Trigonometric background multivariate smooth Poisson-Cauchy singular integrals approximation

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis,
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

In this article we apply the uniform and L_p, $1 \leq p < \infty$ approximation properties of general smooth multivariate singular integral operators over $\mathbb{R}^N$, $N \geq 1$. It is a trigonometric based approach with detailed applications to the corresponding smooth multivariate Poisson-Cauchy singular integral operators. The results are quantitative via Jackson type inequalities involving the first uniform and L_p moduli of continuity.

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1 Introduction

The degree of approximation by univariate and multivariate singular integral operators has been studied extensively in [1]-[3] and [7], [9] and [10]. All these sources motivate our current work. In particular we studied the approximation properties of the smooth singular integral operators in [1]-[3], [9]. These are not in general positive operators. Here we use the uniform and L_p, $p \geq 1$, results of our multivariate general theory [5], [6], to establish approximation properties of the smooth Poisson-Cauchy singular integral operators. The degrees of approximation are given quantitatively by using the uniform and L_p first moduli of continuity. The essential tool here comes from [4], where a multivariate trigonometric Taylor formula is proved.

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2 Background of General Theory

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_j := \alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, ..., r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0. \end{cases} \quad (1)$$

and

$$\delta_k := \delta_{k,r}^{[m]} := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, ..., m \in \mathbb{N}. \quad (2)$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \quad (3)$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}.$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\theta_n (f; x_1, ..., x_N) := \theta_{r,n}^{[m]} (f; x_1, ..., x_N) :=$$

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f (x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) \, d\mu_{\xi_n} (s), \quad (4)$$

where $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers, we take $0 < \xi_n \leq 1$.

Remark 1 *The operators $\theta_{r,n}^{[m]}$ are not in general positive, see [2], p. 2.*

We observe that

Lemma 2 *It holds*

$$\theta_{r,n}^{[m]} (c; x_1, ..., x_n) = c,$$

where c is a constant.

We need

Definition 3 *Let $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$. We define the first uniform modulus of continuity of f as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in\mathbb{R}^N: \\ \|x-y\|_\infty \leq \delta}} |f(x) - f(y)|, \quad \delta > 0, \tag{5}$$

where $\|\cdot\|_\infty$ is the max norm in $\mathbb{R}^N$. The functional $\omega_1(f,\delta)$ is bounded for f being bounded or uniformly continuous, and $\omega_1(f,\delta) \to 0$ as $\delta \to 0$, in the case of f being uniformly continuous.

We mention the main uniform general approximation result regarding the operator θ_n.

Theorem 4 *([5]) Here $f \in C^2\left(\mathbb{R}^N\right)$ and let all $\alpha_i \in \mathbb{Z}^+$, $i = 1,...,N$, $N \geq 1$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = 2$; $x \in \mathbb{R}^N$, and all the partials f_α of order 2, along with $f \in C_B\left(\mathbb{R}^N\right)$ (continuous and bounded functions); or all f_α of order 2, $f \in C_U\left(\mathbb{R}^N\right)$ (uniformly continuous functions). Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$, for $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.*

Suppose that for all $\alpha := (\alpha_1,...,\alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1,...,N$, $|\alpha| = \sum_{i=1}^{N} \alpha_i = 2$, $j = 0,1,...,r$, we have that both

$$I_{1j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right)\left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s), \tag{6}$$

$$I_{2j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) d\mu_{\xi_n}(s), \tag{7}$$

are uniformly bounded in $\xi_n \in (0,1]$.

Denote ($n \in \mathbb{N}$)

$$\Delta_n(x) := \theta_n(f,x) - f(x) - \left(\sum_{j=0}^{r} \alpha_j j\right) \sin(1) \left[\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} \left(\int_{\mathbb{R}^N} s_i d\mu_{\xi_n}(s)\right)\right]$$

$$-2\left(\sum_{j=0}^{r} \alpha_j j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)\right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \tag{8}$$

$$\left. \sum_{\substack{i \neq j^*, \\ i,j^* \in \{1,...,N\}}} \left(\int_{\mathbb{R}^N} s_i s_{j^*} d\mu_{\xi_n}(s)\right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}} \right\}.$$

Then

(i)

$$|\Delta_n(x)| \leq \|\Delta_n(x)\|_\infty \leq \sum_{j=0}^{r} |\alpha_j|$$

$$\left[\left[j^2 \sum_{\substack{\alpha_i \in \mathbb{Z}^+, \\ \alpha : |\alpha| = 2}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_1(f_\alpha, \xi_n) \int_{\mathbb{R}^N} \left(1 + \frac{j \|s\|_\infty}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) d\mu_{\xi_n}(s) \right] + \right.$$

$$\left. \frac{1}{2} \omega_1(f, \xi_n) \int_{\mathbb{R}^N} \left(1 + \frac{j \|s\|_\infty}{3\xi_n}\right) d\mu_{\xi_n}(s) \right] =: \varphi_{\xi_n}. \tag{9}$$

In case of all f_α of order 2 and $f \in C_U(\mathbb{R}^N)$ and $\xi_n \to 0$, as $n \to \infty$, then $\Delta_n(x)$, $\|\Delta_n(x)\|_\infty \to 0$ with rates.

(ii) If $\frac{\partial f(x)}{\partial x_i} = 0$, $i = 1, ..., N$, and $f_\alpha(x) = 0$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, with $|\alpha| = 2$, then

$$|\theta_n(f, x) - f(x)| \leq \varphi_{\xi_n}. \tag{10}$$

And $\theta_n(f, x) \to f(x)$ in the uniformly continuous case.

(iii) Additionally assume all partials of order ≤ 2 are bounded. Hence

$$\|\theta_n(f) - f\|_\infty \leq \left(\sum_{j=0}^{r} |\alpha_j| j \right) (0.8414) \left[\sum_{i=1}^{N} \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \left(\int_{\mathbb{R}^N} |s_i| d\mu_{\xi_n}(s) \right) \right] +$$

$$\left(\sum_{j=0}^{r} |\alpha_j| j^2 \right) (0.4596) \left\{ \sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_\infty + \right.$$

$$\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, ..., N\}}} \left(\int_{\mathbb{R}^N} |s_i| |s_{j^*}| d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_\infty \right\} + \varphi_{\xi_n}. \tag{11}$$

If all $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)$ and $\int_{\mathbb{R}^N} |s_i| |s_{j^*}| d\mu_{\xi_n}(s)$ converge to zero, as $n \to \infty$, with $\xi_n \to 0$, and all f_α of order 2, $f \in C_U(\mathbb{R}^N)$, then

$$\|\theta_n(f) - f\|_\infty \to 0 \text{ with rates, as } \xi_n \to 0,\ n \to +\infty.$$

Next we deal with $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{Z}^+$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m \in \mathbb{Z}^+$, $p \geq 1$; where f_α denotes the mixed partial $\frac{\partial^{\widetilde{j}} f(\cdot, ..., \cdot)}{\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\alpha| := \sum_{j=1}^{N} \alpha_j = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

We need

Definition 5 *(see also [2], p. 20) We call*

$$\Delta_u^r f(x) := \Delta_{u_1,u_2,\ldots,u_N}^r f(x_1,\ldots,x_N) := \tag{12}$$

$$\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x_1+ju_1, x_2+ju_2, \ldots, x_N+ju_N).$$

Let $p \geq 1$, the L_p modulus of smoothness of order r is given by

$$\omega_r(f;h)_p := \sup_{\|u\|_2 \leq h} \|\Delta_u^r(f)\|_p, \tag{13}$$

$h > 0$.

