Trigonometric induced multivariate smooth Gauss-Weierstrass singular integrals approximation

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Abstract

In this article we employ the uniform and L_p, $1 \leq p < \infty$ approximation properties of general smooth multivariate singular integral operators over $\mathbb{R}^N$, $N \geq 1$. It is a trigonometric relied approach with detailed applications to the corresponding smooth multivariate Gauss-Weierstrass singular integral operators. The results are quantitative via Jackson type inequalities involving the first uniform and L_p moduli of continuity.

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1 Introduction

The degree of approximation by univariate and multivariate singular integral operators has been researched extensively in [1]-[3] and [7], [8] and [10]. All these sources motivate our current work. In particular we studied the approximation properties of the smooth singular integral operators in [1]-[3], [8]. These are not in general positive operators. Here we use the uniform and L_p, $p \geq 1$, results of our multivariate general theory [5], [6], to establish approximation properties of the smooth Gauss-Weierstrass singular integral operators. The degrees of approximation are given quantitatively by employing the uniform and L_p first moduli of continuity. The fundamental tool here comes from [4], where a multivariate trigonometric Taylor formula is presented.

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2 Background on General Theory

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_j := \alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, ..., r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0. \end{cases} \tag{1}$$

and

$$\delta_k := \delta_{k,r}^{[m]} := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, ..., m \in \mathbb{N}. \tag{2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}.$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\theta_n (f; x_1, ..., x_N) := \theta_{r,n}^{[m]} (f; x_1, ..., x_N) :=$$

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f (x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) \, d\mu_{\xi_n} (s) , \tag{4}$$

where $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers, we take $0 < \xi_n \leq 1$.

Remark 1 *The operators $\theta_{r,n}^{[m]}$ are not in general positive, see [2], p. 2.*

We observe that

Lemma 2 *It holds*

$$\theta_{r,n}^{[m]} (c; x_1, ..., x_n) = c,$$

where c is a constant.

We need

Definition 3 *Let $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$. We define the first uniform modulus of continuity of f as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in\mathbb{R}^N: \\ \|x-y\|_\infty \leq \delta}} |f(x) - f(y)|, \quad \delta > 0, \tag{5}$$

where $\|\cdot\|_\infty$ is the max norm in $\mathbb{R}^N$. The functional $\omega_1(f,\delta)$ is bounded for f being bounded or uniformly continuous, and $\omega_1(f,\delta) \to 0$ as $\delta \to 0$, in the case of f being uniformly continuous.

We mention the main uniform general approximation result regarding the operator θ_n.

Theorem 4 *([5]) Here $f \in C^2\left(\mathbb{R}^N\right)$ and let all $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $N \geq 1$, $|\alpha| := \sum_{i=1}^N \alpha_i = 2$; $x \in \mathbb{R}^N$, and all the partials f_α of order 2, along with $f \in C_B\left(\mathbb{R}^N\right)$ (continuous and bounded functions); or all f_α of order 2, $f \in C_U\left(\mathbb{R}^N\right)$ (uniformly continuous functions). Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$, for $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.*

Suppose that for all $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\alpha| = \sum_{i=1}^N \alpha_i = 2$, $j = 0, 1, ..., r$, we have that both

$$I_{1j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^N |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s), \tag{6}$$

$$I_{2j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) d\mu_{\xi_n}(s), \tag{7}$$

are uniformly bounded in $\xi_n \in (0,1]$.

Denote ($n \in \mathbb{N}$)

$$\Delta_n(x) := \theta_n(f,x) - f(x) - \left(\sum_{j=0}^r \alpha_j j\right) \sin(1) \left[\sum_{i=1}^N \frac{\partial f(x)}{\partial x_i} \left(\int_{\mathbb{R}^N} s_i d\mu_{\xi_n}(s)\right)\right]$$

$$-2\left(\sum_{j=0}^r \alpha_j j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^N \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)\right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \tag{8}$$

$$\left. \sum_{\substack{i\neq j^*, \\ i,j^*\in\{1,...,N\}}} \left(\int_{\mathbb{R}^N} s_i s_{j^*} d\mu_{\xi_n}(s)\right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}} \right\}.$$

Then

(i)

$$|\Delta_n(x)| \leq \|\Delta_n(x)\|_\infty \leq \sum_{j=0}^{r} |\alpha_j|$$

$$\left[\left[j^2 \sum_{\substack{\alpha_i \in \mathbb{Z}^+, \\ \alpha : |\alpha| = 2}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_1(f_\alpha, \xi_n) \int_{\mathbb{R}^N} \left(1 + \frac{j \|s\|_\infty}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) d\mu_{\xi_n}(s) \right] + \right.$$

$$\left. \frac{1}{2}\omega_1(f, \xi_n) \int_{\mathbb{R}^N} \left(1 + \frac{j \|s\|_\infty}{3\xi_n}\right) d\mu_{\xi_n}(s) \right] =: \varphi_{\xi_n}. \tag{9}$$

In case of all f_α *of order* 2 *and* $f \in C_U(\mathbb{R}^N)$ *and* $\xi_n \to 0$, *as* $n \to \infty$, *then* $\Delta_n(x)$, $\|\Delta_n(x)\|_\infty \to 0$ *with rates.*

(ii) If $\frac{\partial f(x)}{\partial x_i} = 0$, $i = 1, ..., N$, *and* $f_\alpha(x) = 0$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, *with* $|\alpha| = 2$, *then*

$$|\theta_n(f, x) - f(x)| \leq \varphi_{\xi_n}. \tag{10}$$

And $\theta_n(f, x) \to f(x)$ *in the uniformly continuous case.*

(iii) Additionally assume all partials of order ≤ 2 *are bounded. Hence*

$$\|\theta_n(f) - f\|_\infty \leq \left(\sum_{j=0}^{r} |\alpha_j| j \right) (0.8414) \left[\sum_{i=1}^{N} \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \left(\int_{\mathbb{R}^N} |s_i| d\mu_{\xi_n}(s) \right) \right] +$$

$$\left(\sum_{j=0}^{r} |\alpha_j| j^2 \right) (0.4596) \left\{ \sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_\infty + \right.$$

$$\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, ..., N\}}} \left(\int_{\mathbb{R}^N} |s_i| |s_{j^*}| d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_\infty \right\} + \varphi_{\xi_n}. \tag{11}$$

