s\|_2\right)_1 dt\right\}d\mu_{\xi_n}(s)\Big)+ \tag{55}$$

$$\left(\int_{\mathbb{R}^N}\left(\int_0^1\omega_1\left(f,jt\|s\|_2\right)_1 dt\right)d\mu_{\xi_n}(s)\right)\Bigg]\Bigg\}\leq$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|\left[\sum_{|\alpha|=2}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)\right.\right.\right.$$

$$\left.\left.\left.\omega_1\left(f_\alpha,\frac{\xi_n j\|s\|_2}{\xi_n}\right)_1 d\mu_{\xi_n}(s)\right)+\left(\int_{\mathbb{R}^N}\omega_1\left(f,\frac{\xi_n j\|s\|_2}{\xi_n}\right)_1 d\mu_{\xi_n}(s)\right)\right]\right\}\leq \tag{56}$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|\left[\sum_{|\alpha|=2}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\omega_1\left(f_\alpha,\xi_n\right)_1\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}(j|s_i|)^{\alpha_i}\right)\right.\right.\right.$$

$$\left.\left.\left.\left(1+\frac{j\|s\|_2}{\xi_n}\right)d\mu_{\xi_n}(s)\right)+\omega_1\left(f,\xi_n\right)_1\left(\int_{\mathbb{R}^N}\left(1+\frac{j\|s\|_2}{\xi_n}\right)d\mu_{\xi_n}(s)\right)\right]\right\}.$$

Then, it holds

$$\|\Delta_n(x)\|_1\leq\left\{\sum_{j=0}^{r}|\alpha_j|\left[j^2\sum_{|\alpha|=2}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\omega_1\left(f_\alpha,\xi_n\right)_1\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}(|s_i|)^{\alpha_i}\right)\right.\right.\right.$$

$$\left.\left.\left.\left(1+\frac{j\|s\|_2}{\xi_n}\right)d\mu_{\xi_n}(s)\right)+\omega_1\left(f,\xi_n\right)_1\left(\int_{\mathbb{R}^N}\left(1+\frac{j\|s\|_2}{\xi_n}\right)d\mu_{\xi_n}(s)\right)\right]\right\}. \tag{57}$$

We have proved the following trigonometric based alternative L_1 approximation result for θ_n operators.

Theorem 8 *Let $0<\xi_n\leq 1$, $n\in\mathbb{N}$, $x\in\mathbb{R}^N$. Here we deal with $f\in C^2(\mathbb{R}^N)$, $N\geq 1$, with $f_\alpha\in L_1(\mathbb{R}^N)$, $|\alpha|=2$, where $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, and $|\alpha|=\sum_{i=1}^{N}\alpha_i$. Let $\mu_{\xi n}$ be a Borel probability measure on $\mathbb{R}^N$. Suppose that for all $\alpha:|\alpha|=2$, $j=0,1,...,r$, we have that both*

$$I_{1j}^{*}(\alpha):=\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\left(1+\frac{j\|s\|_2}{\xi_n}\right)d\mu_{\xi_n}(s), \tag{58}$$

and

$$I_2^* := \int_{\mathbb{R}^N} \frac{\|s\|_2}{\xi_n} d\mu_{\xi_n}(s), \tag{59}$$

are uniformly bounded in $\xi_n \in (0,1]$.

Here $\Delta_n(x)$ *is as in (51).*

Then, it holds

$$\|\Delta_n(x)\|_1 \le \left\{ \sum_{j=0}^{r} |\alpha_j| \left[j^2 \sum_{|\alpha|=2} \left(\frac{2}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_1(f_\alpha, \xi_n)_1 \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} (|s_i|)^{\alpha_i} \right) \right.\right.\right.$$

$$\left.\left.\left. \left(1 + \frac{j\|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s) \right) + \omega_1(f, \xi_n)_1 \left(\int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s) \right) \right] \right\}. \tag{60}$$

