Trigonometric background multivariate smooth Trigonometric singular integrals approximations

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Abstract

In this article we apply the uniform and L_p, $1 \leq p < \infty$ approximation properties of general smooth multivariate singular integral operators over $\mathbb{R}^N$, $N \geq 1$. It is a trigonometric based approach with detailed applications to the corresponding smooth multivariate Trigonometric singular integral operators. The results are quantitative via Jackson type inequalities involving the first uniform and L_p moduli of continuity.

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1 Introduction

The degree of approximation by univariate and multivariate singular integral operators has been studied extensively in [1]-[3] and [7], [9] and [11]. All these sources motivate our current work. In particular we studied the approximation properties of the smooth singular integral operators in [1]-[3], [9]. These are not in general positive operators. Here we use the uniform and L_p, $p \geq 1$, results of our multivariate general theory [5], [6], to establish approximation properties of the smooth Trigonometric singular integral operators. The degrees of approximation are given quantitatively by using the uniform and L_p first moduli of continuity. The essential tool here comes from [4], where a multivariate trigonometric Taylor formula is proved.

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2 Background of General Theory

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_j := \alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, ..., r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0. \end{cases} \tag{1}$$

and

$$\delta_k := \delta_{k,r}^{[m]} := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, ..., m \in \mathbb{N}. \tag{2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}.$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\theta_n(f; x_1, ..., x_N) := \theta_{r,n}^{[m]}(f; x_1, ..., x_N) :=$$

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) \, d\mu_{\xi_n}(s), \tag{4}$$

where $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers, we take $0 < \xi_n \leq 1$.

Remark 1 *The operators $\theta_{r,n}^{[m]}$ are not in general positive, see [2], p. 2.*

We observe that

Lemma 2 *It holds*

$$\theta_{r,n}^{[m]}(c; x_1, ..., x_n) = c,$$

where c is a constant.

We need

Definition 3 *Let $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$. We define the first uniform modulus of continuity of f as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in\mathbb{R}^N: \\ \|x-y\|_\infty \leq \delta}} |f(x) - f(y)|, \quad \delta > 0, \tag{5}$$

where $\|\cdot\|_\infty$ is the max norm in $\mathbb{R}^N$. The functional $\omega_1(f,\delta)$ is bounded for f being bounded or uniformly continuous, and $\omega_1(f,\delta) \to 0$ as $\delta \to 0$, in the case of f being uniformly continuous.

We mention the main uniform general approximation result regarding the operator θ_n.

Theorem 4 *([5]) Here $f \in C^2\left(\mathbb{R}^N\right)$ and let all $\alpha_i \in \mathbb{Z}^+$, $i = 1,...,N$, $N \geq 1$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = 2$; $x \in \mathbb{R}^N$, and all the partials f_α of order 2, along with $f \in C_B\left(\mathbb{R}^N\right)$ (continuous and bounded functions); or all f_α of order 2, $f \in C_U\left(\mathbb{R}^N\right)$ (uniformly continuous functions). Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$, for $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.*

Suppose that for all $\alpha := (\alpha_1,...,\alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1,...,N$, $|\alpha| = \sum_{i=1}^{N} \alpha_i = 2$, $j = 0,1,...,r$, we have that both

$$I_{1j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right)\left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s), \tag{6}$$

$$I_{2j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) d\mu_{\xi_n}(s), \tag{7}$$

are uniformly bounded in $\xi_n \in (0,1]$.

Denote ($n \in \mathbb{N}$)

$$\Delta_n(x) := \theta_n(f,x) - f(x) - \left(\sum_{j=0}^{r} \alpha_j j\right) \sin(1) \left[\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} \left(\int_{\mathbb{R}^N} s_i d\mu_{\xi_n}(s)\right)\right]$$

$$-2\left(\sum_{j=0}^{r} \alpha_j j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)\right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \tag{8}$$

$$\left. \sum_{\substack{i\neq j^*, \\ i,j^*\in\{1,...,N\}}} \left(\int_{\mathbb{R}^N} s_i s_{j^*} d\mu_{\xi_n}(s)\right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}} \right\}.$$

Then

(i)

$$\left|\Delta_n(x)\right| \leq \left\|\Delta_n(x)\right\|_\infty \leq \sum_{j=0}^{r} |\alpha_j|$$

$$\left[\left[j^2 \sum_{\substack{\alpha_i \in \mathbb{Z}^+, \\ \alpha : |\alpha| = 2}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right) \omega_1(f_\alpha, \xi_n) \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_\infty}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s)\right] + \right.$$

$$\left. \frac{1}{2}\omega_1(f, \xi_n) \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_\infty}{3\xi_n}\right) d\mu_{\xi_n}(s)\right] =: \varphi_{\xi_n}. \tag{9}$$

In case of all f_α of order 2 and $f \in C_U(\mathbb{R}^N)$ and $\xi_n \to 0$, as $n \to \infty$, then $\Delta_n(x)$, $\|\Delta_n(x)\|_\infty \to 0$ with rates.

(ii) If $\frac{\partial f(x)}{\partial x_i} = 0$, $i = 1, ..., N$, and $f_\alpha(x) = 0$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, with $|\alpha| = 2$, then

$$|\theta_n(f, x) - f(x)| \leq \varphi_{\xi_n}. \tag{10}$$

And $\theta_n(f, x) \to f(x)$ in the uniformly continuous case.

(iii) Additionally assume all partials of order ≤ 2 are bounded. Hence

$$\|\theta_n(f) - f\|_\infty \leq \left(\sum_{j=0}^{r} |\alpha_j| j\right)(0.8414)\left[\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty \left(\int_{\mathbb{R}^N} |s_i| \, d\mu_{\xi_n}(s)\right)\right] +$$

$$\left(\sum_{j=0}^{r} |\alpha_j| j^2\right)(0.4596)\left\{\sum_{i=1}^{N}\left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_\infty + \right.$$

$$\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, ..., N\}}} \left(\int_{\mathbb{R}^N} |s_i| \, |s_{j^*}| \, d\mu_{\xi_n}(s)\right)\left\|\frac{\partial^2 f}{\partial x_i \partial x_{j^*}}\right\|_\infty\right\} + \varphi_{\xi_n}. \tag{11}$$

If all $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)$ and $\int_{\mathbb{R}^N} |s_i| \, |s_{j^*}| \, d\mu_{\xi_n}(s)$ converge to zero, as $n \to \infty$, with $\xi_n \to 0$, and all f_α of order 2, $f \in C_U(\mathbb{R}^N)$, then

$$\|\theta_n(f) - f\|_\infty \to 0 \text{ with rates, as } \xi_n \to 0, \; n \to +\infty.$$

Next we deal with $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{Z}^+$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m \in \mathbb{Z}^+$, $p \geq 1$; where f_α denotes the mixed partial $\frac{\partial^{\widetilde{j}} f(\cdot, ..., \cdot)}{\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N : |\alpha| := \sum_{j=1}^{N} \alpha_j = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

We need

Definition 5 *(see also [2], p. 20) We call*

$$\Delta_u^r f(x) := \Delta_{u_1,u_2,...,u_N}^r f(x_1,...,x_N) := \tag{12}$$

$$\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x_1 + ju_1, x_2 + ju_2, ..., x_N + ju_N).$$

Let $p \geq 1$, the L_p modulus of smoothness of order r is given by

$$\omega_r(f;h)_p := \sup_{\|u\|_2 \leq h} \|\Delta_u^r(f)\|_p, \tag{13}$$

$h > 0$.

We mention

Theorem 6 *([6]) Let $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence. Assume for all $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = m$ that we have*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{14}$$

For $\widetilde{j} = 1, ..., m$, and $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \widetilde{j}$, call

$$c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} d\mu_{\xi_n}(s). \tag{15}$$

Then

$$\left\| E_{r,n}^{[m]} \right\|_p := \left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) \right\|_{p,x} \leq \tag{16}$$

$$\left[\binom{m+N-1}{m}^{\frac{p}{q}} \left(\frac{m}{(q(m-1)+1)^{\frac{p}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \cdot \right. \right.$$

$$\left. \left. \left[\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right] \omega_r(f_\alpha, \xi_n)_p^p \right) \right]^{\frac{1}{p}}.$$

As $n \to \infty$ and $\xi_n \to 0$, by (16), we obtain that $\left\| E_{r,n}^{[m]} \right\|_p \to 0$ with rates.