We mention

Theorem 6 *([6]) Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence. Assume for all $\alpha := (\alpha_1,\ldots,\alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1,\ldots,N$, $|\alpha| := \sum_{i=1}^N \alpha_i = m$ that we have*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{14}$$

For $\widetilde{j} = 1,\ldots,m$, and $\alpha := (\alpha_1,\ldots,\alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1,\ldots,N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \widetilde{j}$, call

$$c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} d\mu_{\xi_n}(s). \tag{15}$$

Then

$$\left\| E_{r,n}^{[m]} \right\|_p := \left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^m \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^N \alpha_i! \right)} \right) \right\|_{p,x} \leq \tag{16}$$

$$\left[\binom{m+N-1}{m}^{\frac{p}{q}} \left(\frac{m}{(q(m-1)+1)^{\frac{p}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \cdot \right.\right.$$

$$\left.\left. \left[\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right] \omega_r(f_\alpha, \xi_n)_p^p \right) \right]^{\frac{1}{p}}.$$

As $n \to \infty$ and $\xi_n \to 0$, by (16), we obtain that $\left\| E_{r,n}^{[m]} \right\|_p \to 0$ with rates.

One also gets by (16) that

$$\left\| \theta_{r,n}^{[m]} (f;x) - f(x) \right\|_{p,x} \leq$$

$$\sum_{\widetilde{j}=1}^{m} \left| \delta_{\widetilde{j},r}^{[m]} \right| \left(\sum_{|\alpha| = \widetilde{j}} \frac{\left| c_{\alpha,n,\widetilde{j}} \right|}{\prod_{i=1}^{N} \alpha_i!} \|f_\alpha\|_p \right) + R.H.S.(16), \tag{17}$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

Assuming that $c_{\alpha,n,\widetilde{j}} \to 0$, $\xi_n \to 0$, *as* $n \to \infty$, *we get* $\left\| \theta_{r,n}^{[m]} (f) - f \right\|_p \to 0$, *that is* $\theta_{r,n}^{[m]} \to I$ *the unit operator, in* L_p *norm, with rates.*

We make

Remark 7 *Notice that* $(p > 1)$

$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \leq$$

$$\left[\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} < \infty, \tag{18}$$

by assumption of Theorem 6.

By (18) we get that

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i} d\mu_{\xi_n}(s) < \infty. \tag{19}$$

Hence $c_{\alpha,n,\widetilde{j}} \in \mathbb{R}$.

We mention also the following trigonometric induced alternative L_p approximation result for θ_n operators.

Theorem 8 *([6]) Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Here we deal with* $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, *with* f, $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = 2$, *where* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, *and* $|\alpha| = \sum_{i=1}^{N} \alpha_i$; $x \in \mathbb{R}^N$. *Let* μ_{ξ_n} *be a Borel probability measure on* $\mathbb{R}^N$. *Suppose that for all* $\alpha : |\alpha| = 2$, $j = 0, 1, ..., r$, *we have that both*

$$I_{1j}(\alpha) := \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) d\mu_{\xi_n}(s), \tag{20}$$

$$I_2 := \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p d\mu_{\xi_n}(s), \tag{21}$$

are uniformly bounded in $\xi_n \in (0,1]$.

Denote ($n \in \mathbb{N}$)

$$\Delta_n(x) := \theta_n(f,x) - f(x) - \left(\sum_{j=0}^{r} \alpha_j j \right) \sin(1) \left[\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} \left(\int_{\mathbb{R}^N} s_i d\mu_{\xi_n}(s) \right) \right]$$

$$-2 \left(\sum_{j=0}^{r} \alpha_j j^2 \right) \sin^2\left(\frac{1}{2}\right) \left\{ \sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \tag{22}$$

$$\left. \sum_{\substack{i \neq j^*, \\ i,j^* \in \{1,\dots,N\}}} \left(\int_{\mathbb{R}^N} s_i s_{j^*} d\mu_{\xi_n}(s) \right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}} \right\}.$$

Then, it holds

$$\|\Delta_n(x)\|_p \leq \left(\sum_{j=0}^{r} |\alpha_j|^q \right)^{\frac{1}{q}} \left\{ \frac{2}{(q+1)^{\frac{1}{q}}} \left(\frac{N(N+1)+2}{2} \right)^{\frac{1}{q}} \right\}$$

$$\left\{ \sum_{j=0}^{r} \left[j^{2p} \sum_{|\alpha|=2} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right)^p \omega_1(f_\alpha, \xi_n)_p^p \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) d\mu_{\xi_n}(s) \right) \right. \right.$$

$$\left. \left. +\omega_1(f,\xi_n)_p^p \left(\int_{\mathbb{R}^N} \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) d\mu_{\xi_n}(s) \right) \right] \right\}^{\frac{1}{p}}. \tag{23}$$

As $n \to \infty$ *and* $\xi_n \to 0$, *by (23), we obtain that* $\|\Delta_n\|_p \to 0$ *with rates. One also gets by (23) that*

$$\|\theta_n(f,x) - f(x)\|_{p,x} \leq$$

$$\left(\sum_{j=0}^{r} |\alpha_j| j \right) \sin(1) \left[\sum_{i=1}^{N} \left\| \frac{\partial f}{\partial x_i} \right\|_p \left(\int_{\mathbb{R}^N} |s_i| d\mu_{\xi_n}(s) \right) \right]$$

$$+2 \left(\sum_{j=0}^{r} |\alpha_j| j^2 \right) \sin^2\left(\frac{1}{2}\right) \left\{ \sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_p + \right.$$

$$\left. \sum_{\substack{i \neq j^*, \\ i,j^* \in \{1,\dots,N\}}} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_p \right\} + R.H.S.(23), \tag{24}$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, 2$. *Assuming that* $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_m}(s)$, $\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s)$, $i, j^* \in \{1, ..., N\}$, $i \neq j^*$, *converge to zero as* $\xi_n \to 0$, *we get* $\|\theta_n(f, x) - f(x)\|_p \to 0$, *that is* $\theta_n \to I$ *the unit operator, in* L_p *norm, with rates.*

Furthermore we mention the following trigonometric based alternative L_1 approximation result for θ_n operators.

Theorem 9 *([6]) Let* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$. *Here we deal with* $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, *with* f, $f_\alpha \in L_1(\mathbb{R}^N)$, $|\alpha| = 2$, *where* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, *and* $|\alpha| = \sum_{i=1}^N \alpha_i$. *Let* μ_{ξ_n} *be a Borel probability measure on* $\mathbb{R}^N$. *Suppose that for all* $\alpha : |\alpha| = 2$, $j = 0, 1, ..., r$, *we have that both*

$$I_{1j}^*(\alpha) := \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{j \|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s), \tag{25}$$

and

$$I_2^* := \int_{\mathbb{R}^N} \frac{\|s\|_2}{\xi_n} d\mu_{\xi_n}(s), \tag{26}$$

are uniformly bounded in $\xi_n \in (0, 1]$.