If all $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)$ *and* $\int_{\mathbb{R}^N} |s_i| |s_{j^*}| d\mu_{\xi_n}(s)$ *converge to zero, as* $n \to \infty$, *with* $\xi_n \to 0$, *and all* f_α *of order* 2, $f \in C_U(\mathbb{R}^N)$, *then*

$$\|\theta_n(f) - f\|_\infty \to 0 \text{ with rates, as } \xi_n \to 0,\ n \to +\infty.$$

Next we deal with $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{Z}^+$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m \in \mathbb{Z}^+$, $p \geq 1$; where f_α denotes the mixed partial $\frac{\partial^{\widetilde{j}} f(\cdot, ..., \cdot)}{\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\alpha| := \sum_{j=1}^{N} \alpha_j = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

We need

Definition 5 *(see also [2], p. 20) We call*

$$\Delta_u^r f(x) := \Delta_{u_1,u_2,...,u_N}^r f(x_1,...,x_N) := \tag{12}$$

$$\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x_1 + ju_1, x_2 + ju_2, ..., x_N + ju_N).$$

Let $p \geq 1$, the L_p modulus of smoothness of order r is given by

$$\omega_r(f;h)_p := \sup_{\|u\|_2 \leq h} \|\Delta_u^r(f)\|_p, \tag{13}$$

$h > 0$.

We mention

Theorem 6 *([6]) Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence. Assume for all $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = m$ that we have*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{14}$$

For $\widetilde{j} = 1, ..., m$, and $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$, call

$$c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} d\mu_{\xi_n}(s). \tag{15}$$

Then

$$\left\| E_{r,n}^{[m]} \right\|_p := \left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) \right\|_{p,x} \leq \tag{16}$$

$$\left[\binom{m+N-1}{m}^{\frac{p}{q}} \left(\frac{m}{(q(m-1)+1)^{\frac{p}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \cdot \right. \right.$$

$$\left. \left. \left[\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right] \omega_r(f_\alpha, \xi_n)_p^p \right) \right]^{\frac{1}{p}}.$$

As $n \to \infty$ and $\xi_n \to 0$, by (16), we obtain that $\left\| E_{r,n}^{[m]} \right\|_p \to 0$ with rates.

One also gets by (16) that

$$\left\|\theta_{r,n}^{[m]}\left(f;x\right)-f\left(x\right)\right\|_{p,x}\leq$$

$$\sum_{\widetilde{j}=1}^{m}\left|\delta_{\widetilde{j},r}^{[m]}\right|\left(\sum_{|\alpha|=\widetilde{j}}\frac{\left|c_{\alpha,n,\widetilde{j}}\right|}{\prod_{i=1}^{N}\alpha_i!}\left\|f_\alpha\right\|_p\right)+R.H.S.(16), \tag{17}$$

given that $\|f_\alpha\|_p<\infty$, $|\alpha|=\widetilde{j}$, $\widetilde{j}=1,...,m$.

Assuming that $c_{\alpha,n,\widetilde{j}}\to 0$, $\xi_n\to 0$, *as* $n\to\infty$, *we get* $\left\|\theta_{r,n}^{[m]}(f)-f\right\|_p\to 0$, *that is* $\theta_{r,n}^{[m]}\to I$ *the unit operator, in* L_p *norm, with rates.*

We make

Remark 7 *Notice that* $(p>1)$

$$\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s)\leq$$

$$\left[\int_{\mathbb{R}^N}\left(\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right)^p d\mu_{\xi_n}(s)\right]^{\frac{1}{p}}<\infty, \tag{18}$$

by assumption of Theorem 6.

By (18) we get that

$$\int_{\mathbb{R}^N}\prod_{i=1}^{N}|s_i|^{\alpha_i}\,d\mu_{\xi_n}(s)<\infty. \tag{19}$$

Hence $c_{\alpha,n,\widetilde{j}}\in\mathbb{R}$.

We mention also the following trigonometric induced alternative L_p approximation result for θ_n operators.

Theorem 8 *([6]) Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $0<\xi_n\leq 1$, $n\in\mathbb{N}$. *Here we deal with* $f\in C^2\left(\mathbb{R}^N\right)$, $N\geq 1$, *with* f, $f_\alpha\in L_p\left(\mathbb{R}^N\right)$, $|\alpha|=2$, *where* $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, *and* $|\alpha|=\sum_{i=1}^{N}\alpha_i$; $x\in\mathbb{R}^N$. *Let* μ_{ξ_n} *be a Borel probability measure on* $\mathbb{R}^N$. *Suppose that for all* $\alpha:|\alpha|=2$, $j=0,1,...,r$, *we have that both*

$$I_{1j}(\alpha):=\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}|s_i|^{p\alpha_i}\right)\left(1+\frac{j^p\|s\|_2^p}{\xi_n^p}\right)d\mu_{\xi_n}(s), \tag{20}$$

$$I_2 := \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p d\mu_{\xi_n}(s), \tag{21}$$

are uniformly bounded in $\xi_n \in (0,1]$.

Denote ($n \in \mathbb{N}$)

$$\Delta_n(x) := \theta_n(f,x) - f(x) - \left(\sum_{j=0}^r \alpha_j j \right) \sin(1) \left[\sum_{i=1}^N \frac{\partial f(x)}{\partial x_i} \left(\int_{\mathbb{R}^N} s_i d\mu_{\xi_n}(s) \right) \right]$$

$$-2 \left(\sum_{j=0}^r \alpha_j j^2 \right) \sin^2\left(\frac{1}{2}\right) \left\{ \sum_{i=1}^N \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \tag{22}$$

$$\left. \sum_{\substack{i \neq j^*, \\ i,j^* \in \{1,\dots,N\}}} \left(\int_{\mathbb{R}^N} s_i s_{j^*} d\mu_{\xi_n}(s) \right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}} \right\}.$$

Then, it holds

$$\|\Delta_n(x)\|_p \leq \left(\sum_{j=0}^r |\alpha_j|^q \right)^{\frac{1}{q}} \left\{ \frac{2}{(q+1)^{\frac{1}{q}}} \left(\frac{N(N+1)+2}{2} \right)^{\frac{1}{q}} \right\}$$

$$\left\{ \sum_{j=0}^r \left[j^{2p} \sum_{|\alpha|=2} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right)^p \omega_1(f_\alpha, \xi_n)_p^p \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) d\mu_{\xi_n}(s) \right) \right. \right. \tag{23}$$

$$\left. \left. + \omega_1(f,\xi_n)_p^p \left(\int_{\mathbb{R}^N} \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) d\mu_{\xi_n}(s) \right) \right] \right\}^{\frac{1}{p}}.$$