One also gets by (16) that

$$\left\| \theta_{r,n}^{[m]} (f; x) - f (x) \right\|_{p,x} \leq$$

$$\sum_{\widetilde{j}=1}^{m} \left| \delta_{\widetilde{j},r}^{[m]} \right| \left(\sum_{|\alpha|=\widetilde{j}} \frac{\left| c_{\alpha,n,\widetilde{j}} \right|}{\prod_{i=1}^{N} \alpha_i!} \|f_\alpha\|_p \right) + R.H.S.(16), \tag{17}$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, ..., m$.

Assuming that $c_{\alpha,n,\widetilde{j}} \to 0$, $\xi_n \to 0$, *as* $n \to \infty$, *we get* $\left\| \theta_{r,n}^{[m]} (f) - f \right\|_p \to 0$, *that is* $\theta_{r,n}^{[m]} \to I$ *the unit operator, in* L_p *norm, with rates.*

We make

Remark 7 *Notice that* $(p > 1)$

$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n} (s) \leq$$

$$\left[\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n} (s) \right]^{\frac{1}{p}} < \infty, \tag{18}$$

by assumption of Theorem 6.

By (18) we get that

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i} d\mu_{\xi_n} (s) < \infty. \tag{19}$$

Hence $c_{\alpha,n,\widetilde{j}} \in \mathbb{R}$.

We mention also the following trigonometric induced alternative L_p approximation result for θ_n operators.

Theorem 8 *([6]) Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Here we deal with* $f \in C^2 \left(\mathbb{R}^N\right)$, $N \geq 1$, *with* f, $f_\alpha \in L_p \left(\mathbb{R}^N\right)$, $|\alpha| = 2$, *where* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, *and* $|\alpha| = \sum_{i=1}^{N} \alpha_i$; $x \in \mathbb{R}^N$. *Let* μ_{ξ_n} *be a Borel probability measure on* $\mathbb{R}^N$. *Suppose that for all* $\alpha : |\alpha| = 2$, $j = 0, 1, ..., r$, *we have that both*

$$I_{1j} (\alpha) := \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) d\mu_{\xi_n} (s), \tag{20}$$

$$I_2 := \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p d\mu_{\xi_n}(s), \tag{21}$$

are uniformly bounded in $\xi_n \in (0,1]$.

Denote ($n \in \mathbb{N}$)

$$\Delta_n(x) := \theta_n(f,x) - f(x) - \left(\sum_{j=0}^{r} \alpha_j j \right) \sin(1) \left[\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} \left(\int_{\mathbb{R}^N} s_i d\mu_{\xi_n}(s) \right) \right]$$

$$-2 \left(\sum_{j=0}^{r} \alpha_j j^2 \right) \sin^2\left(\frac{1}{2}\right) \left\{ \sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \tag{22}$$

$$\left. \sum_{\substack{i \neq j^*, \\ i,j^* \in \{1,\dots,N\}}} \left(\int_{\mathbb{R}^N} s_i s_{j^*} d\mu_{\xi_n}(s) \right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}} \right\}.$$

Then, it holds

$$\|\Delta_n(x)\|_p \leq \left(\sum_{j=0}^{r} |\alpha_j|^q \right)^{\frac{1}{q}} \left\{ \frac{2}{(q+1)^{\frac{1}{q}}} \left(\frac{N(N+1)+2}{2} \right)^{\frac{1}{q}} \right\}$$

$$\left\{ \sum_{j=0}^{r} \left[j^{2p} \sum_{|\alpha|=2} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right)^p \omega_1(f_\alpha, \xi_n)_p^p \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) d\mu_{\xi_n}(s) \right) \right. \right. \tag{23}$$

$$\left. \left. +\omega_1(f,\xi_n)_p^p \left(\int_{\mathbb{R}^N} \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) d\mu_{\xi_n}(s) \right) \right] \right\}^{\frac{1}{p}}.$$

As $n \to \infty$ *and* $\xi_n \to 0$, *by (23), we obtain that* $\|\Delta_n\|_p \to 0$ *with rates. One also gets by (23) that*

$$\|\theta_n(f,x) - f(x)\|_{p,x} \leq$$

$$\left(\sum_{j=0}^{r} |\alpha_j| j \right) \sin(1) \left[\sum_{i=1}^{N} \left\| \frac{\partial f}{\partial x_i} \right\|_p \left(\int_{\mathbb{R}^N} |s_i| d\mu_{\xi_n}(s) \right) \right]$$

$$+2 \left(\sum_{j=0}^{r} |\alpha_j| j^2 \right) \sin^2\left(\frac{1}{2}\right) \left\{ \sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_p + \right.$$

$$\left. \sum_{\substack{i \neq j^*, \\ i,j^* \in \{1,\dots,N\}}} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_p \right\} + R.H.S.(23), \tag{24}$$

given that $\|f_\alpha\|_p < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, 2$. *Assuming that* $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_m}(s)$, $\int_{\mathbb{R}^N} |s_i s_{j^*}| \, d\mu_{\xi_n}(s)$, $i, j^* \in \{1, ..., N\}$, $i \neq j^*$, *converge to zero as* $\xi_n \to 0$, *we get* $\|\theta_n(f, x) - f(x)\|_p \to 0$, *that is* $\theta_n \to I$ *the unit operator, in* L_p *norm, with rates.*

Furthermore we mention the following trigonometric based alternative L_1 approximation result for θ_n operators.

Theorem 9 *([6]) Let* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$. *Here we deal with* $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, *with* f, $f_\alpha \in L_1(\mathbb{R}^N)$, $|\alpha| = 2$, *where* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, *and* $|\alpha| = \sum_{i=1}^N \alpha_i$. *Let* μ_{ξ_n} *be a Borel probability measure on* $\mathbb{R}^N$. *Suppose that for all* $\alpha : |\alpha| = 2$, $j = 0, 1, ..., r$, *we have that both*

$$I_{1j}^*(\alpha) := \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{j \|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s), \tag{25}$$

and

$$I_2^* := \int_{\mathbb{R}^N} \frac{\|s\|_2}{\xi_n} d\mu_{\xi_n}(s), \tag{26}$$

are uniformly bounded in $\xi_n \in (0, 1]$.

Here $\Delta_n(x)$ *is as in (22).*

Then, it holds

$$\|\Delta_n(x)\|_1 \leq \left\{ \sum_{j=0}^r |\alpha_j| \left[j^2 \sum_{|\alpha|=2} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \omega_1(f_\alpha, \xi_n)_1 \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N (|s_i|)^{\alpha_i} \right) \right. \right. \right.$$

$$\left. \left. \left. \left(1 + \frac{j \|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s) \right) + \omega_1(f, \xi_n)_1 \left(\int_{\mathbb{R}^N} \left(1 + \frac{j \|s\|_2}{\xi_n} \right) d\mu_{\xi_n}(s) \right) \right] \right\}. \tag{27}$$

As $n \to \infty$ *and* $\xi_n \to 0$, *by (27), we obtain that* $\|\Delta_n\|_1 \to 0$ *with rates. One also obtains by (27) that*

$$\|\theta_n(f) - f\|_1 \leq$$

$$\left(\sum_{j=0}^r |\alpha_j| \, j \right) \sin(1) \left[\sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_1 \left(\int_{\mathbb{R}^N} |s_i| \, d\mu_{\xi_n}(s) \right) \right]$$

$$+2 \left(\sum_{j=0}^r |\alpha_j| \, j^2 \right) \sin^2\left(\frac{1}{2} \right) \left\{ \sum_{i=1}^N \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i^2} \right\|_1 + \right.$$

$$\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, ..., N\}}} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| \, d\mu_{\xi_n}(s) \right) \left\| \frac{\partial^2 f}{\partial x_i \partial x_{j^*}} \right\|_1 \right\} + R.H.S.(27), \tag{28}$$

given that $\|f_\alpha\|_1 < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, 2$. *Assuming that* $\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)$, $\int_{\mathbb{R}^N} |s_i s_{j^*}| d\mu_{\xi_n}(s)$, $i, j^* \in \{1, ..., N\}$, $i \neq j^*$, *converge to zero as* $\xi_n \to 0$, *we derive* $\|\theta_n(f) - f\|_1 \to 0$, *that is* $\theta_n \to I$ *in* L_1 *norm, with rates.*

