

Trigonometric derived rate of convergence of various smooth singular integral operators

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Abstract

In this article we continue the study of approximation of various smooth singular integral operators. This time the foundation of our research is a trigonometric Taylor's formula. We establish the convergence of our operators to the unit operator with rates via Jackson type inequalities engaging the first modulus of continuity. Of interest here is a residual appearing term. Note that our operators are not positive. Our results are pointwise and uniform. The studied operators here are of the following types: Gauss-Weierstrass, Poisson-Cauchy and Trigonometric.

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1 Introduction

We are motivated by [2], [3] chapters 10-14, and [1], [4]. We use a trigonometric new Taylor formula from [1], see also [4]. Here we consider some very general operators, the smooth Gauss-Weierstrass, Poisson-Cauchy and Trigonometric singular integral operators over the real line and we study further their convergence properties quantitatively. We establish related inequalities involving the first modulus of continuity with respect to uniform norm and the estimates are pointwise and uniform. We demonstrate detailed proofs.

2 Results

By [1], [4], for $f \in C^2(\mathbb{R})$ and $a, x \in \mathbb{R}$, we have by trigonometric Taylor formula

$$f(x) - f(a) = f'(a) \sin(x-a) + 2f''(a) \sin^2\left(\frac{x-a}{2}\right) + \quad (1)$$

$$\int_a^x [(f''(t) + f(t)) - (f''(a) + f(a))] \sin(x-t) dt.$$

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 1, \dots, r, \\ 1 - \sum_{j=1}^r (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 0, \end{cases} \quad (2)$$

that is

$$\sum_{j=0}^r \alpha_j = 1. \quad (3)$$

$C_U(\mathbb{R})$ denotes the space of uniformly continuous functions on \mathbb{R} , and $C_B(\mathbb{R})$ denotes the space of bounded continuous functions on \mathbb{R} .

Here we consider both $f, f'' \in C_U(\mathbb{R}) \cup C_B(\mathbb{R})$.

Denote by

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in \mathbb{R} \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0, \quad (4)$$

the first modulus of continuity of f .

1) We define the smooth Gauss-Weierstrass singular integral operators ([5]).

Let $f \in C^2(\mathbb{R})$, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$W_{r,\xi}(f; x) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left(\sum_{j=0}^r \alpha_j f(x+jt) \right) e^{-\frac{t^2}{\xi}} dt. \quad (5)$$

We assume that $W_{r,\xi}(f; x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. We will use also that

$$W_{r,\xi}(f; x) = \frac{1}{\sqrt{\pi\xi}} \sum_{j=0}^r \alpha_j \left(\int_{-\infty}^{\infty} f(x+jt) e^{-\frac{t^2}{\xi}} dt \right), \quad (6)$$

notice by $\frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{\xi}} dt = 1$ that $W_{r,\xi}(c; x) = c$, c constant and

$$W_{r,\xi}(f; x) - f(x) = \frac{1}{\sqrt{\pi\xi}} \left(\sum_{j=0}^r \alpha_j \int_{-\infty}^{\infty} (f(x+jt) - f(x)) e^{-\frac{t^2}{\xi}} dt \right). \quad (7)$$

We set

$$\Delta_2(x) := W_{r,\xi}(f;x) - f(x) - f''(x) \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{j^2}{4}\xi}\right), \quad x \in \mathbb{R}; \quad (8)$$

$j = 0, 1, \dots, r \in \mathbb{N}$.

II) We define the smooth Poisson-Cauchy singular integral operators ([5]).