As $n \to \infty$ *and* $\xi_n \to 0$, *by (23), we obtain that* $\|\Delta_n\|_p \to 0$ *with rates. One also gets by (23) that*

$$\|\theta_n(f,x) - f(x)\|_{p,x} \leq$$

$$\left(\sum_{j=0}^r |\alpha_j| j \right) \sin(1) \left[\sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_p \left(\int_{\mathbb{R}^N} |s_i| d\mu_{\xi_n}(s) \right) \right]$$

$$+2 \left(\sum_{j=0}^r |\alpha_j| j^2 \right) \sin^2\left(\frac{1}{2}\right) \left\{ \sum_{i=1}^N \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_p + \right.$$

$$\left. \sum_{\substack{i \neq j^*, \\ i,j^* \in \{1,\dots,N\}}} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_p \right\} + R.H.S.(23), \tag{24}$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, 2$. *Assuming that* $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_m}(s)$, $\int_{\mathbb{R}^N} |s_i s_{j^*}| \, d\mu_{\xi_n}(s)$, $i, j^* \in \{1, ..., N\}$, $i \neq j^*$, *converge to zero as* $\xi_n \to 0$, *we get* $\|\theta_n(f, x) - f(x)\|_p \to 0$, *that is* $\theta_n \to I$ *the unit operator, in* L_p *norm, with rates.*

Furthermore we mention the following trigonometric based alternative L_1 approximation result for θ_n operators.

Theorem 9 *([6]) Let* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$. *Here we deal with* $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, *with* f, $f_\alpha \in L_1(\mathbb{R}^N)$, $|\alpha| = 2$, *where* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, *and* $|\alpha| = \sum_{i=1}^{N} \alpha_i$. *Let* $\mu_{\xi n}$ *be a Borel probability measure on* $\mathbb{R}^N$. *Suppose that for all* $\alpha : |\alpha| = 2$, $j = 0, 1, ..., r$, *we have that both*

$$I_{1j}^*(\alpha) := \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{j \|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s), \tag{25}$$

and

$$I_2^* := \int_{\mathbb{R}^N} \frac{\|s\|_2}{\xi_n} d\mu_{\xi_n}(s), \tag{26}$$

are uniformly bounded in $\xi_n \in (0, 1]$.

Here $\Delta_n(x)$ *is as in (22).*

Then, it holds

$$\|\Delta_n(x)\|_1 \leq \left\{ \sum_{j=0}^{r} |\alpha_j| \left[j^2 \sum_{|\alpha|=2} \left(\frac{2}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_1(f_\alpha, \xi_n)_1 \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} (|s_i|)^{\alpha_i} \right) \right.\right.\right.$$

$$\left.\left.\left. \left(1 + \frac{j \|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s) \right) + \omega_1(f, \xi_n)_1 \left(\int_{\mathbb{R}^N} \left(1 + \frac{j \|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s) \right) \right] \right\}. \tag{27}$$

As $n \to \infty$ *and* $\xi_n \to 0$, *by (27), we obtain that* $\|\Delta_n\|_1 \to 0$ *with rates. One also obtains by (27) that*

$$\|\theta_n(f) - f\|_1 \leq$$

$$\left(\sum_{j=0}^{r} |\alpha_j| \, j \right) \sin(1) \left[\sum_{i=1}^{N} \left\| \frac{\partial f}{\partial x_i} \right\|_1 \left(\int_{\mathbb{R}^N} |s_i| \, d\mu_{\xi_n}(s) \right) \right]$$

$$+2 \left(\sum_{j=0}^{r} |\alpha_j| \, j^2 \right) \sin^2\left(\frac{1}{2} \right) \left\{ \sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_1 + \right.$$

$$\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, ..., N\}}} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| \, d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_1 \right\} + R.H.S.\,(27), \tag{28}$$

given that $\|f_\alpha\|_1 < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, 2$. *Assuming that* $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)$, $\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s)$, $i, j^* \in \{1, ..., N\}$, $i \neq j^*$, *converge to zero as* $\xi_n \to 0$, *we derive* $\|\theta_n(f) - f\|_1 \to 0$, *that is* $\theta_n \to I$ *in* L_1 *norm, with rates.*

3 Auxiliary Essential Results

We need

Theorem 10 *Let* $N \in \mathbb{N}$; $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\alpha| := \sum_{i=1}^{N} \alpha_i = 2$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$; $j = 0, 1, ..., r \in \mathbb{N}$. *Then*

$$I_{1j}^*(\alpha) := \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N \leq \quad (29)$$

$$\sqrt{\xi_n} \left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1 + jN) + (1 + j)\left(\frac{\lfloor 6e \rfloor}{e}\right)^N\right] \leq$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1 + jN) + (1 + j)\left(\frac{\lfloor 6e \rfloor}{e}\right)^N\right] < +\infty,$$

are uniformly bounded, and $I_{1j}^*(\alpha) \to 0$, *as* $\xi_n \to 0$.

Above $\lfloor \cdot \rfloor$ *denotes the integral part.*

Proof. We have that

$$I_{1j}^*(\alpha) = \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N =$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N \left(\sqrt{\xi_n}\right)^N} \int_{\mathbb{R}_+^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N = \quad (30)$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N \left(\sqrt{\xi_n}\right)^N} \int_{\mathbb{R}_+^N} \left(1 + \frac{j\left(\sum_{i=1}^{N} s_i\right)}{3\xi_n}\right) \left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N \leq$$

$$\frac{2^N \left(\sqrt{\xi_n}\right)^2}{\left(\sqrt{\pi}\right)^N} \int_{\mathbb{R}_+^N} \left(\frac{1}{\sqrt{\xi_n}} + j\frac{1}{\sqrt{\xi_n}}\left(\sum_{i=1}^{N} \frac{s_i}{\sqrt{\xi_n}}\right)\right)$$