Here $\Delta_n(x)$ *is as in (22).*

Then, it holds

$$\|\Delta_n(x)\|_1 \leq \left\{ \sum_{j=0}^r |\alpha_j| \left[j^2 \sum_{|\alpha|=2} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \omega_1(f_\alpha, \xi_n)_1 \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N (|s_i|)^{\alpha_i} \right) \right.\right.\right.$$

$$\left.\left.\left. \left(1 + \frac{j \|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s) \right) + \omega_1(f, \xi_n)_1 \left(\int_{\mathbb{R}^N} \left(1 + \frac{j \|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s) \right) \right] \right\}. \tag{27}$$

As $n \to \infty$ *and* $\xi_n \to 0$, *by (27), we obtain that* $\|\Delta_n\|_1 \to 0$ *with rates. One also obtains by (27) that*

$$\|\theta_n(f) - f\|_1 \leq$$

$$\left(\sum_{j=0}^r |\alpha_j| j \right) \sin(1) \left[\sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_1 \left(\int_{\mathbb{R}^N} |s_i| d\mu_{\xi_n}(s) \right) \right]$$

$$+2 \left(\sum_{j=0}^r |\alpha_j| j^2 \right) \sin^2\left(\frac{1}{2} \right) \left\{ \sum_{i=1}^N \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_1 + \right.$$

$$\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, ..., N\}}} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_1 \right\} + R.H.S.(27), \tag{28}$$

given that $\|f_\alpha\|_1 < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, 2$. *Assuming that* $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)$, $\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s)$, $i, j^* \in \{1, ..., N\}$, $i \neq j^*$, *converge to zero as* $\xi_n \to 0$, *we derive* $\|\theta_n(f) - f\|_1 \to 0$, *that is* $\theta_n \to I$ *in* L_1 *norm, with rates.*

We need

Definition 10 *The general multivariate Poisson-Cauchy singular integral operators are defined as follows:*

$$U_n := U_{r,n}^{[m]}(f; x_1, ..., x_N) := W_n^N \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{29}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, ..., x_N + s_N j) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} ds_1 ... ds_N,$$

with $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$, *and*

$$W_n := \frac{\Gamma(\beta)\, \alpha \xi_n^{2\alpha\beta - 1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)}, \tag{30}$$

see [2], p. 15.

Notice that

$$W_n^N \int_{\mathbb{R}^N} \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} ds_1 ... ds_N = 1, \tag{31}$$

see [8], [11], p. 397, formula 595.

That is, U_n are the θ_n operators applied for the Borel probability measures on $\mathbb{R}^N$, $N \geq 1$,

$$d\mu_{\xi_n}(s) = W_n^N \left(\prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} \right) ds_1 ... ds_N, \quad s \in \mathbb{R}^N, \tag{32}$$

where $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.

Wr will apply to U_n the Theorems 4, 6, 8 and 9, with the help of all of section 3. That is a presentation of approximation properties by U_n.

3 Auxiliary Results

We need

Theorem 11 *Let* $N \in \mathbb{N}$; $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\alpha| := \sum_{i=1}^{N} \alpha_i = 2$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$; $j = 0, 1, ..., r \in \mathbb{N}$; $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$, W_n *as in (30). Then*

$$\overline{I}_{1j}(\alpha) := W_n^N \int_{\mathbb{R}^N} \left(1 + \frac{j \|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^{\beta}} ds_1 ... ds_N \leq \tag{33}$$

$$\frac{\xi_n^{2\alpha\beta(N-1)+2}}{\Gamma^N\left(\frac{1}{2\alpha}\right)\Gamma^N\left(\beta - \frac{1}{2\alpha}\right)}$$

$$\left[(2\alpha\Gamma(\beta))^N \left(1 + \frac{j}{3}N\right) + \left(1 + \frac{j}{3}\right)\Gamma^N\left(\frac{2}{\alpha}\right)\Gamma^N\left(\beta - \frac{2}{\alpha}\right)\right] \leq$$

$$\frac{1}{\Gamma^N\left(\frac{1}{2\alpha}\right)\Gamma^N\left(\beta - \frac{1}{2\alpha}\right)}$$

$$\left[(2\alpha\Gamma(\beta))^N \left(1 + \frac{j}{3}N\right) + \left(1 + \frac{j}{3}\right)\Gamma^N\left(\frac{2}{\alpha}\right)\Gamma^N\left(\beta - \frac{2}{\alpha}\right)\right] < +\infty, \tag{34}$$

are uniformly bounded in ξ_n, *and* $\overline{I}_{1j}(\alpha) \to 0$, *as* $\xi_n \to 0$.

Proof. We have that

$$\overline{I}_{1j}(\alpha) = W_n^N \int_{\mathbb{R}^N} \left(1 + \frac{j \|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^{\beta}} ds_1 ... ds_N =$$

$$2^N W_n^N \int_{\mathbb{R}_+^N} \left(1 + \frac{j\left(\sum_{i=1}^{N} s_i\right)}{3\xi_n}\right) \left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^{\beta}} ds_1 ... ds_N \tag{35}$$

$$\left(c := \frac{\Gamma(\beta)\,\alpha}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)}, \text{ i.e. } W_n = c\xi_n^{2\alpha\beta-1}\right)$$

$$= (2c)^N \xi_n^{2\alpha\beta N - N} \int_{\mathbb{R}_+^N} \left(1 + j\frac{\sum_{i=1}^{N} s_i}{3\xi_n}\right) \left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^{\beta}} \prod_{i=1}^{N} ds_i =$$

$$(2c)^N \xi_n^{2\alpha\beta N - N} \xi_n^2 \xi_n^{-2\alpha\beta} \xi_n^N \int_{\mathbb{R}_+^N} \left(1 + \frac{j}{3}\sum_{i=1}^{N}\left(\frac{s_i}{\xi_n}\right)\right)$$

$$\left(\prod_{i=1}^{N}\left(\frac{s_i}{\xi_n}\right)^{\alpha_i}\right) \prod_{i=1}^{N} \frac{1}{\left(\left(\frac{s_i}{\xi_n}\right)^{2\alpha} + 1\right)^{\beta}} \prod_{i=1}^{N} d\left(\frac{s_i}{\xi_n}\right) =$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)+2} \int_{\mathbb{R}_+^N} \left(1+\frac{j}{3}\sum_{i=1}^N z_i\right)\left(\prod_{i=1}^N z_i^{\alpha_i}\right)\prod_{i=1}^N \frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i =$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)+2}\left[\int_{[0,1]^N}\left(1+\frac{j}{3}\sum_{i=1}^N z_i\right)\left(\prod_{i=1}^N z_i^{\alpha_i}\right)\prod_{i=1}^N \frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i + \right. \tag{36}$$

$$\left.\int_{(\mathbb{R}_+-[0,1])^N}\left(1+\frac{j}{3}\left(\sum_{i=1}^N z_i\right)\right)\left(\prod_{i=1}^N z_i^{\alpha_i}\right)\prod_{i=1}^N \frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i\right] \leq$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)+2}\left[\left(1+\frac{j}{3}N\right)+\right.$$

$$\left.\left(1+\frac{j}{3}\right)\int_{(\mathbb{R}_+-[0,1])^N}\left(\sum_{i=1}^N z_i\right)\left(\prod_{i=1}^N z_i^{\alpha_i}\right)\prod_{i=1}^N \frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i\right] \leq$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)+2}\left[\left(1+\frac{j}{3}N\right)+\right.$$

$$\left.\left(1+\frac{j}{3}\right)\int_{(\mathbb{R}_+-[0,1])^N}\left(\prod_{i=1}^N z_i\right)\left(\prod_{i=1}^N z_i^{\alpha_i}\right)\prod_{i=1}^N \frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i\right] =$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)+2}$$

$$\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\int_{(\mathbb{R}_+-[0,1])^N}\left(\prod_{i=1}^N z_i^{\alpha_i+1}\right)\prod_{i=1}^N \frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i\right] = \tag{37}$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)+2}\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\prod_{i=1}^N\int_1^\infty z_i^{\alpha_i+1}\frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}dz_i\right] \leq$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)+2}\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\left(\int_1^\infty \frac{z^{2+1}}{\left(z^{2\alpha}+1\right)^\beta}dz\right)^N\right] \leq$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)+2}\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\left(\int_0^\infty \frac{z^{3}}{\left(z^{2\alpha}+1\right)^\beta}dz\right)^N\right] \tag{38}$$