As $n \to \infty$ *and* $\xi_n \to 0$, *by (60), we obtain that* $\|\Delta_n\|_1 \to 0$ *with rates. One also obtains by (60) that*

$$\|\theta_n(f) - f\|_1 \le$$

$$\left(\sum_{j=0}^{r} |\alpha_j| j \right) \sin(1) \left[\sum_{i=1}^{N} \left\| \frac{\partial f}{\partial x_i} \right\|_1 \left(\int_{\mathbb{R}^N} |s_i| d\mu_{\xi_n}(s) \right) \right]$$

$$+2 \left(\sum_{j=0}^{r} |\alpha_j| j^2 \right) \sin^2\left(\frac{1}{2}\right) \left\{ \sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_1 + \right.$$

$$\left. \sum_{\substack{i \ne j^*, \\ i, j^* \in \{1,...,N\}}} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_1 \right\} + R.H.S.(60), \tag{61}$$

given that $\|f_\alpha\|_1 < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, 2$. *Assuming that* $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)$, $\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s)$, $i, j^* \in \{1, ..., N\}$, $i \ne j^*$, *converge to zero as* $\xi_n \to 0$, *we derive* $\|\theta_n(f) - f\|_1 \to 0$, *that is* $\theta_n \to I$ *in* L_1 *norm, with rates.*

We make

Remark 9 *Next we will apply Theorems4, 6 and 8 to the specific multivariate smooth Picard singular integral operators*

$$P_n(f; x_1, ..., x_N) := P_{r,n}^{[m]}(f; x_1, ..., x_N) :=$$

$$\frac{1}{(2\xi_n)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N, \tag{62}$$

$r, n \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $0 < \xi_n \leq 1$.

Clearly here it is

$$d\mu_{\xi_n}(s) = \frac{1}{(2\xi_n)^N} e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N, \quad s \in \mathbb{R}^N. \tag{63}$$

We observe that

$$\frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N = 1, \tag{64}$$

see [3], Chap. 22, p. 356.

We need

Theorem 10 *([3], p. 361) Let $r, N \in \mathbb{N}$, $p > 1$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\alpha| := \sum_{i=1}^{N} \alpha_i = m \in \mathbb{N}$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. Then*

$$A_{\xi_n}(\alpha) := \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{j \|s\|_2}{\xi_n} \right)^r \right)^p e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N \leq$$
$$\xi^{mp} \left[(1+N)^{rp} + 2^{rp} \Gamma^N ((m+r)p + 1, 1) \right] \leq$$
$$(1+N)^{rp} + 2^{rp} \Gamma^N ((m+r)p + 1, 1) < +\infty, \tag{65}$$

are uniformly bounded.

Above $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function.

Clearly, it holds

$$B_{\xi_n}(\alpha) := \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N < \infty, \tag{66}$$

uniformly bounded in $\xi_n \in (0, 1]$. And clearly $B_{\xi_n}(\alpha) \to 0$, as $\xi_n \to 0$, $n \to \infty$, by (65). It follows an application of Theorem 4.

Theorem 11 *Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $\xi_n \in (0, 1]$, $n \in \mathbb{N}$. For $\widetilde{j} = 1, ..., m$, and $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$, call*

$$\overline{c}_{\alpha, n, \widetilde{j}} := \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \right) e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N. \tag{67}$$

Then

$$\left\|\overline{E}_{r,n}^{[m]}\right\|_p := \left\|P_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{\overline{c}_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^{N} \alpha_i!\right)} \right) \right\|_{p,x} \leq \quad (68)$$

$$\left[\binom{m+N-1}{m}^{\frac{p}{q}} \left(\frac{m}{(q(m-1)+1)^{\frac{p}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \cdot \right.\right.$$

$$\left.\left. A_{\xi_n}(\alpha)\, \omega_r(f_\alpha, \xi_n)_p^p \right) \right]^{\frac{1}{p}},$$

where $A_{\xi_n}(\alpha)$ *as in (65).*

As $n \to \infty$ *and* $\xi_n \to 0$, *by (68), we obtain that* $\left\|\overline{E}_{r,n}^{[m]}\right\|_p \to 0$ *with rates.*