$$\left(\prod_{i=1}^{N} \left(\frac{s_i}{\sqrt{\xi_n}}\right)^{\alpha_i}\right) e^{-\sum_{i=1}^{N} \left(\frac{s_i}{\sqrt{\xi_n}}\right)^2} \frac{ds_1}{\sqrt{\xi_n}} ... \frac{ds_N}{\sqrt{\xi_n}} =$$

$$\frac{2^N\sqrt{\xi_n}}{\left(\sqrt{\pi}\right)^N}\int_{\mathbb{R}_+^N}\left(1+j\left(\sum_{i=1}^N z_i\right)\right)\left(\prod_{i=1}^N z_i^{\alpha_i}\right)e^{-\left(\sum\limits_{i=1}^N z_i^2\right)}dz_1...dz_N = \tag{31}$$

$$\frac{2^N\sqrt{\xi_n}}{\left(\sqrt{\pi}\right)^N}\left[\int_{[0,1]^N}\left(1+j\left(\sum_{i=1}^N z_i\right)\right)\left(\prod_{i=1}^N z_i^{\alpha_i}\right)e^{-\left(\sum\limits_{i=1}^N z_i^2\right)}dz_1...dz_N+\right.$$

$$\left.\int_{(\mathbb{R}_+-[0,1])^N}\left(1+j\left(\sum_{i=1}^N z_i\right)\right)\left(\prod_{i=1}^N z_i^{\alpha_i}\right)e^{-\left(\sum\limits_{i=1}^N z_i^2\right)}dz_1...dz_N\right]\leq$$

$$\frac{2^N\sqrt{\xi_n}}{\left(\sqrt{\pi}\right)^N}\left[(1+jN)+(1+j)\int_{(\mathbb{R}_+-[0,1])^N}\left(\sum_{i=1}^N z_i\right)\left(\prod_{i=1}^N z_i^{\alpha_i}\right)e^{-\sum\limits_{i=1}^N z_i}dz_1...dz_N\right]\leq$$

$$\frac{2^N\sqrt{\xi_n}}{\left(\sqrt{\pi}\right)^N}\left[(1+jN)+(1+j)\int_{(\mathbb{R}_+-[0,1])^N}\left(\prod_{i=1}^N z_i\right)\left(\prod_{i=1}^N z_i^{\alpha_i}\right)\prod_{i=1}^N e^{-z_i}\prod_{i=1}^N dz_i\right]=$$

$$\frac{2^N\sqrt{\xi_n}}{\left(\sqrt{\pi}\right)^N}\left[(1+jN)+(1+j)\prod_{i=1}^N\int_1^\infty z_i^{(\alpha_i+2)-1}e^{-z_i}dz_i\right]= \tag{32}$$

(by [9], p. 348)

$$\frac{2^N\sqrt{\xi_n}}{\left(\sqrt{\pi}\right)^N}\left[(1+jN)+(1+j)\prod_{i=1}^N\Gamma\left((\alpha_i+2),1\right)\right]=$$

(where $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function)

$$\frac{2^N\sqrt{\xi_n}}{\left(\sqrt{\pi}\right)^N}\left[(1+jN)+(1+j)\prod_{i=1}^N\frac{\lfloor e(\alpha_i+1)!\rfloor}{e}\right]\leq$$

$$\frac{2^N\sqrt{\xi_n}}{\left(\sqrt{\pi}\right)^N}\left[(1+jN)+(1+j)\left(\frac{\lfloor 6e\rfloor}{e}\right)^N\right]<+\infty.$$

That is

$$I_{1j}^*(\alpha)\leq\sqrt{\xi_n}\left(\frac{2}{\sqrt{\pi}}\right)^N\left[(1+jN)+(1+j)\left(\frac{\lfloor 6e\rfloor}{e}\right)^N\right]\leq \tag{33}$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N\left[(1+jN)+(1+j)\left(\frac{\lfloor 6e\rfloor}{e}\right)^N\right]<+\infty,$$

are uniformly bounded, furthermore $I_{1j}^*(\alpha)\to 0$, as $\xi_n\to 0$. ■

We make

Remark 11 *By Theorem 10,* $j = 0, 1, ..., r \in \mathbb{N}$*;* $i, j^* \in \{1, ..., N\}$ *,* $i \neq j^*$*, we have that*

$$\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\int_{\mathbb{R}^N}|s_i|\,|s_{j^*}|\,e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N,$$

$$\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\int_{\mathbb{R}^N}s_i^2e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N \leq I_{1j}^* < \infty, \tag{34}$$

and

$$\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\int_{\mathbb{R}^N}|s_i|\,e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N \leq \tag{35}$$

$$\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\int_{\mathbb{R}^N}s_i^2e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N < \infty,$$

and all these integrals are uniformly bounded in $\xi_n \in (0, 1]$*. And all integrals converge to zero, as* $\xi_n \to 0$*.*

We continue with

Theorem 12 *Let* $N \geq 1$*,* $\xi_n \in (0, 1]$*,* $n \in \mathbb{N}$*. Then*

$$I_2^* := \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\int_{\mathbb{R}^N}\frac{\|s\|_1}{\xi_n}e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N \leq$$

$$\frac{1}{\sqrt{\xi_n}}\left(\frac{2}{\sqrt{\pi}}\right)^N\left[N + \left(\frac{\lfloor e\rfloor}{e}\right)^N\right]. \tag{36}$$

Proof. We observe that

$$I_2^* = \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\int_{\mathbb{R}^N}\frac{\|s\|_1}{\xi_n}e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N =$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N\left(\sqrt{\xi_n}\right)^N}\int_{\mathbb{R}_+^N}\frac{\sum_{i=1}^{N}s_i}{\xi_n}e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N =$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N}\frac{1}{\sqrt{\xi_n}}\int_{\mathbb{R}_+^N}\left(\sum_{i=1}^{N}\left(\frac{s_i}{\sqrt{\xi_n}}\right)\right)e^{-\sum_{i=1}^{N}\left(\frac{s_i}{\sqrt{\xi_n}}\right)^2}d\frac{s_1}{\sqrt{\xi_n}}...d\frac{s_N}{\sqrt{\xi_n}} = \tag{37}$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N}\frac{1}{\sqrt{\xi_n}}\int_{\mathbb{R}_+^N}\left(\sum_{i=1}^{N}z_i\right)e^{-\sum_{i=1}^{N}z_i^2}dz_1...dz_N =$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \frac{1}{\sqrt{\xi_n}} \left[\int_{[0,1]^N} \left(\sum_{i=1}^N z_i\right) e^{-\sum_{i=1}^N z_i^2} dz_1 ... dz_N + \right.$$