(by [11], p. 397, formula 595)

$$= (2c)^N \xi_n^{2\alpha\beta(N-1)+2}\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\left(\frac{\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta-\frac{2}{\alpha}\right)}{2\alpha\Gamma\left(\beta\right)}\right)^N\right]$$

$$= \left(\frac{2\Gamma\left(\beta\right)\alpha}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\right)^N \xi_n^{2\alpha\beta(N-1)+2}$$

$$\left[\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\left(\frac{\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta-\frac{2}{\alpha}\right)}{2\alpha\Gamma\left(\beta\right)}\right)^{N}\right],$$

where $N\in\mathbb{N}$ and $\beta>\frac{1}{2\alpha}$. The claim is proved. ■

Remark 12 *By Theorem 11, with $i,j^{*}\in\{1,...,N\}$, $i\neq j^{*}$, we have that*

$$W_{n}^{N}\int_{\mathbb{R}^{N}}|s_{i}|\,|s_{j^{*}}|\left(\prod_{i=1}^{N}\frac{1}{\left(s_{i}^{2\alpha}+\xi_{n}^{2\alpha}\right)^{\beta}}\right)ds_{1}...ds_{N},$$

$$W_{n}^{N}\int_{\mathbb{R}^{N}}s_{i}^{2}\left(\prod_{i=1}^{N}\frac{1}{\left(s_{i}^{2\alpha}+\xi_{n}^{2\alpha}\right)^{\beta}}\right)ds_{1}...ds_{N}\leq\overline{I}_{1j}\left(\alpha\right)<\infty, \tag{39}$$

$(j=0,1,...,r\in\mathbb{N})$;

and

$$W_{n}^{N}\int_{\mathbb{R}^{N}}|s_{i}|\left(\prod_{i=1}^{N}\frac{1}{\left(s_{i}^{2\alpha}+\xi_{n}^{2\alpha}\right)^{\beta}}\right)ds_{1}...ds_{N}\leq \tag{40}$$

$$(W_{n}^{N}\int_{\mathbb{R}^{N}}s_{i}^{2}\left(\prod_{i=1}^{N}\frac{1}{\left(s_{i}^{2\alpha}+\xi_{n}^{2\alpha}\right)^{\beta}}\right)ds_{1}...ds_{N})^{1/2}<\infty,$$

and all these integrals are uniformly bounded in $\xi_{n}\in(0,1]$. And all integrals converge to zero, as $\xi_{n}\to0$.

We continue.

Theorem 13 *Let $N\geq1$, $\xi_{n}\in(0,1]$, $n\in\mathbb{N}$, $\beta>\frac{1}{\alpha}$. Then*

$$\overline{I}_{2}:=W_{n}^{N}\int_{\mathbb{R}^{N}}\frac{\|s\|_{1}}{\xi_{n}}\prod_{i=1}^{N}\frac{1}{\left(s_{i}^{2\alpha}+\xi_{n}^{2\alpha}\right)^{\beta}}ds_{1}...ds_{N}\leq$$

$$\frac{\xi^{2\alpha\beta(N-1)}}{\Gamma^{N}\left(\frac{1}{2\alpha}\right)\Gamma^{N}\left(\beta-\frac{1}{2\alpha}\right)}\left[\left(2\alpha\Gamma\left(\beta\right)\right)^{N}N+\Gamma^{N}\left(\frac{1}{\alpha}\right)\Gamma^{N}\left(\beta-\frac{1}{\alpha}\right)\right]\leq \tag{41}$$

$$\frac{1}{\Gamma^{N}\left(\frac{1}{2\alpha}\right)\Gamma^{N}\left(\beta-\frac{1}{2\alpha}\right)}\left[\left(2\alpha\Gamma\left(\beta\right)\right)^{N}N+\Gamma^{N}\left(\frac{1}{\alpha}\right)\Gamma^{N}\left(\beta-\frac{1}{\alpha}\right)\right]<+\infty,$$

uniformly bounded in ξ_{n}; and as $\xi_{n}\to0$, $\overline{I}_{2}\to0$, when $N>1$.

Proof. We observe that

$$\overline{I}_{2}=W_{n}^{N}\int_{\mathbb{R}^{N}}\frac{\|s\|_{1}}{\xi_{n}}\prod_{i=1}^{N}\frac{1}{\left(s_{i}^{2\alpha}+\xi_{n}^{2\alpha}\right)^{\beta}}ds_{1}...ds_{N}=$$

$$2^N W_n^N \int_{\mathbb{R}_+^N} \left(\frac{\sum_{i=1}^N s_i}{\xi_n} \right) \prod_{i=1}^N \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} ds_1 ... ds_N =$$

$$\left(c := \frac{\Gamma(\beta)\,\alpha}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)}, \text{ i.e. } W_n = c\xi_n^{2\alpha\beta-1}\right)$$

$$= (2c)^N \xi_n^{2\alpha\beta N - N} \xi_n^{-2\alpha\beta} \xi_n^N \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^N \left(\frac{s_i}{\xi_n} \right) \right) \prod_{i=1}^N \frac{1}{\left(\left(\frac{s_i}{\xi_n} \right)^{2\alpha} + 1 \right)^\beta} \prod_{i=1}^N d\left(\frac{s_i}{\xi_n} \right) = \tag{42}$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^N z_i \right) \left(\prod_{i=1}^N \frac{1}{\left(z_i^{2\alpha} + 1\right)^\beta} \right) \prod_{i=1}^N dz_i =$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)} \left[\int_{[0,1]^N} \left(\sum_{i=1}^N z_i \right) \left(\prod_{i=1}^N \frac{1}{\left(z_i^{2\alpha} + 1\right)^\beta} \right) \prod_{i=1}^N dz_i + \right.$$

$$\left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^N z_i \right) \left(\prod_{i=1}^N \frac{1}{\left(z_i^{2\alpha} + 1\right)^\beta} \right) \prod_{i=1}^N dz_i \right] \leq$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)} \left[N + \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i \right) \left(\prod_{i=1}^N \frac{1}{\left(z_i^{2\alpha} + 1\right)^\beta} \right) \prod_{i=1}^N dz_i \right] = \tag{43}$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)} \left[N + \prod_{i=1}^N \int_1^\infty \frac{z_i}{\left(z_i^{2\alpha} + 1\right)^\beta} dz_i \right] =$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)} \left[N + \left(\int_1^\infty \frac{z}{\left(z^{2\alpha} + 1\right)^\beta} dz \right)^N \right] \leq$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)} \left[N + \left(\int_0^\infty \frac{z}{\left(z^{2\alpha} + 1\right)^\beta} dz \right)^N \right]$$

(by [11], p. 397, formula 595)

$$= (2c)^N \xi_n^{2\alpha\beta(N-1)} \left[N + \left(\frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\beta - \frac{1}{\alpha}\right)}{2\alpha\Gamma(\beta)} \right)^N \right] =$$

$$\left(\frac{2\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)} \right)^N \xi_n^{2\alpha\beta(N-1)} \left[N + \left(\frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\beta - \frac{1}{\alpha}\right)}{2\alpha\Gamma(\beta)} \right)^N \right] = \tag{44}$$

$$\frac{\xi_n^{2\alpha\beta(N-1)}}{\Gamma^N\left(\frac{1}{2\alpha}\right)\Gamma^N\left(\beta - \frac{1}{2\alpha}\right)} \left[(2\alpha\Gamma(\beta))^N N + \Gamma^N\left(\frac{1}{\alpha}\right)\Gamma^N\left(\beta - \frac{1}{\alpha}\right) \right] \leq$$

$$\frac{1}{\Gamma^N\left(\frac{1}{2\alpha}\right)\Gamma^N\left(\beta-\frac{1}{2\alpha}\right)}\left[\left(2\alpha\Gamma\left(\beta\right)\right)^N N+\Gamma^N\left(\frac{1}{\alpha}\right)\Gamma^N\left(\beta-\frac{1}{\alpha}\right)\right]<+\infty,$$

uniformly bounded in ξ_n; and as $\xi_n\to 0$, we get $\overline{I}_2\to 0$, for $N>1$; $\beta>\frac{1}{\alpha}$. ■

We will apply.