One also gets by (68) that

$$\left\|P_{r,n}^{[m]}(f;x) - f(x)\right\|_{p,x} \leq \quad (69)$$

$$\sum_{\widetilde{j}=1}^{m} \left|\delta_{\widetilde{j},r}^{[m]}\right| \left(\sum_{|\alpha|=\widetilde{j}} \frac{\left|\overline{c}_{\alpha,n,\widetilde{j}}\right|}{\prod_{i=1}^{N} \alpha_i!} \|f_\alpha\|_p \right) + R.H.S.(68),$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

Furthermore it holds $\left\|P_{r,n}^{[m]}(f) - f\right\|_p \to 0$, *that is* $P_{r,n}^{[m]} \to I$, *in* L_p *norm, with rates.*

We need the following auxiliary results.

Theorem 12 *Let* $N \in \mathbb{N}$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\alpha| = \sum_{i=1}^{N} \alpha_i = 2$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$, $p \geq 1$, $j = 0, 1, ..., r$. *Then*

$$I_{1jn}^*(\alpha) := \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N \leq \quad (70)$$

$$\xi_n^{2p} \left[(1 + Nj^p) + (1 + j^p)\, \Gamma^N(3p+1, 1) \right] \leq$$

$$\left[(1 + Nj^p) + (1 + j^p)\, \Gamma^N(3p+1, 1) \right] < +\infty,$$

uniformly bounded in $\xi_n \in (0,1]$.

Also $I_{1jn}^*(\alpha) \to 0$, *as* $\xi_n \to 0$, $n \to \infty$.

Above $\Gamma(\cdot,\cdot)$ *denotes the upper incomplete gamma function. Denote* $I_{1jn}^{**}(\alpha)$ *the* $I_{1jn}^*(\alpha)$ *when* $p = 1$.

Proof. We have that

$$I_{1jn}^{*}(\alpha) = \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N =$$

$$\frac{1}{\xi_n^N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} s_i^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) e^{-\frac{\sum_{i=1}^{N} s_i}{\xi_n}} ds_1 ... ds_N \leq \tag{71}$$

$$\frac{1}{\xi_n^N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} s_i^{p\alpha_i} \right) \left(1 + j^p \frac{\left(\sum_{i=1}^{N} s_i \right)^p}{\xi_n^p} \right) e^{-\frac{\sum_{i=1}^{N} s_i}{\xi_n}} ds_1 ... ds_N =$$

$$\xi_n^{2p} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} \left(\frac{s_i}{\xi_n} \right)^{p\alpha_i} \right) \left(1 + j^p \sum_{i=1}^{N} \left(\frac{s_i}{\xi_n} \right)^p \right) e^{-\sum_{i=1}^{N} \frac{s_i}{\xi_n}} d\frac{s_1}{\xi_n} ... d\frac{s_N}{\xi_n} =$$

$$\xi_n^{2p} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) \left(1 + j^p \sum_{i=1}^{N} z_i^p \right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N =$$

$$\xi_n^{2p} \left[\int_{[0,1]^N} \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) \left(1 + j^p \sum_{i=1}^{N} z_i^p \right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N + \right. \tag{72}$$

$$\left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) \left(1 + j^p \sum_{i=1}^{N} z_i^p \right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N \right] \leq$$

$$\xi_n^{2p} \left[(1 + j^p N) + \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) \left(1 + j^p \sum_{i=1}^{N} z_i^p \right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N \right] \leq$$

$$\xi_n^{2p} \left[(1 + j^p N) + (1 + j^p) \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^{N} z_i^p \right) \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N \right] \leq$$

$$\xi_n^{2p} \left[(1 + j^p N) + (1 + j^p) \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^{N} z_i^p \right) \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N \right] = \tag{73}$$

$$\xi_n^{2p} \left[(1 + j^p N) + (1 + j^p) \prod_{i=1}^{N} \int_1^{\infty} z_i^{p(\alpha_i + 1)} e^{-z_i} dz_i \right] =$$

$$\xi_n^{2p} \left[(1 + j^p N) + (1 + j^p) \prod_{i=1}^{N} \int_1^{\infty} z_i^{(p(\alpha_i + 1) + 1) - 1} e^{-z_i} dz_i \right] =$$