$$\left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^N z_i\right) e^{-\sum_{i=1}^N z_i^2} dz_1 ... dz_N \right] \leq$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \frac{1}{\sqrt{\xi_n}} \left[N + \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i\right) e^{-\sum_{i=1}^N z_i} dz_1 ... dz_N\right] =$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \frac{1}{\sqrt{\xi_n}} \left[N + \prod_{i=1}^N \int_1^\infty z_i e^{-z_i} dz_i\right] =$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \frac{1}{\sqrt{\xi_n}} \left[N + \left(\int_1^\infty z^{2-1} e^{-z} dz\right)^N\right] = \tag{38}$$

(by [9], p. 348)

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \frac{1}{\sqrt{\xi_n}} \left[N + \Gamma^N(2,1)\right] =$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \frac{1}{\sqrt{\xi_n}} \left[N + \left(\frac{\lfloor e \rfloor}{e}\right)^N\right],$$

proving the claim. ■

We need the following.

Theorem 13 *([3], p. 403) Let $r, N, m \in \mathbb{N}$, with $m > r$; $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$: $|\alpha| := \sum_{i=1}^N \alpha_i = m$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$; $p > 1$. Then*

$$\widetilde{A}_{\xi_n}(\alpha) := \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i}\right) \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r\right)^p e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 ... ds_N \leq \tag{39}$$

$$\sqrt{\xi_n}^{p(m-r)} \left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p+1, 1)\right] \leq$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1+N)^{rp} + 2^{rp} \Gamma^N((m+r)p+1, 1)\right] < +\infty,$$

are uniformly bounded, where $m > r$.

Also $\widetilde{A}_{\xi_n}(\alpha) \to 0$, as $\xi_n \to 0$, $n \to +\infty$.

Above $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function.

Clearly, it holds $(m > r)$

$$B_{\xi_n}(\alpha) := \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 ... ds_N < \infty, \tag{40}$$

uniformly bounded in $\xi_n \in (0,1]$*. And, clearly* $B_{\xi_n}(\alpha) \to 0$*, as* $\xi_n \to 0$*,* $n \to \infty$*, by (39).*

We need the following

Theorem 14 *Let* $\alpha_i \in \mathbb{Z}^+$*,* $i = 1, ..., N \in \mathbb{N} : |\alpha| := \sum_{i=1}^N \alpha_i = 2$*,* $\xi_n \in (0,1]$*,* $n \in \mathbb{N}$*,* $p \geq 1$*, and* $j = 0, 1, ..., r \in \mathbb{N}$*. Then*

$$\widetilde{A}_{j\xi_n}(\alpha) := \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{p\alpha_i}\right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p}\right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 ... ds_N \leq$$

$$\left(\sqrt{\xi_n}\right)^p \left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1 + j^p N^p) + (1 + j^p) \Gamma^N (3p + 1, 1)\right] \leq \tag{41}$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1 + j^p N^p) + (1 + j^p) \Gamma^N (3p + 1, 1)\right] < +\infty,$$

are uniformly bounded, furthermore $\widetilde{A}_{j\xi_n}(\alpha) \to 0$*, as* $\xi_n \to 0$*,* $n \to \infty$*.*

Call $\widetilde{A}_{j\xi_n}(\alpha)$ *as* $\widetilde{\widetilde{A}}_{j\xi_n}(\alpha)$ *when* $p = 1$*.*

Proof. We estimate $(p \geq 1)$

$$\widetilde{A}_{j\xi_n}(\alpha) = \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{p\alpha_i}\right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p}\right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 ... ds_N =$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N \left(\sqrt{\xi_n}\right)^N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N s_i^{p\alpha_i}\right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p}\right) e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1 ... ds_N \leq \tag{42}$$

$$\frac{2^N \left(\sqrt{\xi_n}\right)^{2p}}{\left(\sqrt{\pi}\right)^N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}}\right)^{p\alpha_i}\right)$$

$$\left(\frac{1}{\left(\sqrt{\xi_n}\right)^p} + \left(\frac{j}{\sqrt{\xi_n}}\right)^p \left(\sum_{i=1}^N \frac{s_i}{\sqrt{\xi_n}}\right)^p\right) e^{-\sum_{i=1}^N \left(\frac{s_i}{\sqrt{\xi_n}}\right)^2} d\frac{s_1}{\sqrt{\xi_n}} ... d\frac{s_N}{\sqrt{\xi_n}} =$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \left(\sqrt{\xi_n}\right)^p \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N z_i^{p\alpha_i}\right) \left(1 + j^p \left(\sum_{i=1}^N z_i\right)^p\right) e^{-\sum_{i=1}^N z_i^2} dz_1 ... dz_N =$$

$$\left(\sqrt{\xi_n}\right)^p \left(\frac{2}{\sqrt{\pi}}\right)^N \left[\int_{[0,1]^N} \left(\prod_{i=1}^N z_i^{p\alpha_i}\right)\left(1+j^p\left(\sum_{i=1}^N z_i\right)^p\right) e^{-\sum\limits_{i=1}^N z_i^2} dz_1...dz_N + \right. \tag{43}$$

$$\left. \int_{(\mathbb{R}_+-[0,1])^N} \left(\prod_{i=1}^N z_i^{p\alpha_i}\right)\left(1+j^p\left(\sum_{i=1}^N z_i\right)^p\right) e^{-\sum\limits_{i=1}^N z_i^2} dz_1...dz_N\right] \leq$$

$$\left(\sqrt{\xi_n}\right)^p \left(\frac{2}{\sqrt{\pi}}\right)^N$$

$$\left[(1+j^pN^p) + (1+j^p)\int_{(\mathbb{R}_+-[0,1])^N} \left(\prod_{i=1}^N z_i^{p\alpha_i}\right)\left(\sum_{i=1}^N z_i\right)^p e^{-\sum\limits_{i=1}^N z_i} dz_1...dz_N\right] \leq$$