We need

Definition 10 *The general multivariate Trigonometric singular integral operators are:*

$$T_n(f; x_1, ..., x_N) := T_{r,n}^{[m]}(f; x_1, ..., x_N) := \lambda_n^{-N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{29}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, ..., x_N + s_N j) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} ds_1 ... ds_N,$$

with $\beta \in \mathbb{N}$, *and*

$$\lambda_n := 2\xi_n^{1-2\beta} \pi (-1)^\beta \beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta - k)!(\beta + k)!}, \tag{30}$$

see [2, 10], p. 210, item 1033.

Notice that

$$\lambda_n^{-N} \int_{\mathbb{R}^N} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} ds_1 ... ds_N = 1, \tag{31}$$

see also [2, 10], p. 210, item 1033.

So here it is

$$d\mu_{\xi_n}(s) = \lambda_n^{-N} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} ds_1 ... ds_N, \tag{32}$$

$s \in \mathbb{R}^N$, $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.

We will apply to T_n the Theorems 4, 6, 8 and 9, with the help of section 3. That is a study of approximation by T_n.

3 Auxiliary Results

We mention

Lemma 11 *([3], p. 486) Let $N \in \mathbb{N}$, $r > 0$, $z_i \in \mathbb{R}_+$, $i = 1, ..., N$. Then*

$$\left(1 + \sum_{i=1}^{N} z_i\right)^r \leq \prod_{i=1}^{N} (1 + z_i)^r . \tag{33}$$

In Definition 10, we call

$$\gamma := 2\pi (-1)^\beta \beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta - k)! (\beta + k)!}, \tag{34}$$

that is

$$\lambda_n = \gamma \xi_n^{1-2\beta}. \tag{35}$$

We need.

Theorem 12 *Let $N \in \mathbb{N}$; $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\alpha| := \sum_{i=1}^{N} \alpha_i = 2$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$; $j = 0, 1, ..., r \in \mathbb{N}$; $\beta \in \mathbb{N} - \{1, 2\}$. We define*

$$\psi_{10} := \frac{\pi (-1)^\beta (2\beta)!}{2 (2\beta - 1)!} \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta - k)! (\beta + k)!}, \tag{36}$$

$$\psi_{12} := \frac{\pi (-1)^{\beta-1} (2\beta)!}{8 (2\beta - 3)!} \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-3}}{(\beta - k)! (\beta + k)!}, \tag{37}$$

$$\psi_{21} := \frac{(2\beta)!}{2 (2\beta - 2)!} \sum_{k=1}^{\beta} (-1)^{\beta-k} \frac{k^{2(\beta-1)} \ln 2k}{(\beta - k)! (\beta + k)!}, \tag{38}$$

and

$$\psi_{23} := \frac{- (2\beta)!}{8 (2\beta - 4)!} \sum_{k=1}^{\beta} (-1)^{\beta-k} \frac{k^{2(\beta-2)} \ln 2k}{(\beta - k)! (\beta + k)!}. \tag{39}$$

Then

$$J_{1j} (\alpha) := \lambda_n^{-N} \int_{\mathbb{R}^N} \left(1 + \frac{j \|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1 ... ds_N \leq \tag{40}$$

$$\xi_n^2 2^N \gamma^{-N} \left[\psi_{10} + \psi_{12} + \frac{j}{3} (\psi_{21} + \psi_{23})\right]^N \leq$$

$$2^N \gamma^{-N} \left[\psi_{10} + \psi_{12} + \frac{j}{3} (\psi_{21} + \psi_{23})\right]^N < +\infty, \tag{41}$$

is uniformly bounded with respect to ξ_n, and $J_{1j} (\alpha) \to 0$, as $\xi_n \to 0$, $n \to \infty$.

Proof. We have that

$$J_{1j}(\alpha) = \lambda_n^{-N} \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1...ds_N =$$

$$2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i =$$

$$2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(1 + \frac{j\sum_{i=1}^{N} s_i}{3\xi_n}\right) \left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^{N} ds_i = \quad (42)$$

$$2^N \lambda_n^{-N} \xi_n^2 \xi_n^{-2\beta N} \xi_n^N \int_{\mathbb{R}_+^N} \left(1 + \frac{j}{3} \sum_{i=1}^{N} \left(\frac{s_i}{\xi_n}\right)\right)$$

$$\left(\prod_{i=1}^{N} \left(\frac{s_i}{\xi_n}\right)^{\alpha_i}\right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{\frac{s_i}{\xi_n}}\right)^{2\beta} \prod_{i=1}^{N} d\left(\frac{s_i}{\xi_n}\right) =$$

$$2^N \lambda_n^{-N} \xi_n^{(1-2\beta)N+2} \int_{\mathbb{R}_+^N} \left(1 + \frac{j}{3} \sum_{i=1}^{N} z_i\right) \left(\prod_{i=1}^{N} z_i^{\alpha_i}\right) \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^{N} dz_i =$$

(by $\lambda_n = \gamma \xi_n^{1-2\beta}$)

$$2^N \gamma^{-N} \xi_n^2 \int_{\mathbb{R}_+^N} \left(1 + \frac{j}{3} \sum_{i=1}^{N} z_i\right) \left(\prod_{i=1}^{N} z_i^{\alpha_i}\right) \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^{N} dz_i \leq$$

$$2^N \gamma^{-N} \xi_n^2 \int_{\mathbb{R}_+^N} \prod_{i=1}^{N} \left(1 + \frac{j}{3} z_i\right) \left(\prod_{i=1}^{N} z_i^{\alpha_i}\right) \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} \prod_{i=1}^{N} dz_i =$$

$$2^N \gamma^{-N} \xi_n^2 \prod_{i=1}^{N} \left(\int_0^\infty \left(1 + \frac{j}{3} z_i\right) z_i^{\alpha_i} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} dz_i\right) = \quad (43)$$

$$2^N \gamma^{-N} \xi_n^2 \prod_{i=1}^{N} \left[\int_0^\infty z_i^{\alpha_i} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} dz_i + \frac{j}{3} \int_0^\infty z_i^{1+\alpha_i} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} dz_i\right] =$$

$$2^N \gamma^{-N} \xi_n^2 \prod_{i=1}^{N} \left[\int_0^1 z_i^{\alpha_i} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} dz_i + \int_1^\infty z_i^{\alpha_i} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} dz_i + \right.$$

$$\left. \frac{j}{3} \int_0^1 z_i^{1+\alpha_i} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} dz_i + \frac{j}{3} \int_1^\infty z_i^{1+\alpha_i} \left(\frac{\sin z_i}{z_i}\right)^{2\beta} dz_i\right] \leq$$

$$2^N\gamma^{-N}\xi_n^2\left[\int_0^1\left(\frac{\sin z}{z}\right)^{2\beta}dz+\frac{j}{3}\int_0^1 z\left(\frac{\sin z}{z}\right)^{2\beta}dz+\right.$$

$$\left.\int_1^\infty z^2\left(\frac{\sin z}{z}\right)^{2\beta}dz+\frac{j}{3}\int_1^\infty z^3\left(\frac{\sin z}{z}\right)^{2\beta}dz\right]^N\leq \tag{44}$$

$$2^N\gamma^{-N}\xi_n^2\left[\int_0^\infty\left(\frac{\sin z}{z}\right)^{2\beta}dz+\frac{j}{3}\int_0^\infty z\left(\frac{\sin z}{z}\right)^{2\beta}dz+\right.$$

$$\left.\int_0^\infty z^2\left(\frac{\sin z}{z}\right)^{2\beta}dz+\frac{j}{3}\int_0^\infty z^3\left(\frac{\sin z}{z}\right)^{2\beta}dz\right]^N=:I.$$