Theorem 14 *([3], p. 447) Let* $r,m\in\mathbb{N}$, $N\in\mathbb{N}$-$\{1\}$, $\alpha\in\mathbb{N}$, $\beta>\frac{(m+r)p+1}{2\alpha}$, $\alpha_j\in\mathbb{Z}^+$, $p>1$, $j=1,...,N:|\overline{\alpha}|:=\sum_{j=1}^{N}\alpha_j=m$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$. *Denote* $W_n:=\frac{\alpha\Gamma(\beta)\xi_n^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}$. *Then*

$$\overline{\theta}_{\xi_n}\left(\overline{\alpha}\right):=W_n^N\int_{\mathbb{R}^N}\left(\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right)^p\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}\prod_{i=1}^{N}ds_i\leq$$

$$\xi_n^{2\alpha\beta(N-1)+mp}\left(\frac{2\alpha\Gamma\left(\beta\right)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\right)^N$$

$$\left[\left(1+N\right)^{rp}+2^{rp}\left(\frac{\Gamma\left(\frac{(m+r)p+1}{2\alpha}\right)\Gamma\left(\beta-\left(\frac{(m+r)p+1}{2\alpha}\right)\right)}{2\alpha\Gamma\left(\beta\right)}\right)^N\right]\leq$$

$$\left(\frac{2\alpha\Gamma\left(\beta\right)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\right)^N$$

$$\left[\left(1+N\right)^{rp}+2^{rp}\left(\frac{\Gamma\left(\frac{(m+r)p+1}{2\alpha}\right)\Gamma\left(\beta-\left(\frac{(m+r)p+1}{2\alpha}\right)\right)}{2\alpha\Gamma\left(\beta\right)}\right)^N\right]<+\infty, \tag{45}$$

uniformly bounded in ξ_n.

Also $\overline{\theta}_{\xi_n}\left(\overline{\alpha}\right)\to 0$, *as* $\xi_n\to 0$, $n\to+\infty$.

Remark 15 *(on Theorem 14) Clearly, it holds*

$$\overline{B}_{\xi_n}\left(\overline{\alpha}\right):=W_n^N\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}\prod_{i=1}^{N}ds_i<\infty, \tag{46}$$

uniformly bounded in $\xi_n\in(0,1]$. *And, clearly* $\overline{B}_{\xi_n}\left(\overline{\alpha}\right)\to 0$, *as* $\xi_n\to 0$, $n\to\infty$, *by (45).*

We need the following

Theorem 16 *Let* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N \in \mathbb{N} : |\alpha| := \sum_{i=1}^{N} \alpha_i = 2$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$, $p \geq 1$, *and* $j = 0, 1, ..., r \in \mathbb{N}$. *We assume that* $\beta > \frac{3p+1}{2\alpha}$. *Then*

$$\overline{A}_{j\xi_n}(\alpha) := W_n^N \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha} \right)^\beta} \prod_{i=1}^{N} ds_i \leq \tag{47}$$

$$\frac{\xi_n^{2\alpha\beta(N-1)+2p}}{\Gamma^N\left(\frac{1}{2\alpha}\right)\Gamma^N\left(\beta - \frac{1}{2\alpha}\right)}$$

$$\left[(2\alpha\Gamma(\beta))^N (1 + j^p N^p) + (1 + j^p) \Gamma^N \left(\frac{3p+1}{2\alpha} \right) \Gamma^N \left(\beta - \left(\frac{3p+1}{2\alpha} \right) \right) \right] \leq$$

$$\frac{1}{\Gamma^N\left(\frac{1}{2\alpha}\right)\Gamma^N\left(\beta - \frac{1}{2\alpha}\right)} \tag{48}$$

$$\left[(2\alpha\Gamma(\beta))^N (1 + j^p N^p) + (1 + j^p) \Gamma^N \left(\frac{3p+1}{2\alpha} \right) \Gamma^N \left(\beta - \left(\frac{3p+1}{2\alpha} \right) \right) \right] < +\infty,$$

uniformly bounded in $\xi_n \in (0,1]$, *and* $\overline{A}_{j\xi_n}(\alpha) \to 0$, *as* $\xi_n \to 0$, $n \to \infty$. *When* $p = 1$, *we denote* $\overline{A}_{j\xi_n}(\alpha)$ *as* $\overline{\overline{A}}_{j\xi_n}(\alpha)$.

Proof. We estimate ($p \geq 1$)

$$\overline{A}_{j\xi_n}(\alpha) = W_n^N \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha} \right)^\beta} \prod_{i=1}^{N} ds_i =$$

$$2^N W_n^N \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} s_i^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha} \right)^\beta} \prod_{i=1}^{N} ds_i \leq$$

$$2^N W_n^N \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} s_i^{p\alpha_i} \right) \left(1 + j^p \left(\sum_{i=1}^{N} \frac{s_i}{\xi_n} \right)^p \right) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha} \right)^\beta} \prod_{i=1}^{N} ds_i \tag{49}$$

$$\left(c := \frac{\Gamma(\beta)\alpha}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)}, \text{ i.e. } W_n = c\xi_n^{2\alpha\beta - 1}\right)$$

$$= (2c)^N \xi_n^{2\alpha\beta N - N} \xi_n^{2p} \xi_n^{-2\alpha\beta} \xi_n^N$$

$$\int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} \left(\frac{s_i}{\xi_n} \right)^{p\alpha_i} \right) \left(1 + j^p \left(\sum_{i=1}^{N} \frac{s_i}{\xi_n} \right)^p \right) \prod_{i=1}^{N} \frac{1}{\left(\left(\frac{s_i}{\xi_n} \right)^{2\alpha} + 1 \right)^\beta} \prod_{i=1}^{N} d\left(\frac{s_i}{\xi_n} \right) =$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)+2p} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) \left(1 + j^p \left(\sum_{i=1}^{N} z_i \right)^p \right) \prod_{i=1}^{N} \frac{1}{\left(z_i^{2\alpha} + 1 \right)^\beta} \prod_{i=1}^{N} dz_i =$$

$$(2c)^N\,\xi_n^{2\alpha\beta(N-1)+2p}\left[\int_{[0,1]^N}\left(\prod_{i=1}^N z_i^{p\alpha_i}\right)\left(1+j^p\left(\sum_{i=1}^N z_i\right)^p\right)\prod_{i=1}^N\frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i+\right. \tag{50}$$

$$\left.\int_{(\mathbb{R}_+-[0,1])^N}\left(\prod_{i=1}^N z_i^{p\alpha_i}\right)\left(1+j^p\left(\sum_{i=1}^N z_i\right)^p\right)\prod_{i=1}^N\frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i\right]\le$$