(by [7], p. 348)

$$\xi_n^{2p}\left[(1+j^pN)+(1+j^p)\prod_{i=1}^{N}\Gamma\left(\left(p\left(\alpha_i+1\right)+1\right),1\right)\right]\leq$$

(where $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function) (by [7], p. 909)

$$\xi_n^{2p}\left[(1+j^pN)+(1+j^p)\,\Gamma^N\left(3p+1,1\right)\right]. \tag{74}$$

We have proved that

$$I_{1jn}^*(\alpha)\leq\xi_n^{2p}\left[(1+Nj^p)+(1+j^p)\,\Gamma^N\left(3p+1,1\right)\right]\leq$$
$$\left[(1+Nj^p)+(1+j^p)\,\Gamma^N\left(3p+1,1\right)\right]<+\infty, \tag{75}$$

uniformly bounded in $\xi_n\in(0,1]$.

Above we used ([7], p. 909) the formula

$$\Gamma(\alpha,xy)=y^{\alpha}e^{-xy}\int_0^{\infty}e^{-ty}\left(t+x\right)^{\alpha-1}dt, \tag{76}$$

where $\mathrm{Re}y>0$, $x>0$, $\mathrm{Re}\alpha>1$.

We notice that ($x=y=1$, $\alpha_i\leq 2$)

$$\Gamma\left(\left(\alpha_i+1\right)p+1,1\right)=e^{-1}\int_0^{\infty}e^{-t}\left(t+1\right)^{(\alpha_i+1)p}dt\leq$$
$$e^{-1}\int_0^{\infty}e^{-t}\left(t+1\right)^{(2+1)p}dt=\Gamma\left(3p+1,1\right). \tag{77}$$

That is

$$\Gamma\left(\left(\alpha_i+1\right)p+1,1\right)\leq\Gamma\left(3p+1,1\right), \tag{78}$$

for all $i=1,...,N$.

The theorem is proved. ■

Corollary 13 *(to Theorem 12) It holds*

$$K_{1n}(\alpha):=\frac{1}{\left(2\xi_n\right)^N}\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}\left|s_i\right|^{\alpha_i}\right)e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}}ds_1...ds_N<+\infty, \tag{79}$$

uniformly bounded in $\xi_n\in(0,1]$. *Also* $K_{1n}(\alpha)\to 0$, *as* $\xi_n\to 0$, $n\to\infty$.

Proof. By (70). ■

Proposition 14 *Let $N \in \mathbb{N}$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\alpha| = \sum_{i=1}^{N} \alpha_i = 2$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$, $p \geq 1$. Then*

$$M_n := \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N \leq \tag{80}$$

$$N^p + \Gamma^N (p + 1, 1) < +\infty,$$

uniformly bounded in ξ_n. Denote M_n^* the M_n when $p = 1$.

Proof. We have

$$M_n = \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N =$$

$$\frac{1}{\xi_n^N} \int_{\mathbb{R}_+^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p e^{-\frac{\sum_{i=1}^{N} s_i}{\xi_n}} ds_1 ... ds_N \leq$$

$$\frac{1}{\xi_n^N} \int_{\mathbb{R}_+^N} \left(\frac{\sum_{i=1}^{N} s_i}{\xi_n} \right)^p e^{-\frac{\sum_{i=1}^{N} s_i}{\xi_n}} ds_1 ... ds_N = \tag{81}$$

$$\int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N} \left(\frac{s_i}{\xi_n} \right)^p \right) e^{-\sum_{i=1}^{N} \frac{s_i}{\xi_n}} d\frac{s_1}{\xi_n} ... d\frac{s_N}{\xi_n} =$$

$$\int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N} z_i \right)^p e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N =$$