Based on [10], p. 210, item 1033 and [8], and by [3], p. 389, we have that

$$\int_0^\infty\left(\frac{\sin z}{z}\right)^{2\beta}dz=\frac{\pi\,(-1)^\beta\,(2\beta)!}{2\,(2\beta-1)!}\sum_{k=1}^{\beta}(-1)^k\,\frac{k^{2\beta-1}}{(\beta-k)!\,(\beta+k)!}=\psi_{10}, \tag{45}$$

$$\int_0^\infty z^2\left(\frac{\sin z}{z}\right)^{2\beta}dz=\frac{\pi\,(-1)^{\beta-1}\,(2\beta)!}{8\,(2\beta-3)!}\sum_{k=1}^{\beta}(-1)^k\,\frac{k^{2\beta-3}}{(\beta-k)!\,(\beta+k)!}=\psi_{12}, \tag{46}$$

$$\int_0^\infty z\left(\frac{\sin z}{z}\right)^{2\beta}dz=\frac{(2\beta)!}{2\,(2\beta-2)!}\sum_{k=1}^{\beta}(-1)^{\beta-k}\,\frac{k^{2\beta-2}\ln(2k)}{(\beta-k)!\,(\beta+k)!}=\psi_{21}, \tag{47}$$

and

$$\int_0^\infty z^3\left(\frac{\sin z}{z}\right)^{2\beta}dz=\frac{-\,(2\beta)!}{8\,(2\beta-4)!}\sum_{k=1}^{\beta}\frac{(-1)^{\beta-k}\,k^{2\beta-4}\ln(2k)}{(\beta-k)!\,(\beta+k)!}=\psi_{23}. \tag{48}$$

Hence, it holds

$$I=2^N\gamma^{-N}\xi_n^2\left[\psi_{10}+\frac{j}{3}\psi_{21}+\psi_{12}+\frac{j}{3}\psi_{23}\right]^N\leq \tag{49}$$

$$2^N\gamma^{-N}\left[\psi_{10}+\psi_{12}+\frac{j}{3}\left(\psi_{21}+\psi_{23}\right)\right]^N<+\infty.$$

I.e. $J_{1j}(\alpha)$ is uniformly bounded with respect to ξ_n, and $J_{1j}(\alpha)\to 0$, as $\xi_n\to 0$, $n\to\infty$. The claim is proved. ■

Remark 13 *(to Theorem 12) If $i,j^*\in\{1,...,N\}$, $i\neq j^*$, we have that*

$$\lambda_n^{-N}\int_{\mathbb{R}^N}|s_i|\,|s_{j^*}|\prod_{i=1}^N\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^N ds_i,$$

$$\lambda_n^{-N} \int_{\mathbb{R}^N} s_i^2 \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i \leq J_{1j}(\alpha) < \infty, \tag{50}$$

($j = 0, 1, ..., r \in \mathbb{N}$);

and

$$\lambda_n^{-N} \int_{\mathbb{R}^N} |s_i| \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i \leq \tag{51}$$

$$\left(\lambda_n^{-N} \int_{\mathbb{R}^N} s_i^2 \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i \right)^{1/2} < \infty,$$

and all these integrals are uniformly bounded in $\xi_n \in (0,1]$. And all integrals converge to zero, as $\xi_n \to 0$, $n \to \infty$.

Remark 14 *(to Theorem 12) By (50), (51) we obtain ($j = 0, 1, ..., r \in \mathbb{N}$)*

$$J_2 := \lambda_n^{-N} \int_{\mathbb{R}^N} \frac{\|s\|_1}{\xi_n} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i =$$

$$\frac{1}{\xi_n} \sum_{i=1}^{N} \left(\lambda_n^{-N} \int_{\mathbb{R}^N} |s_i| \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i \right) \leq$$

$$\frac{1}{\xi_n} \sum_{i=1}^{N} (J_{1j}(\alpha))^{\frac{1}{2}} = \frac{N}{\xi_n} (J_{1j}(\alpha))^{\frac{1}{2}} \overset{(40)}{\leq}$$

$$\frac{N}{\xi_n} \xi_n 2^{\frac{N}{2}} \gamma^{-\frac{N}{2}} \left[\psi_{10} + \psi_{12} + \frac{j}{3} (\psi_{21} + \psi_{23}) \right]^{\frac{N}{2}} = \tag{52}$$

$$N 2^{\frac{N}{2}} \gamma^{-\frac{N}{2}} \left[\psi_{10} + \psi_{12} + \frac{j}{3} (\psi_{21} + \psi_{23}) \right]^{\frac{N}{2}} < \infty,$$

uniformly bounded with respect to ξ_n.

We have proved the following result.

Theorem 15 *All as in Theorem 12. Then ($j = 0, 1, ..., r \in \mathbb{N}$)*

$$J_2 := \lambda_n^{-N} \int_{\mathbb{R}^N} \frac{\|s\|_1}{\xi_n} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i \leq \tag{53}$$

$$N 2^{\frac{N}{2}} \gamma^{-\frac{N}{2}} \left[\psi_{10} + \psi_{12} + \frac{j}{3} (\psi_{21} + \psi_{23}) \right]^{\frac{N}{2}} < \infty,$$

uniformly bounded with respect to $\xi_n \in (0,1]$.

We need the following result.

Theorem 16 *([3], p. 491) Let $p > 1$; $r, \beta, N \in \mathbb{N}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, ..., N$: $|\overline{\alpha}| := \sum_{j=1}^{N} \alpha_j = m$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$. Here we take $\beta > \frac{\lceil rp \rceil + \lceil mp \rceil + 1}{2}$, where $\lceil \cdot \rceil$ is the ceiling of the number, and γ, λ_n are as in (34) and (35), respectively, and λ runs as $\lambda = 0, 1, ..., \lceil rp \rceil$. When λ is even we define*

$$\psi_{1\lambda} := \frac{\pi (-1)^{\frac{2\beta - \lambda}{2}} (2\beta)!}{2^{\lambda+1} (2\beta - \lambda - 1)!} \left(\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-\lambda-1}}{(\beta - k)! (\beta + k)!} \right), \tag{54}$$

and when λ is odd we define

$$\psi_{2\lambda} := \frac{(-1)^{\frac{\lambda-1}{2}} (2\beta)!}{2^{\lambda} (2\beta - \lambda - 1)!} \left(\sum_{k=1}^{\beta} (-1)^{\beta-k} \frac{k^{2\beta-\lambda-1} \ln (2k)}{(\beta - k)! (\beta + k)!} \right), \tag{55}$$

and we set

$$\psi_\lambda := \begin{cases} \psi_{1\lambda}, \text{ if } \lambda \text{ is even} \\ \psi_{2\lambda}, \text{ if } \lambda \text{ is odd.} \end{cases} \tag{56}$$

Similarly, it is defined $\psi_{\lambda + \lceil mp \rceil}$ with $\lambda + \lceil mp \rceil$ in the place of λ.

Then

$$C_{\xi_n} (\overline{\alpha}) := \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p$$

$$\prod_{i=1}^{N} \left(\frac{\sin \left(\frac{s_i}{\xi_n} \right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i \leq \tag{57}$$

$$\xi_n^{2\beta(N-1)+mp} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left(\psi_\lambda + \psi_{\lambda + \lceil mp \rceil} \right) \right\}^N \leq \tag{58}$$

$$2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil rp \rceil} \binom{\lceil rp \rceil}{\lambda} \left(\psi_\lambda + \psi_{\lambda + \lceil mp \rceil} \right) \right\}^N < \infty,$$

uniformly bounded with respect to $\xi_n \in (0,1]$, and convergent to zero as $\xi_n \to 0$, when $n \to \infty$.