Let $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$ and $f \in C^2(\mathbb{R})$. We define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$M_{r,\xi}(f;x) = W \int_{-\infty}^{\infty} \frac{\sum_{j=0}^r \alpha_j f(x+jt)}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt, \quad (9)$$

where the constant is defined as

$$W = \frac{\Gamma(\beta) \alpha \beta^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)}. \quad (10)$$

We assume that $M_{r,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. We will use also that

$$M_{r,\xi}(f;x) = W \sum_{j=0}^r \alpha_j \left(\int_{-\infty}^{\infty} f(x+jt) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right). \quad (11)$$

We notice by $W \int_{-\infty}^{\infty} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt = 1$ that $M_{r,\xi}(c;x) = c$, c constant and

$$M_{r,\xi}(f;x) - f(x) = W \left(\sum_{j=0}^r \alpha_j \int_{-\infty}^{\infty} [f(x+jt) - f(x)] \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right). \quad (12)$$

We set

$$\Delta_3(x) := M_{r,\xi}(f;x) - f(x) - 4f''(x) \frac{\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \sum_{j=0}^r \alpha_j \int_0^{\infty} \frac{\sin^2\left(\frac{j\xi}{2}t\right)}{(1+t^{2\alpha})^\beta} dt, \quad (13)$$

$\xi > 0$, $x \in \mathbb{R}$; $\beta > \frac{1}{2\alpha}$, $\alpha \in \mathbb{N}$; $j = 0, 1, \dots, r \in \mathbb{N}$.

III) We define the smooth Trigonometric singular integral operators ([5]) as follows:

Let $\xi > 0$, $f \in C^2(\mathbb{R})$, $x \in \mathbb{R}$, $\beta \in \mathbb{N}$; we set

$$T_{r,\xi}(f;x) := \frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\sum_{j=0}^r \alpha_j f(x+jt) \right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt, \quad (14)$$

where

$$\lambda := \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt = 2\xi^{1-2\beta} \int_0^{\infty} \left(\frac{\sin t}{t} \right)^{2\beta} dt \quad (15)$$

(by [6], p. 210, item 1033)

$$= 2\xi^{1-2\beta} \pi (-1)^\beta \beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)! (\beta+k)!}.$$

Denote

$$\lambda_1 := 2\pi (-1)^\beta \beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)! (\beta+k)!}; \quad (16)$$

that is

$$\lambda = \lambda_1 \xi^{1-2\beta}. \quad (17)$$

We suppose that $T_{r,\xi}(f; x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. Clearly, again it is

$$\frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt = 1, \quad (18)$$

and $T_{r,\xi}(c; x) = c$, c constant.

We set

$$\Delta_4(x) := T_{r,\xi}(f; x) - f(x) - 4f''(x) \lambda_1^{-1} \sum_{j=0}^r \alpha_j \int_0^{\infty} \sin^2\left(\frac{j\xi}{2}t\right) \left(\frac{\sin t}{t}\right)^{2\beta} dt, \quad (19)$$

where $\xi > 0$, $x \in \mathbb{R}$; $\beta \in \mathbb{N} - \{1, 2\}$; $j = 0, 1, \dots, r \in \mathbb{N}$.

Next we present uniform approximation results regarding the smooth singular operators $W_{r,\xi}$, $M_{r,\xi}$ and $T_{r,\xi}$.

Theorem 1 *It holds*

$$|\Delta_2(x)| \leq \|\Delta_2\|_{\infty} \leq \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{\xi}{4} + \frac{j\sqrt{\xi}}{6\sqrt{\pi}} \right] =: A_2; \quad \xi > 0, \quad x \in \mathbb{R}. \quad (20)$$

And $\|\Delta_2\|_{\infty} \rightarrow 0$, as $\xi \rightarrow 0$. If $f''(x) = 0$, then $|W_{r,\xi}(f, x) - f(x)| \leq A_2$ and $W_{r,\xi}(f, x) \rightarrow f(x)$, as $\xi \rightarrow 0$.