$$\int_{[0,1]^N} \left(\sum_{i=1}^{N} z_i \right)^p e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N +$$

$$\left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^{N} z_i \right)^p e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N \right] \leq$$

$$N^p + \int_{(\mathbb{R}_+ - [0,1])^N} \prod_{i=1}^{N} z_i^p \prod_{i=1}^{N} e^{-z_i} \prod_{i=1}^{N} dz_i =$$

$$N^p + \prod_{i=1}^{N} \int_1^{\infty} z_i^p e^{-z_i} dz_i = N^p + \prod_{i=1}^{N} \int_1^{\infty} z_i^{(p+1)-1} e^{-z_i} dz_i = \tag{82}$$

$$N^p + \Gamma^N (p + 1, 1) < +\infty,$$

uniformly bounded in $\xi_n \in (0, 1]$. ∎

Next, we apply Theorem 6 to multivariate Picard operators P_n, L_p approximation $p > 1$.

Theorem 15 *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $0<\xi_n\leq 1$, $n\in\mathbb{N}$. *We consider* $f\in C^2\left(\mathbb{R}^N\right)$, $N\geq 1$, *with* $f_\alpha\in L_p\left(\mathbb{R}^N\right)$, $|\alpha|=2$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, *and* $|\alpha|=\sum_{i=1}^N\alpha_i$; $x\in\mathbb{R}^N$; $j=0,1,...,r$. *Denote by*

$$\psi_{in}^{(1)}:=\frac{1}{\left(2\xi_n\right)^N}\int_{\mathbb{R}^N}|s_i|\,e^{-\frac{\sum_{i=1}^N|s_i|}{\xi_n}}ds_1...ds_N, \tag{83}$$

$$\psi_{in}^{(2)}:=\frac{1}{\left(2\xi_n\right)^N}\int_{\mathbb{R}^N}s_i^2e^{-\frac{\sum_{i=1}^N|s_i|}{\xi_n}}ds_1...ds_N, \tag{84}$$

and

$$\psi_{ij^*n}:=\frac{1}{\left(2\xi_n\right)^N}\int_{\mathbb{R}^N}|s_is_{j^*}|\,e^{-\frac{\sum_{i=1}^N|s_i|}{\xi_n}}ds_1...ds_N, \tag{85}$$

where $i,j^*\in\{1,...,N\}$, $i\neq j^*$; $n\in\mathbb{N}$.

Also denote ($n\in\mathbb{N}$)

$$\overline{\Delta}_n(x):=P_n(f,x)-f(x)-$$

$$\left(\sum_{j=0}^r\alpha_jj\right)\sin(1)\left[\sum_{i=1}^N\frac{\partial f(x)}{\partial x_i}\left(\frac{1}{\left(2\xi_n\right)^N}\int_{\mathbb{R}^N}s_ie^{-\frac{\sum_{i=1}^N|s_i|}{\xi_n}}ds_1...ds_N\right)\right] \tag{86}$$

$$-2\left(\sum_{j=0}^r\alpha_jj^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^N\left(\frac{1}{\left(2\xi_n\right)^N}\int_{\mathbb{R}^N}s_i^2e^{-\frac{\sum_{i=1}^N|s_i|}{\xi_n}}ds_1...ds_N\right)\frac{\partial^2f(x)}{\partial x_i^2}\right.$$

$$\left.+\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}}\left(\frac{1}{\left(2\xi_n\right)^N}\int_{\mathbb{R}^N}s_is_{j^*}e^{-\frac{\sum_{i=1}^N|s_i|}{\xi_n}}ds_1...ds_N\right)\frac{\partial^2f(x)}{\partial x_i\partial x_{j^*}}\right\}.$$