$$\left(\sqrt{\xi_n}\right)^p \left(\frac{2}{\sqrt{\pi}}\right)^N$$

$$\left[(1+j^pN^p) + (1+j^p)\int_{(\mathbb{R}_+-[0,1])^N} \left(\prod_{i=1}^N z_i^{p\alpha_i}\right)\left(\prod_{i=1}^N z_i^p\right)\left(\prod_{i=1}^N e^{-z_i}\right)\prod_{i=1}^N dz_i\right] =$$

$$\left(\sqrt{\xi_n}\right)^p \left(\frac{2}{\sqrt{\pi}}\right)^N$$

$$\left[(1+j^pN^p) + (1+j^p)\int_{(\mathbb{R}_+-[0,1])^N} \left(\prod_{i=1}^N z_i^{p(\alpha_i+1)}\right)\left(\prod_{i=1}^N e^{-z_i}\right)\prod_{i=1}^N dz_i\right] =$$

$$\left(\sqrt{\xi_n}\right)^p \left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1+j^pN^p) + (1+j^p)\prod_{i=1}^N \left(\int_1^\infty z_i^{p(\alpha_i+1)} e^{-z_i} dz_i\right)\right] = \tag{44}$$

(by [9], p. 348)

$$\left(\sqrt{\xi_n}\right)^p \left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1+j^pN^p) + (1+j^p)\prod_{i=1}^N \Gamma\left(\left(p\left(\alpha_i+1\right)+1\right),1\right)\right] \leq$$

(by [9], p. 909)

$$\left(\sqrt{\xi_n}\right)^p \left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1+j^pN^p) + (1+j^p)\,\Gamma^N\left(3p+1,1\right)\right] \leq \tag{45}$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^N \left[(1+j^pN^p) + (1+j^p)\,\Gamma^N\left(3p+1,1\right)\right] < +\infty,$$

are uniformly bounded. ■

We need the following.

Theorem 15 *Let $p \geq 1$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$, $N \geq 1$. Then*

$$K_{p\xi_n} := \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N} \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n}\right)^p e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N \leq \tag{46}$$

$$\frac{1}{\left(\sqrt{\xi_n}\right)^p} \left(\frac{2}{\sqrt{\pi}}\right)^N \left[N^p + \Gamma^N(p+1,1)\right].$$

We set $K_{p\xi_n}$ as $K^*_{\xi_n}$ when $p = 1$.

Proof. We have ($p \geq 1$)

$$K_{p\xi_n} = \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N} \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n}\right)^p e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N =$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N \left(\sqrt{\xi_n}\right)^N} \int_{\mathbb{R}_+^N} \left(\frac{\|s\|_2}{\xi_n}\right)^p e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N \leq$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N \left(\sqrt{\xi_n}\right)^N} \int_{\mathbb{R}_+^N} \left(\frac{\sum_{i=1}^{N} s_i}{\xi_n}\right)^p e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N = \tag{47}$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N} \frac{1}{\left(\sqrt{\xi_n}\right)^p} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N} \left(\frac{s_i}{\sqrt{\xi_n}}\right)\right)^p e^{-\sum_{i=1}^{N}\left(\frac{s_i}{\sqrt{\xi_n}}\right)^2} d\frac{s_1}{\sqrt{\xi_n}} ... d\frac{s_N}{\sqrt{\xi_n}} =$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N} \left(\sqrt{\xi_n}\right)^{-p} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N} z_i\right)^p e^{-\sum_{i=1}^{N} z_i^2} dz_1 ... dz_N =$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N} \left(\sqrt{\xi_n}\right)^{-p} \left[\int_{[0,1]^N} \left(\sum_{i=1}^{N} z_i\right)^p e^{-\sum_{i=1}^{N} z_i^2} dz_1 ... dz_N + \right.$$

$$\left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^{N} z_i\right)^p e^{-\sum_{i=1}^{N} z_i^2} dz_1 ... dz_N \right] \leq$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N} \left(\sqrt{\xi_n}\right)^{-p} \left[N^p + \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^{N} z_i^p\right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N\right] =$$

$$\frac{2^N}{\left(\sqrt{\pi}\right)^N} \left(\sqrt{\xi_n}\right)^{-p} \left[N^p + \prod_{i=1}^{N} \int_1^{\infty} z_i^p e^{-z_i} dz_i\right] = \tag{48}$$

$$\frac{2^N}{(\sqrt{\pi})^N}\left(\sqrt{\xi_n}\right)^{-p}\left[N^p+\left(\int_1^{\infty} z^p e^{-z}dz\right)^N\right]=$$

(by [9], p. 348)

$$\frac{2^N}{(\sqrt{\pi})^N}\left(\sqrt{\xi_n}\right)^{-p}\left[N^p+\Gamma^N\left(p+1,1\right)\right].$$

■

4 Main Results

The general smooth multivariate Gauss-Weierstass singular integral operators are defined as:

$$W_n\left(f;x_1,...,x_N\right):=W_{r,n}^{[m]}\left(f;x_1,...,x_N\right):=$$

$$\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\int_{\mathbb{R}^N} f\left(x_1+s_1j,x_2+s_2j,...,x_N+s_Nj\right)e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N. \tag{49}$$

Observe that

$$\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\int_{\mathbb{R}^N} e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N=1, \tag{50}$$

see [2], p. 15.

That is, $W_{r,n}^{[m]}$ are the $\theta_{r,n}^{[m]}$ operators applied for the Borel probability measures on $\mathbb{R}^N$, $N\geq 1$,

$$d\mu_{\xi_n}\left(s\right)=\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N,\quad s\in\mathbb{R}^N, \tag{51}$$

where $0<\xi_n\leq 1$, $n\in\mathbb{N}$.

We will apply to $W_{r,n}^{[m]}$ the Theorems 4, 6, 8 and 9, with the help of all section 3. That is a presentation of approximation properties by W_n.

Here apply first Theorem 4 to W_n operators.