Proof. By (1) we get that

$$f(x+jt) - f(x) = f'(x) \sin(jt) + 2f''(x) \sin^2\left(\frac{jt}{2}\right) + \int_x^{x+jt} [(f''(s) + f(s)) - (f''(x) + f(x))] \sin(x+jt-s) ds, \quad (21)$$

or better

$$f(x+jt) - f(x) = f'(x) \sin(jt) + 2f''(x) \sin^2\left(\frac{jt}{2}\right) +$$

$$\int_0^{jt} [(f''(x+z) + f(x+z)) - (f''(x) + f(x))] \sin(jt-z) dz. \quad (22)$$

Furthermore, it holds

$$\begin{aligned} & \sum_{j=0}^r \alpha_j [f(x+jt) - f(x)] = \\ & f'(x) \sum_{j=0}^r \alpha_j \sin(jt) + 2f''(x) \sum_{j=0}^r \alpha_j \sin^2\left(\frac{jt}{2}\right) + \\ & \sum_{j=0}^r \alpha_j \int_0^{jt} [(f''(x+z) + f(x+z)) - (f''(x) + f(x))] \sin(jt-z) dz, \end{aligned} \quad (23)$$

or better

$$\begin{aligned} & \sum_{j=0}^r \alpha_j [f(x+jt) - f(x)] = \\ & f'(x) \sum_{j=0}^r \alpha_j \sin(jt) + 2f''(x) \sum_{j=0}^r \alpha_j \sin^2\left(\frac{jt}{2}\right) + \\ & \sum_{j=0}^r \alpha_j j \int_0^t [(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))] \sin j(t-w) dw. \end{aligned} \quad (24)$$

Call

$$R := R(t) := \sum_{j=0}^r \alpha_j j \int_0^t [(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))] \sin j(t-w) dw, \quad (25)$$

$\forall t \in \mathbb{R}$.

Then, for $t \geq 0$,

$$|R| \leq \sum_{j=0}^r |\alpha_j| j \int_0^t |(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))| |\sin j(t-w)| dw \leq$$

$$\sum_{j=0}^r |\alpha_j| j \int_0^t \omega_1(f'' + f, jw) j(t-w) dw =$$

($\xi > 0$)

$$\sum_{j=0}^r |\alpha_j| j^2 \int_0^t \omega_1\left(f'' + f, \frac{\xi jw}{\xi}\right) (t-w) dw \leq \quad (26)$$

$$\sum_{j=0}^r |\alpha_j| j^2 \omega_1(f'' + f, \xi) \int_0^t \left(1 + \frac{jw}{\xi}\right) (t-w) dw =$$

$$\begin{aligned}
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\int_0^t (t-w) dw + \frac{j}{\xi} \int_0^t w^{2-1} (t-w)^{2-1} dw \right] = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{(t-w)^2}{2} \Big|_t^0 + \frac{j}{\xi} \frac{1}{6} t^3 \right] = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{t^2}{2} + j \frac{t^3}{6\xi} \right].
\end{aligned}$$

Hence ($t \geq 0$)

$$|R| \leq \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{t^2}{2} + j \frac{t^3}{6\xi} \right]. \quad (27)$$

Let now $t < 0$, then

$$|R| \leq \sum_{j=0}^r |\alpha_j| j \left| \int_0^t [(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))] \sin j(t-w) dw \right| \leq$$

$$\sum_{j=0}^r |\alpha_j| j \int_t^0 |(f''(x+jw) + f(x+jw)) - (f''(x) + f(x))| |\sin j(t-w)| dw \leq$$

$$\begin{aligned}
& \sum_{j=0}^r |\alpha_j| j \int_t^0 \omega_1(f'' + f, -jw) j(w-t) dw = \\
& \sum_{j=0}^r |\alpha_j| j \int_t^0 \omega_1\left(f'' + f, -jw \frac{\xi}{\xi}\right) j(w-t) dw = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \int_t^0 \left(1 - \frac{j}{\xi} w\right) (w-t) dw = \quad (28)
\end{aligned}$$

$$\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\int_t^0 (w-t) dw + \frac{j}{\xi} \int_t^0 (0-w)^{2-1} (w-t)^{2-1} dw \right] =$$

$$\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{(w-t)^2}{2} \Big|_t^0 + \frac{j}{\xi} \frac{1}{6} (-t)^3 \right] =$$

$$\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{t^2}{2} + j \frac{(-t)^3}{6\xi} \right].$$