Remark 17 *(to Theorem 16) Clearly, it holds*

$$\widetilde{B}_{\xi_n} (\overline{\alpha}) := \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \prod_{i=1}^{N} \left(\frac{\sin \left(\frac{s_i}{\xi_n} \right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i < \infty, \tag{59}$$

uniformly bounded in $\xi_n \in (0,1]$. And, clearly $\widetilde{B}_{\xi_n} (\overline{\alpha}) \to 0$, as $\xi_n \to 0$, $n \to \infty$, by (57).

We need the following result.

Theorem 18 *Let* $N \in \mathbb{N}$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\alpha| := \sum_{i=1}^{N} \alpha_i = 2$, $\xi_n \in (0, 1]$, $n \in \mathbb{N}$; $j = 0, 1, ..., r \in \mathbb{N}$; $p \geq 1$; $\beta > \frac{3\lceil p \rceil + 1}{2}$ *and* $\beta \in \mathbb{N}$; $\lambda = 0, 1, ..., \lceil p \rceil$.

When λ *is even define*

$$\psi_{1\lambda} := \frac{\pi (-1)^{\frac{2\beta-\lambda}{2}} (2\beta)!}{2^{\lambda+1} (2\beta - \lambda - 1)!} \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-\lambda-1}}{(\beta - k)! (\beta + k)!}, \tag{60}$$

for λ *odd define*

$$\psi_{2\lambda} := \frac{(-1)^{\frac{\lambda-1}{2}} (2\beta)!}{2^{\lambda} (2\beta - \lambda - 1)!} \sum_{k=1}^{\beta} (-1)^{\beta-k} \frac{k^{2\beta-\lambda-1} \ln (2k)}{(\beta - k)! (\beta + k)!}, \tag{61}$$

and

$$\psi_\lambda := \begin{cases} \psi_{1\lambda}, \text{ when } \lambda \text{ is even} \\ \psi_{2\lambda}, \text{ when } \lambda \text{ is odd.} \end{cases} \tag{62}$$

When $\lambda + 2\lceil p \rceil$ *is even define*

$$\psi_{1,\lambda+2\lceil p \rceil} := \frac{\pi (-1)^{\frac{2\beta-\lambda-2\lceil p \rceil}{2}} (2\beta)!}{2^{\lambda+2\lceil p \rceil+1} (2\beta - \lambda - 2\lceil p \rceil - 1)!} \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-\lambda-2\lceil p \rceil-1}}{(\beta - k)! (\beta + k)!}, \tag{63}$$

for $\lambda + 2\lceil p \rceil$ *odd define*

$$\psi_{2,\lambda+2\lceil p \rceil} := \frac{(-1)^{\frac{\lambda+2\lceil p \rceil-1}{2}} (2\beta)!}{2^{\lambda+2\lceil p \rceil} (2\beta - \lambda - 2\lceil p \rceil - 1)!} \sum_{k=1}^{\beta} (-1)^{\beta-k} \frac{k^{2\beta-\lambda-2\lceil p \rceil-1} \ln (2k)}{(\beta - k)! (\beta + k)!}, \tag{64}$$

and

$$\psi_{\lambda+2\lceil p \rceil} := \begin{cases} \psi_{1,\lambda+2\lceil p \rceil}, \text{ when } \lambda + 2\lceil p \rceil \text{ is even} \\ \psi_{2,\lambda+2\lceil p \rceil}, \text{ when } \lambda + 2\lceil p \rceil \text{ is odd.} \end{cases} \tag{65}$$

Then

$$\widetilde{A}_{j\xi_n} (\alpha) := \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i \leq \tag{66}$$

$$\xi_n^{2p} 2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil p \rceil} \binom{\lceil p \rceil}{\lambda} j^\lambda \left(\psi_\lambda + \psi_{\lambda+2\lceil p \rceil} \right) \right\}^N \leq$$

$$2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil p \rceil} \binom{\lceil p \rceil}{\lambda} j^\lambda \left(\psi_\lambda + \psi_{\lambda+2\lceil p \rceil} \right) \right\}^N < \infty. \tag{67}$$

That is $\widetilde{A}_{j\xi_n}(\alpha)$ *is uniformly bounded with respect to* $\xi_n \in (0,1]$*, and* $\widetilde{A}_{j\xi_n}(\alpha) \to 0,$ *as* $\xi_n \to 0,\ n \to \infty.$

When $p = 1,$ *we denote* $\widetilde{A}_{j\xi_n}(\alpha)$ *as* $\widetilde{\widetilde{A}}_{j\xi_n}(\alpha).$

Proof. We have that $(p \geq 1)$

$$\widetilde{A}_{j\xi_n}(\alpha) = \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i =$$

$$2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} s_i^{p\alpha_i} \right) \left(1 + \frac{j^p \|s\|_2^p}{\xi_n^p} \right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i \leq$$

$$2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} s_i^{p\alpha_i} \right) \left(1 + j^p \left(\sum_{i=1}^{N} \frac{s_i}{\xi_n} \right)^p \right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^{N} ds_i \quad (68)$$

$$= 2^N \lambda_n^{-N} \xi_n^{2p} \xi_n^{-2\beta N} \xi_n^N$$

$$\int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} \left(\frac{s_i}{\xi_n} \right)^{p\alpha_i} \right) \left(1 + j^p \left(\sum_{i=1}^{N} \frac{s_i}{\xi_n} \right)^p \right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{\frac{s_i}{\xi_n}} \right)^{2\beta} \prod_{i=1}^{N} d\left(\frac{s_i}{\xi_n} \right) =$$

$$2^N \lambda_n^{-N} \xi_n^{(1-2\beta)N+2p} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) \left(1 + j^p \left(\sum_{i=1}^{N} z_i \right)^p \right) \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^{N} dz_i =$$

(by $\lambda_n = \gamma \xi_n^{1-2\beta}$)

$$2^N \gamma^{-N} \xi_n^{2p} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) \left(1 + j^p \left(\sum_{i=1}^{N} z_i \right)^p \right) \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^{N} dz_i = \quad (69)$$

$$2^N \gamma^{-N} \xi_n^{2p} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) \left(1 + \left(\sum_{i=1}^{N} j z_i \right)^p \right) \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^{N} dz_i \leq$$

$$2^N \gamma^{-N} \xi_n^{2p} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) \left(1 + \sum_{i=1}^{N} j z_i \right)^p \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^{N} dz_i \overset{\text{(by (33))}}{\leq}$$

$$2^N \gamma^{-N} \xi_n^{2p} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^{N} z_i^{p\alpha_i} \right) \prod_{i=1}^{N} (1 + j z_i)^p \prod_{i=1}^{N} \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^{N} dz_i =$$

$$2^N \gamma^{-N} \xi_n^{2p} \prod_{i=1}^{N} \left(\int_0^\infty z_i^{p\alpha_i} (1 + j z_i)^p \left(\frac{\sin z_i}{z_i} \right)^{2\beta} dz_i \right) \leq \quad (70)$$

$$2^N\gamma^{-N}\xi_n^{2p}\prod_{i=1}^{N}\left(\int_0^\infty z_i^{p\alpha_i}\left(1+jz_i\right)^{\lceil p\rceil}\left(\frac{\sin z_i}{z_i}\right)^{2\beta}dz_i\right)=$$

$$2^N\gamma^{-N}\xi_n^{2p}\prod_{i=1}^{N}\left[\sum_{\lambda=0}^{\lceil p\rceil}\binom{\lceil p\rceil}{\lambda}j^\lambda\left(\int_0^\infty z_i^{\lambda+p\alpha_i}\left(\frac{\sin z_i}{z_i}\right)^{2\beta}dz_i\right)\right]=$$

$$2^N\gamma^{-N}\xi_n^{2p}\prod_{i=1}^{N}\left\{\sum_{\lambda=0}^{\lceil p\rceil}\binom{\lceil p\rceil}{\lambda}j^\lambda\right.$$