$$(2c)^N\,\xi_n^{2\alpha\beta(N-1)+2p}\left[(1+j^pN^p)+\right.$$

$$\left.\int_{(\mathbb{R}_+-[0,1])^N}\left(\prod_{i=1}^N z_i^{p\alpha_i}\right)\left(\left(\sum_{i=1}^N z_i\right)^p+j^p\left(\sum_{i=1}^N z_i\right)^p\right)\prod_{i=1}^N\frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i\right]=$$

$$(2c)^N\,\xi_n^{2\alpha\beta(N-1)+2p}$$

$$\left[(1+j^pN^p)+(1+j^p)\int_{(\mathbb{R}_+-[0,1])^N}\left(\prod_{i=1}^N z_i^{p\alpha_i}\right)\left(\sum_{i=1}^N z_i\right)^p\prod_{i=1}^N\frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i\right]\le$$

$$(2c)^N\,\xi_n^{2\alpha\beta(N-1)+2p}$$

$$\left[(1+j^pN^p)+(1+j^p)\int_{(\mathbb{R}_+-[0,1])^N}\left(\prod_{i=1}^N z_i^{p\alpha_i}\right)\left(\prod_{i=1}^N z_i^p\right)\prod_{i=1}^N\frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i\right]=$$

$$(2c)^N\,\xi_n^{2\alpha\beta(N-1)+2p}$$

$$\left[(1+j^pN^p)+(1+j^p)\int_{(\mathbb{R}_+-[0,1])^N}\prod_{i=1}^N z_i^{p(\alpha_i+1)}\prod_{i=1}^N\frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}\prod_{i=1}^N dz_i\right]= \tag{51}$$

$$(2c)^N\,\xi_n^{2\alpha\beta(N-1)+2p}\left[(1+j^pN^p)+(1+j^p)\prod_{i=1}^N\int_1^\infty z_i^{p(\alpha_i+1)}\frac{1}{\left(z_i^{2\alpha}+1\right)^\beta}dz_i\right]\le$$

$$(2c)^N\,\xi_n^{2\alpha\beta(N-1)+2p}\left[(1+j^pN^p)+(1+j^p)\left(\int_1^\infty\frac{z^{3p}}{\left(z^{2\alpha}+1\right)^\beta}dz\right)^N\right]\le$$

$$(2c)^N\,\xi_n^{2\alpha\beta(N-1)+2p}\left[(1+j^pN^p)+(1+j^p)\left(\int_0^\infty\frac{z^{3p}}{\left(z^{2\alpha}+1\right)^\beta}dz\right)^N\right] \tag{52}$$

(by [11], p. 397, formula 595, by assuming $\beta>\frac{3p+1}{2\alpha}$)

$$=(2c)^N\,\xi_n^{2\alpha\beta(N-1)+2p}\left[(1+j^pN^p)+(1+j^p)\left(\frac{\Gamma\left(\frac{3p+1}{2\alpha}\right)\Gamma\left(\beta-\left(\frac{3p+1}{2\alpha}\right)\right)}{2\alpha\Gamma(\beta)}\right)^N\right]=$$

$$\left(\frac{2\Gamma(\beta)\,\alpha}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\right)^N\xi_n^{2\alpha\beta(N-1)+2p}$$

$$\left[(1+j^pN^p)+(1+j^p)\left(\frac{\Gamma\left(\frac{3p+1}{2\alpha}\right)\Gamma\left(\beta-\left(\frac{3p+1}{2\alpha}\right)\right)}{2\alpha\Gamma(\beta)}\right)^N\right]=$$

$$\frac{\xi_n^{2\alpha\beta(N-1)+2p}}{\Gamma^N\left(\frac{1}{2\alpha}\right)\Gamma^N\left(\beta-\frac{1}{2\alpha}\right)}$$

$$\left[(2\alpha\Gamma(\beta))^N(1+j^pN^p)+(1+j^p)\Gamma^N\left(\frac{3p+1}{2\alpha}\right)\Gamma^N\left(\beta-\left(\frac{3p+1}{2\alpha}\right)\right)\right]. \quad (53)$$

The claim is proved. ■

We need the following.

Theorem 17 *Let $p\geq 1$, $\xi_n\in(0,1]$, $n\in\mathbb{N}$, $N\geq 1$; and $\beta>\frac{p+1}{2\alpha}$. Then*

$$\overline{K}_{p\xi_n}:=W_n^N\int_{\mathbb{R}^N}\left(\frac{\|s\|_2}{\xi_n}\right)^p\prod_{i=1}^N\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^\beta}\prod_{i=1}^N ds_i\leq \quad (54)$$

$$\frac{\xi^{2\alpha\beta(N-1)}}{\Gamma^N\left(\frac{1}{2\alpha}\right)\Gamma^N\left(\beta-\frac{1}{2\alpha}\right)}\left[(2\alpha\Gamma(\beta))^NN^p+\Gamma^N\left(\frac{p+1}{2\alpha}\right)\Gamma^N\left(\beta-\left(\frac{p+1}{2\alpha}\right)\right)\right]\leq$$

$$\frac{1}{\Gamma^N\left(\frac{1}{2\alpha}\right)\Gamma^N\left(\beta-\frac{1}{2\alpha}\right)}\left[(2\alpha\Gamma(\beta))^NN^p+\Gamma^N\left(\frac{p+1}{2\alpha}\right)\Gamma^N\left(\beta-\left(\frac{p+1}{2\alpha}\right)\right)\right]<+\infty, \quad (55)$$

uniformly bounded in $\xi_n\in(0,1]$, and $\overline{K}_{p\xi_n}\to 0$, as $\xi_n\to 0$, $n\to\infty$, for $N>1$. When $p=1$ we denote $\overline{K}_{p\xi_n}$ as $\overline{\overline{K}}_{1\xi_n}$.

Proof. We estimate ($p\geq 1$)

$$\overline{K}_{p\xi_n}=W_n^N\int_{\mathbb{R}^N}\left(\frac{\|s\|_2}{\xi_n}\right)^p\prod_{i=1}^N\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^\beta}\prod_{i=1}^N ds_i=$$

$$2^NW_n^N\int_{\mathbb{R}_+^N}\left(\frac{\|s\|_2}{\xi_n}\right)^p\prod_{i=1}^N\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^\beta}\prod_{i=1}^N ds_i\leq$$

$$2^NW_n^N\int_{\mathbb{R}_+^N}\left(\frac{\sum_{i=1}^N s_i}{\xi_n}\right)^p\prod_{i=1}^N\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^\beta}\prod_{i=1}^N ds_i= \quad (56)$$

$$\left(c:=\frac{\Gamma(\beta)\alpha}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)},\text{ i.e. } W_n=c\xi_n^{2\alpha\beta-1}\right)$$

$$(2c)^N\xi_n^{2\alpha\beta N-N}\xi_n^{-2\alpha\beta}\xi_n^N\int_{\mathbb{R}_+^N}\left(\sum_{i=1}^N\left(\frac{s_i}{\xi_n}\right)\right)^p\prod_{i=1}^N\frac{1}{\left(\left(\frac{s_i}{\xi_n}\right)^{2\alpha}+1\right)^\beta}\prod_{i=1}^N d\left(\frac{s_i}{\xi_n}\right)=$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^N z_i \right)^p \prod_{i=1}^N \frac{1}{(z_i^{2\alpha}+1)^\beta} \prod_{i=1}^N dz_i =$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)} \left[\int_{[0,1]^N} \left(\sum_{i=1}^N z_i \right)^p \prod_{i=1}^N \frac{1}{(z_i^{2\alpha}+1)^\beta} \prod_{i=1}^N dz_i + \right.$$

$$\left. \int_{(\mathbb{R}_+-[0,1])^N} \left(\sum_{i=1}^N z_i \right)^p \prod_{i=1}^N \frac{1}{(z_i^{2\alpha}+1)^\beta} \prod_{i=1}^N dz_i \right] \leq$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)} \left[N^p + \int_{(\mathbb{R}_+-[0,1])^N} \left(\prod_{i=1}^N z_i^p \right) \prod_{i=1}^N \frac{1}{(z_i^{2\alpha}+1)^\beta} \prod_{i=1}^N dz_i \right] = \quad (57)$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)} \left[N^p + \left(\int_1^\infty z^p \frac{1}{(z^{2\alpha}+1)^\beta} dz \right)^N \right] \leq$$

$$(2c)^N \xi_n^{2\alpha\beta(N-1)} \left[N^p + \left(\int_0^\infty \frac{z^p}{(z^{2\alpha}+1)^\beta} dz \right)^N \right]$$