Then, it holds

$$\left\|\overline{\Delta}_n(x)\right\|_p\leq\left(\sum_{j=0}^r|\alpha_j|^q\right)^{\frac{1}{q}}\left\{\frac{2}{(q+1)^{\frac{1}{q}}}\left(\frac{N(N+1)+2}{2}\right)^{\frac{1}{q}}\right\}$$

$$\left\{\sum_{j=0}^r\left[j^{2p}\sum_{|\alpha|=2}\left(\frac{1}{\prod_{i=1}^N\alpha_i!}\right)^p\omega_1\left(f_\alpha,\xi_n\right)_p^pI_{1jn}^*(\alpha)++\omega_1\left(f,\xi_n\right)_p^p\left(1+j^pM_n\right)\right]\right\}^{\frac{1}{p}}, \tag{87}$$

where $I_{1jn}^*(\alpha)$ *as in (70), and* M_n *as in (80).*

As $n\to\infty$ and $\xi_n\to0$, by (87), we obtain that $\left\|\overline{\Delta}_n\right\|_p\to0$ with rates.

One also gets by (87) that

$$\|P_n(f)-f\|_p\leq$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\,j\right)\sin(1)\left[\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_p\psi_{in}^{(1)}\right]$$

$$+2\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{N}\psi_{in}^{(2)}\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_p+\right.$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,\ldots,N\}}}\psi_{ij^*n}\left\|\frac{\partial^2 f}{\partial x_i\partial x_{j^*}}\right\|_p\right\}+R.H.S.(87), \tag{88}$$

given that $\|f_\alpha\|_p<\infty$, $|\alpha|=\widetilde{j}$, $\widetilde{j}=1,2$. Also we obtain $\|P_n(f)-f\|_p\to0$, i.e. $P_n\to I$ the unit operator, in L_p norm, with rates.

Proof. By Theorems 6, 12, Corollary 13 and Proposition 14. ∎

Finally, we apply Theorem 8 to an alternative L_1 approximation by the P_n multivariate operators, $p=1$ case.

Theorem 16 *Let $0<\xi_n\leq1$, $n\in\mathbb{N}$, $x\in\mathbb{R}^N$. Here we consider $f\in C^2\left(\mathbb{R}^N\right)$, $N\geq1$, with $f_\alpha\in L_1\left(\mathbb{R}^N\right)$, $|\alpha|=2$, where $\alpha_i\in\mathbb{Z}^+$, $i=1,\ldots,N$, and $|\alpha|=\sum_{i=1}^{N}\alpha_i$; $j=0,1,\ldots,r$. Here $\overline{\Delta}_n(x)$ is as in (86), $\psi_{in}^{(1)}$ as in (83), $\psi_{in}^{(2)}$ as in (84), ψ_{ij^*n} as in (85).*

Then

$$\left\|\overline{\Delta}_n(x)\right\|_1\leq$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|\left[j^2\sum_{|\alpha|=2}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\omega_1(f_\alpha,\xi_n)_1\,I_{1jn}^{**}(\alpha)+\omega_1(f,\xi_n)_1(1+jM_n^*)\right]\right\}, \tag{89}$$

where $I_{1jn}^{**}(\alpha)$ as in Theorem 12, and M_n^* as in Proposition 14.

As $n\to\infty$ and $\xi_n\to0$, by (89), we obtain that $\left\|\overline{\Delta}_n\right\|_1\to0$ with rates.

One also obtains by (89) that

$$\|P_n(f)-f\|_1\leq$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\,j\right)\sin(1)\left[\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_1\psi_{in}^{(1)}\right]$$

$$+2\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{N}\psi_{in}^{(2)}\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_1+\right. \quad (90)$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,\ldots,N\}}}\psi_{ij^*n}\left\|\frac{\partial^2 f}{\partial x_i\partial x_{j^*}}\right\|_1\right\}+R.H.S.\,(89)\,,$$

given that $\|f_\alpha\|_1<\infty$, $|\alpha|=\widetilde{j}$, $\widetilde{j}=1,2$. *Also we obtain* $\|P_n(f)-f\|_1\to 0$, *i.e.* $P_n\to I$ *in* L_1 *norm with rates.*

Proof. By Theorems 8, 12, Corollary 13 and Proposition 14. ■

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