$$\left.\left[\int_0^1 z_i^{\lambda+p\alpha_i}\left(\frac{\sin z_i}{z_i}\right)^{2\beta}dz_i+\int_1^\infty z_i^{\lambda+p\alpha_i}\left(\frac{\sin z_i}{z_i}\right)^{2\beta}dz_i\right]\right\}\leq$$

$$2^N\gamma^{-N}\xi_n^{2p}\left\{\sum_{\lambda=0}^{\lceil p\rceil}\binom{\lceil p\rceil}{\lambda}j^\lambda\right.$$

$$\left.\left[\int_0^1 z^\lambda\left(\frac{\sin z}{z}\right)^{2\beta}dz+\int_1^\infty z_i^{\lambda+2\lceil p\rceil}\left(\frac{\sin z}{z}\right)^{2\beta}dz\right]\right\}^N\leq \tag{71}$$

$$2^N\gamma^{-N}\xi_n^{2p}\left\{\sum_{\lambda=0}^{\lceil p\rceil}\binom{\lceil p\rceil}{\lambda}j^\lambda\right.$$

$$\left.\left[\int_0^\infty z^\lambda\left(\frac{\sin z}{z}\right)^{2\beta}dz+\int_0^\infty z_i^{\lambda+2\lceil p\rceil}\left(\frac{\sin z}{z}\right)^{2\beta}dz\right]\right\}^N=:I.$$

Based on [10], p. 210, item 1033 and [8], we have the following calculations for $\lambda=0,1,...,\lceil p\rceil$:

Let λ be even, then

$$\int_0^\infty z^\lambda\left(\frac{\sin z}{z}\right)^{2\beta}dz=\frac{\pi\left(-1\right)^{\frac{2\beta-\lambda}{2}}\left(2\beta\right)!}{2^{\lambda+1}\left(2\beta-\lambda-1\right)!}\sum_{k=1}^{\beta}\left(-1\right)^k\frac{k^{2\beta-\lambda-1}}{\left(\beta-k\right)!\left(\beta+k\right)!}=\psi_{1\lambda}. \tag{72}$$

Let λ be odd, then

$$\int_0^\infty z^\lambda\left(\frac{\sin z}{z}\right)^{2\beta}dz=\frac{\left(-1\right)^{\frac{\lambda-1}{2}}\left(2\beta\right)!}{2^{\lambda}\left(2\beta-\lambda-1\right)!}\sum_{k=1}^{\beta}\left(-1\right)^{\beta-k}\frac{k^{2\beta-\lambda-1}\ln\left(2k\right)}{\left(\beta-k\right)!\left(\beta+k\right)!}=\psi_{2\lambda}. \tag{73}$$

Therefore it holds

$$\int_0^\infty z^\lambda\left(\frac{\sin z}{z}\right)^{2\beta}dz=\psi_\lambda=\begin{cases}\psi_{1\lambda}, \text{ when } \lambda \text{ is even,}\\ \psi_{2\lambda}, \text{ when } \lambda \text{ is odd.}\end{cases} \tag{74}$$

Similarly, for $\lambda + 2\lceil p \rceil$ being even, we get

$$\int_0^\infty z^{\lambda+2\lceil p \rceil} \left(\frac{\sin z}{z}\right)^{2\beta} dz = \frac{\pi (-1)^{\frac{2\beta-\lambda-2\lceil p \rceil}{2}} (2\beta)!}{2^{\lambda+2\lceil p \rceil+1} (2\beta - \lambda - 2\lceil p \rceil - 1)!}$$

$$\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-\lambda-2\lceil p \rceil-1}}{(\beta-k)!\,(\beta+k)!} = \psi_{1,\lambda+2\lceil p \rceil}. \tag{75}$$

When $\lambda + 2\lceil p \rceil$ is odd we obtain

$$\int_0^\infty z^{\lambda+2\lceil p \rceil} \left(\frac{\sin z}{z}\right)^{2\beta} dz = \frac{(-1)^{\frac{\lambda+2\lceil p \rceil-1}{2}} (2\beta)!}{2^{\lambda+2\lceil p \rceil} (2\beta - \lambda - 2\lceil p \rceil - 1)!}$$

$$\sum_{k=1}^{\beta} (-1)^{\beta-k} \frac{k^{2\beta-\lambda-2\lceil p \rceil-1} \ln(2k)}{(\beta-k)!\,(\beta+k)!} = \psi_{2,\lambda+2\lceil p \rceil}. \tag{76}$$

Therefore it holds

$$\int_0^\infty z^{\lambda+2\lceil p \rceil} \left(\frac{\sin z}{z}\right)^{2\beta} dz = \psi_{\lambda+2\lceil p \rceil} =$$

$$\begin{cases} \psi_{1,\lambda+2\lceil p \rceil}, \text{ when } \lambda + 2\lceil p \rceil \text{ is even,} \\ \psi_{2,\lambda+2\lceil p \rceil}, \text{ when } \lambda + 2\lceil p \rceil \text{ is odd.} \end{cases} \tag{77}$$

That is

$$I = 2^N \gamma^{-N} \xi_n^{2p} \left\{ \sum_{\lambda=0}^{\lceil p \rceil} \binom{\lceil p \rceil}{\lambda} j^\lambda \left[\psi_\lambda + \psi_{\lambda+2\lceil p \rceil}\right] \right\}^N \leq \tag{78}$$

$$2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil p \rceil} \binom{\lceil p \rceil}{\lambda} j^\lambda \left[\psi_\lambda + \psi_{\lambda+2\lceil p \rceil}\right] \right\}^N < \infty. \tag{79}$$

I.e. $\widetilde{A}_{j\xi_n}(\alpha)$ is uniformly bounded with respect to $\xi_n \in (0,1]$, and $\widetilde{A}_{j\xi_n}(\alpha) \to 0$, as $\xi_n \to 0$, $n \to \infty$.

The claim is proved. ∎

The last auxiliary result follows.

Theorem 19 *Let $N \in \mathbb{N}$; $\xi_n \in (0,1]$, $n \in \mathbb{N}$; $p \geq 1$; $\lambda = 0, 1, ..., \lceil p \rceil$; $\beta > \frac{\lceil p \rceil + 1}{2}$ and $\beta \in \mathbb{N}$. Here $\psi_{1\lambda}$ is as in (60), $\psi_{2\lambda}$ as in (61) and ψ_λ as in (62). Then*

$$\widetilde{K}_{p\xi_n} := \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n}\right)^p \prod_{i=1}^N \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} \prod_{i=1}^N ds_i \leq$$

$$2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil p \rceil} \binom{\lceil p \rceil}{\lambda} \psi_\lambda \right\}^N < \infty, \tag{80}$$

are uniformly bounded with respect to $\xi_n \in (0,1]$. *When* $p=1$, *we denote* $\widetilde{K}_{p\xi_n}$ *as* $\widetilde{\widetilde{K}}_{1\xi_n}$.