(by [11], p. 397, formula 595, by $\beta > \frac{p+1}{2\alpha}$)

$$= (2c)^N \xi_n^{2\alpha\beta(N-1)} \left[N^p + \left(\frac{\Gamma\left(\frac{p+1}{2\alpha}\right) \Gamma\left(\beta - \left(\frac{p+1}{2\alpha}\right)\right)}{2\alpha\Gamma(\beta)} \right)^N \right] =$$

$$\left(\frac{2\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \right)^N \xi_n^{2\alpha\beta(N-1)} \left[N^p + \left(\frac{\Gamma\left(\frac{p+1}{2\alpha}\right) \Gamma\left(\beta - \left(\frac{p+1}{2\alpha}\right)\right)}{2\alpha\Gamma(\beta)} \right)^N \right] = \quad (58)$$

$$\frac{\xi_n^{2\alpha\beta(N-1)}}{\Gamma^N\left(\frac{1}{2\alpha}\right) \Gamma^N\left(\beta - \frac{1}{2\alpha}\right)} \left[(2\alpha\Gamma(\beta))^N N^p + \Gamma^N\left(\frac{p+1}{2\alpha}\right) \Gamma^N\left(\beta - \left(\frac{p+1}{2\alpha}\right)\right) \right],$$

proving the claim. ■

4 Main Results

Here we apply Theorem 4 to U_n operators, regarding uniform convergence.

Theorem 18 *We consider $f \in C^2(\mathbb{R}^N)$ and let all $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $N \geq 1$, $|\alpha| := \sum_{i=1}^N \alpha_i = 2$; $x \in \mathbb{R}^N$, and all the partials f_α of order 2, along with $f \in C_B(\mathbb{R}^N)$; or all f_α of order 2, $f \in C_U(\mathbb{R}^N)$; $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. Assume that $\beta > \frac{1}{\alpha}$.*

Denote $(n \in \mathbb{N})$

$$\overline{\Delta}_n(x) := U_n(f,x) - f(x) - \left(\sum_{j=0}^{r} \alpha_j j\right) \sin(1)$$

$$\left[\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} W_n^N \left(\int_{\mathbb{R}^N} s_i \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^{\beta}} ds_1 ... ds_N\right)\right]$$

$$-2\left(\sum_{j=0}^{r} \alpha_j j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^{N} W_n^N \left(\int_{\mathbb{R}^N} s_i^2 \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^{\beta}} ds_1 ... ds_N\right) \frac{\partial^2 f(x)}{\partial x_i^2}\right.$$

$$\left. + \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1,...,N\}}} W_n^N \left(\int_{\mathbb{R}^N} s_i s_{j^*} \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^{\beta}} ds_1 ... ds_N\right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}}\right\}. \quad (59)$$

Then

(i)

$$\left|\overline{\Delta}_n(x)\right| \leq \left\|\overline{\Delta}_n\right\|_{\infty} \leq \sum_{j=0}^{r} |\alpha_j|$$

$$\left[\left[j^2 \sum_{\substack{\alpha_i \in \mathbb{Z}^+, \\ \alpha : |\alpha| = 2}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right) \omega_1(f_\alpha, \xi_n) \overline{I}_{1j}(\alpha)\right] + \right.$$

$$\left. \frac{1}{2} \omega_1(f, \xi_n) \left[1 + \frac{j}{3} \overline{I}_2\right]\right] =: \overline{\varphi}_{\xi_n}, \quad (60)$$

where $\overline{I}_{1j}(\alpha)$ *is as in (33), and* $\overline{I}_2$ *is as in (41).*

In case of all f_α *of order* 2 *and* $f \in C_U\left(\mathbb{R}^N\right)$ *and* $\xi_n \to 0$, *as* $n \to \infty$, *then* $\overline{\Delta}_n(x)$, $\left\|\overline{\Delta}_n\right\| \to 0$ *with rates.*

(ii) If $\frac{\partial f}{\partial x_i} = 0$, $i = 1,...,N$, *and* $f_\alpha(x) = 0$, $\alpha_i \in \mathbb{Z}^+$, $i = 1,...,N$, *with* $|\alpha| = 2$, *then*

$$|U_n(f,x) - f(x)| \leq \overline{\varphi}_{\xi_n}. \quad (61)$$

And $U_n(f,x) \to f(x)$ *in the uniformly continuous case.*

(iii) Additionally assume all partials of order ≤ 2 *are bounded. Hence*

$$\|U_n(f) - f\|_{\infty} \leq \left(\sum_{j=0}^{r} |\alpha_j| j\right) (0.8414)$$

$$\left[\sum_{i=1}^{N} \left\|\frac{\partial f}{\partial x_i}\right\|_{\infty} W_n^N \left(\int_{\mathbb{R}^N} |s_i| \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^{\beta}} ds_1 ... ds_N\right)\right] +$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)(0.4596)\left\{\sum_{i=1}^{N}W_n^N\left(\int_{\mathbb{R}^N}s_i^2\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_{\infty}+\right.$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}}W_n^N\left(\int_{\mathbb{R}^N}|s_i|\,|s_{j^*}|\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i\partial x_{j^*}}\right\|_{\infty}\right\}+\overline{\varphi}_{\xi_n}. \tag{62}$$

If all f_α of order 2, $f\in C_U\left(\mathbb{R}^N\right)$, then

$$\|U_n(f)-f\|_{\infty}\to 0 \text{ with rates, as } \xi_n\to 0,\ n\to+\infty.$$

Proof. By Theorems 4, 11, 13 and Remark 12. ■

Next we apply Theorem 6 to U_n operators.

Theorem 19 *Let $f\in C^m\left(\mathbb{R}^N\right)$, $m\in\mathbb{N}$, $N>1$, with $f_{\overline{\alpha}}\in L_p\left(\mathbb{R}^N\right)$, $|\overline{\alpha}|=m$, $x\in\mathbb{R}^N$. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$; $0<\xi_n\leq 1$, $n\in\mathbb{N}$. For $\overline{j}=1,...,m$, and $\overline{\alpha}:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, $|\overline{\alpha}|:=\sum_{i=1}^{N}\alpha_i=\overline{j}$, call*

$$\overline{c}_{\overline{\alpha},n,\overline{j}}:=W_n^N\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\right)\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}ds_1...ds_N. \tag{63}$$

Here we assume $\beta>\frac{(m+r)p+1}{2\alpha}$.