Theorem 16 *We consider $f\in C^2\left(\mathbb{R}^N\right)$ and let all $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $N\geq 1$, $|\alpha|:=\sum_{i=1}^{N}\alpha_i=2$; $x\in\mathbb{R}^N$, and all the partials f_α of order 2, along with $f\in C_B\left(\mathbb{R}^N\right)$; or all f_α of order 2, $f\in C_U\left(\mathbb{R}^N\right)$; $0<\xi_n\leq 1$, $n\in\mathbb{N}$.*

Denote ($n\in\mathbb{N}$)

$$\Delta_n^*\left(x\right):=W_n\left(f,x\right)-f\left(x\right)-\left(\sum_{j=0}^{r}\alpha_j j\right)\sin\left(1\right)$$

$$\left[\sum_{i=1}^{N}\frac{\partial f(x)}{\partial x_i}\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\left(\int_{\mathbb{R}^N}s_i e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N\right)\right]$$

$$-2\left(\sum_{j=0}^{r}\alpha_j j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{N}\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\left(\int_{\mathbb{R}^N}s_i^2 e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N\right)\frac{\partial^2 f(x)}{\partial x_i^2}\right.$$

$$\left.+\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}}\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\left(\int_{\mathbb{R}^N}s_i s_{j^*} e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N\right)\frac{\partial^2 f(x)}{\partial x_i\partial x_{j^*}}\right\}. \tag{52}$$

Then

(i)

$$|\Delta_n^*(x)|\leq\|\Delta_n^*\|_\infty\leq\sum_{j=0}^{r}|\alpha_j|$$

$$\left[\left[j^2\sum_{\substack{\alpha_i\in\mathbb{Z}^+,\\ \alpha:|\alpha|=2}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\omega_1(f_\alpha,\xi_n)I_{1j}^*(\alpha)\right]+\right.$$

$$\left.\frac{1}{2}\omega_1(f,\xi_n)\left[1+\frac{j}{3}I_2^*\right]\right]=:\varphi_{\xi_n}^*, \tag{53}$$

where $I_{1j}^*(\alpha)$ *is as in (29), and* I_2^* *is as in (36).*

In case of all f_α *of order* 2 *and* $f\in C_U\left(\mathbb{R}^N\right)$ *and* $\xi_n\to 0$*, as* $n\to\infty$*, then* $\Delta_n^*(x)$*,* $\|\Delta_n^*\|_\infty\to 0$ *with rates.*

(ii) If $\frac{\partial f(x)}{\partial x_i}=0$*,* $i=1,...,N$*, and* $f_\alpha(x)=0$*,* $\alpha_i\in\mathbb{Z}^+$*,* $i=1,...,N$*, with* $|\alpha|=2$*, then*

$$|W_n(f,x)-f(x)|\leq\varphi_{\xi_n}^*. \tag{54}$$

And $W_n(f,x)\to f(x)$ *in the uniformly continuous case.*

(iii) Additionally assume all partials of order ≤ 2 *are bounded. Hence*

$$\|W_n(f)-f\|_\infty\leq\left(\sum_{j=0}^{r}|\alpha_j|j\right)(0.8414)$$

$$\left[\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\left(\int_{\mathbb{R}^N}|s_i|e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N\right)\right]+$$

$$\left(\sum_{j=0}^{r}|\alpha_j|j^2\right)(0.4596)\left\{\sum_{i=1}^{N}\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\left(\int_{\mathbb{R}^N}s_i^2 e^{-\frac{\sum_{i=1}^{N}s_i^2}{\xi_n}}ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_\infty+\right.$$

$$\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}} \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\left(\int_{\mathbb{R}^N}|s_i|\,|s_{j^*}|\,e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}}ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i\partial x_{j^*}}\right\|_\infty\Bigg\}+\varphi^*_{\xi_n}. \tag{55}$$

If all f_α *of order 2,* $f\in C_U\left(\mathbb{R}^N\right)$*, then*

$$\|W_n(f)-f\|_\infty\to 0 \text{ with rates, as } \xi_n\to 0,\ n\to+\infty.$$

Proof. By Theorems 4, 10, 12 and Remark 11. ■

Next we apply Theorem 6 to W_n operators.

Theorem 17 *Let* $f\in C^m\left(\mathbb{R}^N\right)$*,* $m\in\mathbb{N}$*,* $m>r$*,* $N\geq 1$*, with* $f_\alpha\in L_p\left(\mathbb{R}^N\right)$*,* $|\alpha|=m$*,* $x\in\mathbb{R}^N$*. Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$*;* $0<\xi_n\leq 1$*,* $n\in\mathbb{N}$*. For* $\widetilde{j}=1,...,m$*, and* $\alpha:=(\alpha_1,...,\alpha_N)$*,* $\alpha_i\in\mathbb{Z}^+$*,* $i=1,...,N$*,* $|\alpha|:=\sum_{i=1}^N\alpha_i=\widetilde{j}$*, call*

$$c^*_{\alpha,n,\widetilde{j}}:=\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\int_{\mathbb{R}^N}\left(\prod_{i=1}^N s_i^{\alpha_i}\right)e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}}ds_1...ds_N. \tag{56}$$

Then

$$\left\|E^{*[m]}_{r,n}\right\|_p:=\left\|W_{r,n}(f;x)-f(x)-\sum_{\widetilde{j}=1}^m\delta^{[m]}_{\widetilde{j},r}\left(\sum_{|\alpha|=\widetilde{j}}\frac{c_{\alpha,n,\widetilde{j}}f_\alpha(x)}{\left(\prod_{i=1}^N\alpha_i!\right)}\right)\right\|_{p,x}\leq$$

$$\left[\binom{m+N-1}{m}^{\frac{p}{q}}\left(\frac{m}{(q(m-1)+1)^{\frac{p}{q}}}\right)\left(\sum_{|\alpha|=m}\frac{1}{\prod_{i=1}^N\alpha_i!}\widetilde{A}_{\xi_n}(\alpha)\,\omega_r(f_\alpha,\xi_n)_p^p\right)\right]^{\frac{1}{p}}, \tag{57}$$

where $\widetilde{A}_{\xi_n}(\alpha)$ *as in (39).*

As $n\to\infty$ *and* $\xi_n\to 0$*, by (57), we obtain that* $\left\|E^{*[m]}_{r,n}\right\|_p\to 0$ *with rates.*

One also obtains by (57) that

$$\|W_n(f;x)-f(x)\|_{p,x}\leq$$

$$\sum_{\widetilde{j}=1}^m\left|\delta^{[m]}_{\widetilde{j},r}\right|\left(\sum_{|\alpha|=\widetilde{j}}\frac{\left|c^*_{\alpha,n,\widetilde{j}}\right|}{\prod_{i=1}^N\alpha_i!}\|f_\alpha\|_p\right)+R.H.S.(57), \tag{58}$$

given that $\|f_\alpha\|_p<\infty$*,* $|\alpha|=\widetilde{j}$*,* $\widetilde{j}=1,...,m$*.*

Finally we get $\|W_n(f)-f\|_p\to 0$ *as* $\xi_n\to 0$*, and* $n\to\infty$*. That is* $W_n\to I$ *the unit operator, in* L_p *norm, with rates.*

Proof. By Theorems 6, 13 and (40). ■

Next we apply Theorem 8 to W_n operators.