Proof. We estimate ($p \geq 1$)

$$\widetilde{K}_{p\xi_n} = \lambda_n^{-N} \int_{\mathbb{R}^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p \prod_{i=1}^N \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^N ds_i =$$

$$2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(\frac{\|s\|_2}{\xi_n} \right)^p \prod_{i=1}^N \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^N ds_i \leq$$

$$2^N \lambda_n^{-N} \int_{\mathbb{R}_+^N} \left(\frac{\sum_{i=1}^N s_i}{\xi_n} \right)^p \prod_{i=1}^N \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \prod_{i=1}^N ds_i = \tag{81}$$

$$2^N \lambda_n^{-N} \xi_n^{-2\beta N} \xi_n^N \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^N \frac{s_i}{\xi_n} \right)^p \prod_{i=1}^N \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{\frac{s_i}{\xi_n}} \right)^{2\beta} \prod_{i=1}^N d\frac{s_i}{\xi_n} =$$

$$2^N \lambda_n^{-N} \xi_n^{N(1-2\beta)} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^N z_i \right)^p \prod_{i=1}^N \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^N dz_i =$$

(by $\lambda_n = \gamma \xi_n^{1-2\beta}$)

$$2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^N z_i \right)^p \prod_{i=1}^N \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^N dz_i \leq$$

$$2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \left(1 + \sum_{i=1}^N z_i \right)^p \prod_{i=1}^N \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^N dz_i \overset{\text{(by (33))}}{\leq}$$

$$2^N \gamma^{-N} \int_{\mathbb{R}_+^N} \prod_{i=1}^N (1+z_i)^p \prod_{i=1}^N \left(\frac{\sin z_i}{z_i} \right)^{2\beta} \prod_{i=1}^N dz_i =$$

$$2^N \gamma^{-N} \left(\int_0^\infty (1+z)^p \left(\frac{\sin z}{z} \right)^{2\beta} dz \right)^N \leq$$

$$2^N \gamma^{-N} \left(\int_0^\infty (1+z)^{\lceil p \rceil} \left(\frac{\sin z}{z} \right)^{2\beta} dz \right)^N = \tag{82}$$

$$2^N \gamma^{-N} \left\{ \sum_{\lambda=0}^{\lceil p \rceil} \binom{\lceil p \rceil}{\lambda} \int_0^\infty z^\lambda \left(\frac{\sin z}{z} \right)^{2\beta} dz \right\}^N =$$

$$2^{N}\gamma^{-N}\left\{\sum_{\lambda=0}^{\lceil p\rceil}\binom{\lceil p\rceil}{\lambda}\psi_{\lambda}\right\}^{N}<\infty,$$

ψ_{λ} is as in (74), (62).

Thus $\widetilde{K}_{p\xi_n}$ are uniformly bounded with respect to $\xi_n\in(0,1]$. ■

4 Main Results

Here we apply Theorem 4 to T_n operators, regarding uniform approximation.

Theorem 20 *We consider $f\in C^{2}\left(\mathbb{R}^{N}\right)$, $N\in\mathbb{N}$, and let all $\alpha_i\in\mathbb{Z}^{+}$, $i=1,...,N$, $|\alpha|:=\sum_{i=1}^{N}\alpha_i=2$; $x\in\mathbb{R}^{N}$, and all the partials f_{α} of order 2, along with $f\in C_B\left(\mathbb{R}^{N}\right)$; or all f_{α} of order 2, $f\in C_U\left(\mathbb{R}^{N}\right)$; $0<\xi_n\leq 1$, $n\in\mathbb{N}$; $\beta\in\mathbb{N}-\{1,2\}$; $j=0,1,...,r\in\mathbb{N}$.*

Denote ($n\in\mathbb{N}$)

$$\widetilde{\Delta}_n(x):=T_n(f,x)-f(x)-\left(\sum_{j=0}^{r}\alpha_j j\right)\sin(1)$$

$$\left[\sum_{i=1}^{N}\frac{\partial f(x)}{\partial x_i}\lambda_n^{-N}\left(\int_{\mathbb{R}^N}s_i\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}ds_1...ds_N\right)\right]$$

$$-2\left(\sum_{j=0}^{r}\alpha_j j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{N}\lambda_n^{-N}\left(\int_{\mathbb{R}^N}s_i^2\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}ds_1...ds_N\right)\frac{\partial^2 f(x)}{\partial x_i^2}\right.$$

$$\left.+\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}}\lambda_n^{-N}\left(\int_{\mathbb{R}^N}s_i s_{j^*}\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}ds_1...ds_N\right)\frac{\partial^2 f(x)}{\partial x_i\partial x_{j^*}}\right\}. \tag{83}$$

Then

(i)

$$\left|\widetilde{\Delta}_n(x)\right|\leq\left\|\widetilde{\Delta}_n\right\|_{\infty}\leq\sum_{j=0}^{r}|\alpha_j|$$

$$\left[\left[j^2\sum_{\substack{\alpha_i\in\mathbb{Z}^+,\\ \alpha:|\alpha|=2}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\omega_1(f_{\alpha},\xi_n)J_{1j}(\alpha)\right]+\right.$$

$$\frac{1}{2}\omega_1(f,\xi_n)\left[1+\frac{j}{3}J_2\right]\Bigg] =: \widetilde{\varphi}_{\xi_n}, \tag{84}$$

where $J_{1j}(\alpha)$ *is as in (40), and* J_2 *is as in (53).*

In case of all f_α *of order* 2 *and* $f \in C_U(\mathbb{R}^N)$ *and* $\xi_n \to 0$*, as* $n \to \infty$*, then* $\widetilde{\Delta}_n(x)$, $\left\|\widetilde{\Delta}_n\right\|_\infty \to 0$ *with rates.*

(ii) If $\frac{\partial f(x)}{\partial x_i} = 0$, $i = 1, ..., N$*, and* $f_\alpha(x) = 0$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$*, with* $|\alpha| = 2$*, then*

$$|T_n(f,x) - f(x)| \leq \widetilde{\varphi}_{\xi_n}. \tag{85}$$

And $T_n(f,x) \to f(x)$ *in the uniformly continuous case, as* $\xi_n \to 0$, $n \to \infty$.

(iii) Additionally assume all partials of order ≤ 2 *are bounded. Hence*

$$\|T_n(f) - f\|_\infty \leq \left(\sum_{j=0}^{r} |\alpha_j|\, j\right)(0.8414)$$

$$\left[\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty \lambda_n^{-N}\left(\int_{\mathbb{R}^N}|s_i|\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i\right)\right]+$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\, j^2\right)(0.4596)\left\{\sum_{i=1}^{N}\lambda_n^{-N}\left(\int_{\mathbb{R}^N}s_i^2\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_\infty+\right.$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}}\lambda_n^{-N}\left(\int_{\mathbb{R}^N}|s_i|\,|s_{j^*}|\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta}\prod_{i=1}^{N}ds_i\right)\left\|\frac{\partial^2 f}{\partial x_i\partial x_{j^*}}\right\|_\infty\right\}+\widetilde{\varphi}_{\xi_n}. \tag{86}$$

If all f_α *of order* 2, $f \in C_U(\mathbb{R}^N)$*, then*

$$\|T_n(f) - f\|_\infty \to 0 \text{ with rates, as } \xi_n \to 0,\ n \to +\infty.$$

Proof. By Theorems 4, 12, 15 and Remark 13. ■

Next we apply Theorem 6 to T_n operators.

Theorem 21 *Let* $f \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N > 1$*, with* $f_{\overline{\alpha}} \in L_p(\mathbb{R}^N)$, $|\overline{\alpha}| = m$, $x \in \mathbb{R}^N$*. Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $0 < \xi_n \leq 1$, $n \in \mathbb{N}$*. For* $\overline{j} = 1, ..., m$*, and* $\overline{\alpha} := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\overline{\alpha}| := \sum_{i=1}^{N}\alpha_i = \overline{j}$*, call*

$$\widetilde{c}_{\overline{\alpha},n,\overline{j}} := \lambda_n^{-N}\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}s_i^{\alpha_i}\right)\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1...ds_N. \tag{87}$$

Here we assume $\beta > \frac{\lceil rp \rceil + \lceil mp \rceil + 1}{2}$, $\beta \in \mathbb{N}$.