We found that ($t < 0$)

$$|R| \leq \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{t^2}{2} + j \frac{(-t)^3}{6\xi} \right]. \quad (29)$$

Consequently, for $t \in \mathbb{R}$, we obtain

$$|R(t)| \leq \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{t^2}{2} + \frac{j}{6\xi} |t|^3 \right], \quad \xi > 0. \quad (30)$$

So, we have

$$\sum_{j=0}^r \alpha_j [f(x+jt) - f(x)] - f'(x) \sum_{j=0}^r \alpha_j \sin(jt) - 2f''(x) \sum_{j=0}^r \alpha_j \sin^2\left(\frac{jt}{2}\right) = R(t). \quad (31)$$

We call

$$\begin{aligned} \bar{\Delta}_2(x) &:= W_{r,\xi}(f, x) - f(x) - f'(x) \sum_{j=0}^r \alpha_j \frac{1}{\sqrt{\pi\xi}} \left(\int_{-\infty}^{\infty} \sin(jt) e^{-\frac{t^2}{\xi}} dt \right) \\ &\quad - 2f''(x) \sum_{j=0}^r \alpha_j \frac{1}{\sqrt{\pi\xi}} \left(\int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) e^{-\frac{t^2}{\xi}} dt \right) = \\ &\quad \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} R(t) e^{-\frac{t^2}{\xi}} dt. \end{aligned} \quad (32)$$

Hence we get that

$$\begin{aligned} |\bar{\Delta}_2(x)| &\leq \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} R(t) e^{-\frac{t^2}{\xi}} dt \leq \\ &\frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left[\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{t^2}{2} + \frac{j}{6\xi} |t|^3 \right] \right] e^{-\frac{t^2}{\xi}} dt = \\ &\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left[\frac{t^2}{2} + \frac{j}{6\xi} |t|^3 \right] e^{-\frac{t^2}{\xi}} dt = \\ &\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{1}{2\sqrt{\pi\xi}} \int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{\xi}} dt + \frac{j}{6\xi\sqrt{\pi\xi}} \int_{-\infty}^{\infty} |t|^3 e^{-\frac{t^2}{\xi}} dt \right] = \\ &\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{1}{\sqrt{\pi\xi}} \int_0^{\infty} t^2 e^{-\frac{t^2}{\xi}} dt + \frac{j}{3\xi\sqrt{\pi\xi}} \int_0^{\infty} t^3 e^{-\frac{t^2}{\xi}} dt \right] = \\ &\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{(\sqrt{\xi})^3}{\sqrt{\pi\xi}} \int_0^{\infty} \left(\frac{t}{\sqrt{\xi}} \right)^2 e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\left(\frac{t}{\sqrt{\xi}} \right) \right. \\ &\quad \left. + \frac{j(\sqrt{\xi})^4}{3\xi\sqrt{\pi\xi}} \int_0^{\infty} \left(\frac{t}{\sqrt{\xi}} \right)^3 e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\left(\frac{t}{\sqrt{\xi}} \right) \right] = \end{aligned} \quad (34)$$

$$\begin{aligned} &\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{(\sqrt{\xi})^3}{\sqrt{\pi\xi}} \int_0^{\infty} \left(\frac{t}{\sqrt{\xi}} \right)^2 e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\left(\frac{t}{\sqrt{\xi}} \right) \right. \\ &\quad \left. + \frac{j(\sqrt{\xi})^4}{3\xi\sqrt{\pi\xi}} \int_0^{\infty} \left(\frac{t}{\sqrt{\xi}} \right)^3 e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\left(\frac{t}{\sqrt{\xi}} \right) \right] = \end{aligned} \quad (35)$$

$$\begin{aligned}
\omega_1(f'' + f, \xi) & \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{\xi}{\sqrt{\pi}} \int_0^\infty z^2 e^{-z^2} dz + \frac{j(\sqrt{\xi})^3 \xi^{-1}}{3\sqrt{\pi}} \int_0^\infty z^3 e^{-z^2} dz \right] = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{\xi}{\sqrt{\pi}} \frac{\sqrt{\pi}}{4} + \frac{j\sqrt{\xi}}{3\sqrt{\pi}} \frac{1}{2} \right] = \\
& = \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{\xi}{4} + \frac{j\sqrt{\xi}}{6\sqrt{\pi}} \right]. \tag{36}
\end{aligned}$$