Theorem 18 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. Here we deal with $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, with f, $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = 2$, where $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, and $|\alpha| = \sum_{i=1}^{N} \alpha_i$; $x \in \mathbb{R}^N$; $j = 0, 1, ..., r$.*

Denote ($n \in \mathbb{N}$)

$$\overline{\Delta}_n(x) := W_n(f, x) - f(x) - \left(\sum_{j=0}^{r} \alpha_j j\right) \sin(1)$$

$$\left[\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \left(\int_{\mathbb{R}^N} s_i e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N\right)\right]$$

$$-2\left(\sum_{j=0}^{r} \alpha_j j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^{N} \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \left(\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N\right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \tag{59}$$

$$\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, ..., N\}}} \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \left(\int_{\mathbb{R}^N} s_i s_{j^*} e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N\right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}}\right\}.$$

Then

$$\left\|\overline{\Delta}_n\right\|_p \leq \left(\sum_{j=0}^{r} |\alpha_j|^q\right)^{\frac{1}{q}} \left\{\frac{2}{(q+1)^{\frac{1}{q}}} \left(\frac{N(N+1)+2}{2}\right)^{\frac{1}{q}}\right\}$$

$$\left\{\sum_{j=0}^{r} \left[j^{2p} \sum_{|\alpha|=2} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right)^p \omega_1(f_\alpha, \xi_n)_p^p \widetilde{A}_{j\xi_n}(\alpha) + \omega_1(f, \xi_n)_p^p \left[1 + j^p K_{p\xi_n}\right]\right]\right\}^{\frac{1}{p}}, \tag{60}$$

where $\widetilde{A}_{j\xi_n}(\alpha)$ as in (41), and $K_{p\xi_n}$ as in (46).

One also obtains from (60) that

$$\|W_n(f) - f\|_p \leq$$

$$\left(\sum_{j=0}^{r} |\alpha_j| j\right) \sin(1) \left[\sum_{i=1}^{N} \left\|\frac{\partial f}{\partial x_i}\right\|_p \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \left(\int_{\mathbb{R}^N} |s_i| e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N\right)\right]$$

$$+2\left(\sum_{j=0}^{r} |\alpha_j| j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^{N} \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \left(\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^{N} s_i^2}{\xi_n}} ds_1 ... ds_N\right) \left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_p + \right.$$

$$\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,\dots,N\}}} \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\left(\int_{\mathbb{R}^N}|s_i s_{j^*}|\, e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i \partial x_{j^*}}\right\|_p\Bigg\} + R.H.S.\,(60), \tag{61}$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, 2$.

Additionally assume that $\omega_1(f,\xi_n)_p \leq \lambda\xi_n$, $\lambda > 0$, *then as* $n \to \infty$ *and* $\xi_n \to 0$, *by (60), we obtain that* $\left\|\overline{\Delta}_n\right\|_p \to 0$ *with rates, and furthermore by (61), we derive that* $\|W_n(f) - f\|_p \to 0$, *that is* $W_n \to I$ *the unit operator, in* L_p *norm, with rates.*

Proof. By Theorems 8, 14, 15 and Remark 11. ■

We finish with an application of Theorem 9 to W_n operators.

Theorem 19 *Let* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$. *Here we deal with* $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, *with* f, $f_\alpha \in L_1(\mathbb{R}^N)$, $|\alpha| = 2$, *where* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, *and* $|\alpha| = \sum_{i=1}^N \alpha_i$; $j = 0, 1, ..., r$. *Here* $\overline{\Delta}_n$ *as in (59). Then*

$$\left\|\overline{\Delta}_n\right\|_1 \leq \left\{\sum_{j=0}^r |\alpha_j|\left[j^2\sum_{|\alpha|=2}\left(\frac{2}{\prod_{i=1}^N \alpha_i!}\right)\omega_1(f_\alpha,\xi_n)_1\,\widetilde{\widetilde{A}}_{j\xi_n}(\alpha)\right.\right.$$

$$\left.\left. +\omega_1(f,\xi_n)_1\left[1 + jK^*_{\xi_n}\right]\right]\right\}, \tag{62}$$

where $\widetilde{\widetilde{A}}_{j\xi_n}(\alpha)$ *as in Theorem 14, and* $K^*_{\xi_n}$ *as in Theorem 15.*

We derive also that

$$\|W_n(f) - f\|_1 \leq$$

$$\left(\sum_{j=0}^r |\alpha_j|\, j\right)\sin(1)\left[\sum_{i=1}^N\left\|\frac{\partial f}{\partial x_i}\right\|_1 \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\left(\int_{\mathbb{R}^N}|s_i|\, e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1...ds_N\right)\right]$$

$$+2\left(\sum_{j=0}^r |\alpha_j|\, j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^N \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\left(\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_1 + \right. \tag{63}$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,\dots,N\}}} \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\left(\int_{\mathbb{R}^N}|s_i s_{j^*}|\, e^{-\frac{\sum_{i=1}^N s_i^2}{\xi_n}} ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i \partial x_{j^*}}\right\|_1\right\} + R.H.S.\,(62),$$

given that $\|f_\alpha\|_1 < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, 2$. *Additionally assume that* $\omega_1(f,\xi_n)_1 \leq \lambda^*\xi_n$, $\lambda^* > 0$, *then as* $n \to \infty$ *and* $\xi_n \to 0$, *by (62), we obtain that* $\left\|\overline{\Delta}_n\right\|_1 \to 0$ *with rates, and furthermore by (63), we derive that* $\|W_n(f) - f\|_1 \to 0$, *that is* $W_n \to I$ *the unit operator, in* L_1 *norm, with rates.*

Proof. By Theorems 9, 14, 15 and Remark 11. ■

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