Then

$$\left\| \widetilde{E}_{r,n}^{[m]} \right\|_p := \left\| T_{r,n}(f;x) - f(x) - \sum_{\overline{j}=1}^{m} \delta_{\overline{j},r}^{[m]} \left(\sum_{|\overline{\alpha}|=\overline{j}} \frac{\widetilde{c}_{\overline{\alpha},n,\overline{j}} f_{\overline{\alpha}}(x)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) \right\|_{p,x} \leq$$

$$\left[\binom{m+N-1}{m}^{\frac{p}{q}} \left(\frac{m}{(q(m-1)+1)^{\frac{p}{q}}} \right) \left(\sum_{|\overline{\alpha}|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} C_{\xi_n}(\overline{\alpha}) \omega_r (f_{\overline{\alpha}}, \xi_n)_p^p \right) \right]^{\frac{1}{p}}, \tag{88}$$

where $C_{\xi_n}(\overline{\alpha})$ *as in (57).*

As $n \to \infty$ *and* $\xi_n \to 0$, *by (88), we obtain that* $\left\| \widetilde{E}_{r,n}^{[m]} \right\|_p \to 0$ *with rates.*

One also obtains by (88) that

$$\left\| T_n(f;x) - f(x) \right\|_{p,x} \leq$$

$$\sum_{\overline{j}=1}^{m} \left| \delta_{\overline{j},r}^{[m]} \right| \left(\sum_{|\overline{\alpha}|=\overline{j}} \frac{\left| \widetilde{c}_{\overline{\alpha},n,\overline{j}} \right|}{\prod_{i=1}^{N} \alpha_i!} \left\| f_{\overline{\alpha}} \right\|_p \right) + R.H.S.(88), \tag{89}$$

given that $\|f_{\overline{\alpha}}\|_p < \infty$, $|\overline{\alpha}| = \overline{j}$, $\overline{j} = 1, ..., m$.

Finally we get $\|T_n(f) - f\|_p \to 0$ *as* $\xi_n \to 0$, *and* $n \to \infty$. *That is* $T_n \to I$ *the unit operator, in* L_p *norm, with rates.*

Proof. By Theorems 6, 16 and (59). ■

Next we apply Theorem 8 to T_n operators.

Theorem 22 *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $0 < \xi_n \leq 1$, $n \in \mathbb{N}$. *Here we deal with* $f \in C^2(\mathbb{R}^N)$, $N \geq 1$, *with* f, $f_\alpha \in L_p(\mathbb{R}^N)$, $|\alpha| = 2$, *where* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, *and* $|\alpha| = \sum_{i=1}^{N} \alpha_i$; $x \in \mathbb{R}^N$; $j = 0, 1, ..., r$. *Assume* $\beta > \frac{3\lceil p \rceil + 1}{2}$, $\beta \in \mathbb{N}$.

Denote $(n \in \mathbb{N})$

$$\widetilde{\Delta}_n(x) := T_n(f,x) - f(x) - \left(\sum_{j=0}^{r} \alpha_j j \right) \sin(1)$$

$$\left[\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} \lambda_n^{-N} \left(\int_{\mathbb{R}^N} s_i \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} ds_1 ... ds_N \right) \right]$$

$$-2\left(\sum_{j=0}^{r}\alpha_j j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{N}\lambda_n^{-N}\left(\int_{\mathbb{R}^N} s_i^2\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1...ds_N\right)\frac{\partial^2 f(x)}{\partial x_i^2}+\right.$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}}\lambda_n^{-N}\left(\int_{\mathbb{R}^N} s_i s_{j^*}\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1...ds_N\right)\frac{\partial^2 f(x)}{\partial x_i\partial x_{j^*}}\right\}. \tag{90}$$

Then

$$\left\|\widetilde{\Delta}_n\right\|_p \leq \left(\sum_{j=0}^{r}|\alpha_j|^q\right)^{\frac{1}{q}}\left\{\frac{2}{(q+1)^{\frac{1}{q}}}\left(\frac{N(N+1)+2}{2}\right)^{\frac{1}{q}}\right\}$$

$$\left\{\sum_{j=0}^{r}\left[j^{2p}\sum_{|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)^p\omega_1(f_\alpha,\xi_n)_p^p\widetilde{A}_{j\xi_n}(\alpha)+\omega_1(f,\xi_n)_p^p\left[1+j^p\widetilde{K}_{p\xi_n}\right]\right]\right\}^{\frac{1}{p}}, \tag{91}$$

where $\widetilde{A}_{j\xi_n}(\alpha)$ *as in (66), and* $\widetilde{K}_{p\xi_n}$ *as in (80). Note that* $\left\|\widetilde{\Delta}_n\right\|_p \to 0$ *with rates as* $\xi_n \to 0$, $n\to\infty$, *by (91).*

One also obtains from (91) that

$$\|T_n(f)-f\|_p \leq$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\,j\right)\sin(1)\left[\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_p\lambda_n^{-N}\left(\int_{\mathbb{R}^N}|s_i|\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1...ds_N\right)\right]$$

$$+2\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)\sin^2\left(\frac{1}{2}\right)\left\{\sum_{i=1}^{N}\lambda_n^{-N}\left(\int_{\mathbb{R}^N} s_i^2\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_p+\right.$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,...,N\}}}\lambda_n^{-N}\left(\int_{\mathbb{R}^N}|s_i s_{j^*}|\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1...ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i\partial x_{j^*}}\right\|_p\right\}+R.H.S.(91), \tag{92}$$

given that $\|f_\alpha\|_p<\infty$, $|\alpha|=\widetilde{j}$, $\widetilde{j}=1,2$.

Clearly $\|T_n(f)-f\|_p\to 0$, *as* $n\to\infty$ *and* $\xi_n\to 0$. *That is* $T_n\to I$, *the unit operator, in* L_p *norm with rates.*

Proof. By Theorems 8, 18, 19 and Remark 13. ■

We finish with an application of Theorem 9 to T_n operators.

Theorem 23 *Let $0 < \xi_n \leq 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$. Here we deal with $f \in C^2\left(\mathbb{R}^N\right)$, $N \geq 1$, with f, $f_\alpha \in L_1\left(\mathbb{R}^N\right)$, $|\alpha| = 2$, where $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, and $|\alpha| = \sum_{i=1}^{N} \alpha_i$; $j = 0, 1, ..., r$. Here $\widetilde{\Delta}_n$ is as in (90), $\beta \in \mathbb{N} - \{1, 2\}$. Then*

$$\left\|\widetilde{\Delta}_n\right\|_1 \leq \left\{\sum_{j=0}^{r} |\alpha_j| \left[j^2 \sum_{|\alpha|=2} \left(\frac{2}{\prod_{i=1}^{N} \alpha_i!}\right) \omega_1\left(f_\alpha, \xi_n\right)_1 \widetilde{\widetilde{A}}_{j\xi_n}(\alpha)\right.\right.$$

$$\left.\left. + \omega_1\left(f, \xi_n\right)_1 \left[1 + j\widetilde{\widetilde{K}}_{1\xi_n}\right]\right]\right\}, \tag{93}$$

where $\widetilde{\widetilde{A}}_{j\xi_n}(\alpha)$ as in Theorem 18, and $\widetilde{\widetilde{K}}_{\xi_n}$ as in Theorem 19.

Note that $\left\|\widetilde{\Delta}_n\right\|_1 \to 0$ with rates as $\xi_n \to 0$, $n \to \infty$, by (93).

We derive also that

$$\left\|T_n(f) - f\right\|_1 \leq$$

$$\left(\sum_{j=0}^{r} |\alpha_j| \, j\right) \sin(1) \left[\sum_{i=1}^{N} \left\|\frac{\partial f}{\partial x_i}\right\|_1 \lambda_n^{-N} \left(\int_{\mathbb{R}^N} |s_i| \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1 ... ds_N\right)\right]$$

$$+2\left(\sum_{j=0}^{r} |\alpha_j| \, j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^{N} \lambda_n^{-N} \left(\int_{\mathbb{R}^N} s_i^2 \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1 ... ds_N\right) \left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_1 + \right. \tag{94}$$

$$\left. \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1, ..., N\}}} \lambda_n^{-N} \left(\int_{\mathbb{R}^N} |s_i s_{j^*}| \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1 ... ds_N\right) \left\|\frac{\partial^2 f}{\partial x_i \partial x_{j^*}}\right\|_1 \right\} + R.H.S.(93),$$

given that $\|f_\alpha\|_1 < \infty$, $|\alpha| = \widetilde{j}$, $\widetilde{j} = 1, 2$. As $n \to \infty$ and $\xi_n \to 0$, by (94), we derive that $\|T_n(f) - f\|_1 \to 0$, that is $T_n \to I$ the unit operator, in L_1 norm, with rates.

Proof. By Theorems 9, 18, 19 and Remark 13. ■

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