Then

$$\left\|\overline{E}_{r,n}^{[m]}\right\|_p:=\left\|U_{r,n}(f;x)-f(x)-\sum_{\overline{j}=1}^{m}\delta_{\overline{j},r}^{[m]}\left(\sum_{|\overline{\alpha}|=\overline{j}}\frac{\overline{c}_{\overline{\alpha},n,\overline{j}}f_{\overline{\alpha}}(x)}{\left(\prod_{i=1}^{N}\alpha_i!\right)}\right)\right\|_{p,x}\leq$$

$$\left[\binom{m+N-1}{m}^{\frac{p}{q}}\left(\frac{m}{(q(m-1)+1)^{\frac{p}{q}}}\right)\left(\sum_{|\overline{\alpha}|=m}\frac{1}{\prod_{i=1}^{N}\alpha_i!}\overline{\theta}_{\xi_n}(\overline{\alpha})\,\omega_r\left(f_{\overline{\alpha}},\xi_n\right)_p^p\right)\right]^{\frac{1}{p}}, \tag{64}$$

where $\overline{\theta}_{\xi_n}(\overline{\alpha})$ as in (45).

As $n\to\infty$ and $\xi_n\to 0$, by (64), we obtain that $\left\|\overline{E}_{r,n}^{[m]}\right\|_p\to 0$ with rates.

One also obtains by (64) that

$$\|U_n(f;x)-f(x)\|_{p,x}\leq$$

$$\sum_{\overline{j}=1}^{m}\left|\delta_{\overline{j},r}^{[m]}\right|\left(\sum_{|\overline{\alpha}|=\overline{j}}\frac{\left|\overline{c}_{\overline{\alpha},n,\overline{j}}\right|}{\prod_{i=1}^{N}\alpha_i!}\|f_{\overline{\alpha}}\|_p\right)+R.H.S.(64), \tag{65}$$

given that $\|f_{\overline{\alpha}}\|_p < \infty$, $|\overline{\alpha}| = \overline{j}$, $\overline{j} = 1, ..., m$.

Finally we get $\|U_n(f) - f\|_p \to 0$ *as* $\xi_n \to 0$, *and* $n \to \infty$. *That is* $U_n \to I$ *the unit operator, in* L_p *norm, with rates.*

Proof. By Theorems 6, 14 and (46). ■

Next we apply Theorem 8 to U_n operators.

Theorem 20 *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Here we deal with* $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, *with* f, $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = 2$, *where* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, *and* $|\alpha| = \sum_{i=1}^{N} \alpha_i$; $x \in \mathbb{R}^N$; $j = 0, 1, ..., r$. *Assume* $\beta > \frac{3p+1}{2\alpha}$.

Denote ($n \in \mathbb{N}$)

$$\overline{\Delta}_n(x) := U_n(f, x) - f(x) - \left(\sum_{j=0}^{r} \alpha_j j\right) \sin(1)$$

$$\left[\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} W_n^N \left(\int_{\mathbb{R}^N} s_i \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} ds_1 ... ds_N\right)\right]$$

$$-2\left(\sum_{j=0}^{r} \alpha_j j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^{N} W_n^N \left(\int_{\mathbb{R}^N} s_i^2 \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} ds_1 ... ds_N\right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \tag{66}$$

$$\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, ..., N\}}} W_n^N \left(\int_{\mathbb{R}^N} s_i s_{j^*} \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} ds_1 ... ds_N\right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}} \right\}.$$

Then

$$\left\|\overline{\Delta}_n\right\|_p \leq \left(\sum_{j=0}^{r} |\alpha_j|^q\right)^{\frac{1}{q}} \left\{\frac{2}{(q+1)^{\frac{1}{q}}} \left(\frac{N(N+1)+2}{2}\right)^{\frac{1}{q}}\right\}$$

$$\left\{\sum_{j=0}^{r} \left[j^{2p} \sum_{|\alpha|=2} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right)^p \omega_1(f_\alpha, \xi_n)_p^p \overline{A}_{j\xi_n}(\alpha) + \omega_1(f, \xi_n)_p^p \left[1 + j^p \overline{K}_{p\xi_n}\right]\right]\right\}^{\frac{1}{p}}, \tag{67}$$

where $\overline{A}_{j\xi_n}(\alpha)$ *as in (47), and* $\overline{K}_{p\xi_n}$ *as in (55). Note that* $\left\|\overline{\Delta}_n\right\|_p \to 0$ *with rates as* $\xi_n \to 0$, $n \to \infty$, *by (67).*

One also obtains from (67) that

$$\|U_n(f) - f\|_p \leq$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\,j\right)\sin(1)\left[\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_p W_n^N\left(\int_{\mathbb{R}^N}|s_i|\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}ds_1...ds_N\right)\right]$$

$$+2\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{N}W_n^N\left(\int_{\mathbb{R}^N}s_i^2\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_p+\right.$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}}W_n^N\left(\int_{\mathbb{R}^N}|s_i s_{j^*}|\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i\partial x_{j^*}}\right\|_p\right\}+R.H.S.\,(67)\,, \tag{68}$$

given that $\|f_\alpha\|_p<\infty$, $|\alpha|=\widetilde{j}$, $\widetilde{j}=1,2$.

Clearly $\|U_n(f)-f\|_p\to 0$, *as* $n\to\infty$ *and* $\xi_n\to 0$. *That is* $U_n\to I$, *the unit operator, in* L_p *norm with rates.*

Proof. By Theorems 8, 16, 17 and Remark 12. ■

We finish with an application of Theorem 9 to U_n operators.

Theorem 21 *Let* $0<\xi_n\leq 1$, $n\in\mathbb{N}$, $x\in\mathbb{R}^N$. *Here we deal with* $f\in C^2\left(\mathbb{R}^N\right)$, $N\geq 1$, *with* f, $f_\alpha\in L_1\left(\mathbb{R}^N\right)$, $|\alpha|=2$, *where* $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, *and* $|\alpha|=\sum_{i=1}^{N}\alpha_i$; $j=0,1,...,r$. *Assume that* $\beta>\frac{2}{\alpha}$. *Here* $\overline{\Delta}_n$ *as in (66). Then*

$$\left\|\overline{\Delta}_n\right\|_1\leq\left\{\sum_{j=0}^{r}|\alpha_j|\left[j^2\sum_{|\alpha|=2}\left(\frac{2}{\prod_{i=1}^{N}\alpha_i!}\right)\omega_1\left(f_\alpha,\xi_n\right)_1\overline{\overline{A}}_{j\xi_n}(\alpha)\right.\right.$$

$$\left.\left.+\omega_1\left(f,\xi_n\right)_1\left[1+j\overline{\overline{K}}_{\xi_n}\right]\right]\right\}, \tag{69}$$

where $\overline{\overline{A}}_{j\xi_n}(\alpha)$ *as in Theorem 16, and* $\overline{\overline{K}}_{\xi_n}$ *as in Theorem 17.*

Note that $\left\|\overline{\Delta}_n\right\|_1\to 0$ *with rates as* $\xi_n\to 0$, $n\to\infty$, *by (69).*

We derive also that

$$\|U_n(f)-f\|_1\leq$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\,j\right)\sin(1)\left[\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_1 W_n^N\left(\int_{\mathbb{R}^N}|s_i|\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}ds_1...ds_N\right)\right]$$

$$+2\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{N}W_n^N\left(\int_{\mathbb{R}^N}s_i^2\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_1+\right. \tag{70}$$

$$\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,\ldots,N\}}} W_n^N \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}} ds_1 \ldots ds_N\right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_1 \Bigg\} + R.H.S.\,(69)\,,$$

given that $\|f_\alpha\|_1 < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, 2$. *As* $n \to \infty$ *and* $\xi_n \to 0$, *by (70), we derive that* $\|U_n(f) - f\|_1 \to 0$, *that is* $U_n \to I$ *the unit operator, in* L_1 *norm, with rates.*

Proof. By Theorems 9, 16, 17 and Remark 12. ■

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