We have proved that

$$|\bar{\Delta}_2(x)| \leq \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{\xi}{4} + \frac{j\sqrt{\xi}}{6\sqrt{\pi}} \right]; \quad \xi > 0, x \in \mathbb{R}. \tag{37}$$

Notice that $\bar{\Delta}_2(x) \rightarrow 0$, as $\xi \rightarrow 0$.

Next we simplify the left hand side of (32).

We observe that:

Clearly it is

$$\int_{-\infty}^{\infty} \sin(jt) e^{-\frac{t^2}{\xi}} dt = 0, \quad j = 0, 1, \dots, r. \tag{38}$$

Furthermore, we have that

$$\begin{aligned}
& \int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) e^{-\frac{t^2}{\xi}} dt = 2 \int_0^{\infty} \sin^2\left(\frac{jt}{2}\right) e^{-\frac{t^2}{\xi}} dt = \\
& 2\sqrt{\xi} \int_0^{\infty} \sin^2\left(\left(\frac{j\sqrt{\xi}}{2}\right) \frac{t}{\sqrt{\xi}}\right) e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2} d\frac{t}{\sqrt{\xi}} \\
& \left(\frac{t}{\sqrt{\xi}} =: x \text{ and } \frac{j\sqrt{\xi}}{2} =: \beta_1\right) \\
& = 2\sqrt{\xi} \int_0^{\infty} \sin^2(\beta_1 x) e^{-x^2} dx \tag{39} \\
& = \sqrt{\xi} \frac{1}{2} \sqrt{\pi} e^{-\beta_1^2} \left(e^{\beta_1^2} - 1\right) \\
& = \frac{\sqrt{\xi} \sqrt{\pi}}{2} e^{-\frac{j^2}{4}\xi} \left(e^{\frac{j^2}{4}\xi} - 1\right).
\end{aligned}$$

Consequently we derive

$$\bar{\Delta}_2(x) \stackrel{(32)}{=} W_{r,\xi}(f, x) - f(x) - f''(x) \sum_{j=0}^r \alpha_j \left(1 - e^{-\frac{j^2}{4}\xi}\right) = \Delta_2(x). \tag{40}$$

The theorem is proved. ■

We derive the following.

Corollary 2 (to Theorem 1) *Additionally assume $f'' \in C_B(\mathbb{R})$. It follows ($\xi > 0, x \in \mathbb{R}$)*

$$\begin{aligned} \|W_{r,\xi}(f) - f\|_\infty &\leq \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{\xi}{4} + \frac{j\sqrt{\xi}}{6\sqrt{\pi}} \right] + \\ \|f''\|_\infty &\left(\sum_{j=0}^r |\alpha_j| \left(1 - e^{-\frac{j^2}{4}\xi} \right) \right) \rightarrow 0, \quad \text{as } \xi \rightarrow 0. \end{aligned} \quad (41)$$

Proof. By (20) and (8). ■

We continue with the following result.

Theorem 3 *It holds*

$$\begin{aligned} |\Delta_3(x)| &\leq \|\Delta_3\|_\infty \leq \frac{\omega_1(f'' + f, \xi) \xi^2}{2\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \\ &\sum_{j=0}^r |\alpha_j| j^2 \left[\Gamma\left(\frac{3}{2\alpha}\right) \Gamma\left(\beta - \frac{3}{2\alpha}\right) + \frac{j}{3} \Gamma\left(\frac{2}{\alpha}\right) \Gamma\left(\beta - \frac{2}{\alpha}\right) \right] =: A_3; \end{aligned} \quad (42)$$

$\xi > 0, x \in \mathbb{R}$; and $\beta > \frac{2}{\alpha}, \alpha \in \mathbb{N}$.

Notice that $\|\Delta_3\|_\infty \rightarrow 0$, as $\xi \rightarrow 0$. If $f''(x) = 0$, then $|M_{r,\xi}(f, x) - f(x)| \leq A_3$ and $M_{r,\xi}(f, x) \rightarrow f(x)$, as $\xi \rightarrow 0$.

Proof. The proof is the same as the proof of Theorem 1 from start till equation (31).

We call

$$\begin{aligned} \bar{\Delta}_3(x) &:= M_{r,\xi}(f, x) - f(x) - f'(x) \sum_{j=0}^r \alpha_j W \left(\int_{-\infty}^{\infty} \sin(jt) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right) \\ &- 2f''(x) \sum_{j=0}^r \alpha_j W \left(\int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right) = \\ &W \int_{-\infty}^{\infty} R(t) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt. \end{aligned} \quad (43)$$

Hence we get

$$\begin{aligned} |\bar{\Delta}_3(x)| &\leq W \int_{-\infty}^{\infty} |R(t)| \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \leq \\ &W \int_{-\infty}^{\infty} \left[\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{t^2}{2} + \frac{j}{6\xi} |t|^3 \right] \right] \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt = \end{aligned} \quad (44)$$

$$\begin{aligned}
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 W \int_{-\infty}^{\infty} \left[\frac{t^2}{2} + \frac{j}{6\xi} |t|^3 \right] \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{W}{2} \int_{-\infty}^{\infty} t^2 \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt + \frac{jW}{6\xi} \int_{-\infty}^{\infty} |t|^3 \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right] = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[W \int_0^{\infty} t^2 \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt + \frac{jW}{3\xi} \int_0^{\infty} t^3 \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \right] = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[W \xi^{-2\alpha\beta} \xi^2 \int_0^{\infty} \left(\frac{t}{\xi} \right)^2 \frac{1}{\left(\left(\frac{t}{\xi} \right)^{2\alpha} + 1 \right)^\beta} d\left(\frac{t}{\xi} \right) \right. \\
& \quad \left. + \frac{jW}{3\xi} \xi^{-2\alpha\beta} \xi^4 \int_0^{\infty} \left(\frac{t}{\xi} \right)^3 \frac{1}{\left(\left(\frac{t}{\xi} \right)^{2\alpha} + 1 \right)^\beta} d\left(\frac{t}{\xi} \right) \right] = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[W \xi^{3-2\alpha\beta} \int_0^{\infty} \frac{z^2}{(z^{2\alpha} + 1)^\beta} dz + \frac{jW}{3} \xi^{3-2\alpha\beta} \int_0^{\infty} \frac{z^3}{(z^{2\alpha} + 1)^\beta} dz \right] = \\
& \text{(by [7], p. 397, formula 595)} \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 W \xi^{3-2\alpha\beta} \left[\frac{\Gamma\left(\frac{3}{2\alpha}\right) \Gamma\left(\beta - \left(\frac{3}{2\alpha}\right)\right)}{2\alpha\Gamma(\beta)} + \frac{j}{3} \frac{\Gamma\left(\frac{2}{\alpha}\right) \Gamma\left(\beta - \frac{2}{\alpha}\right)}{2\alpha\Gamma(\beta)} \right] = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \frac{\Gamma(\beta) \alpha \xi^{2\alpha\beta-1} \xi^{3-2\alpha\beta}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \\
& \quad \left[\frac{\Gamma\left(\frac{3}{2\alpha}\right) \Gamma\left(\beta - \left(\frac{3}{2\alpha}\right)\right)}{2\alpha\Gamma(\beta)} + \frac{j}{3} \frac{\Gamma\left(\frac{2}{\alpha}\right) \Gamma\left(\beta - \frac{2}{\alpha}\right)}{2\alpha\Gamma(\beta)} \right] = \\
& \frac{\omega_1(f'' + f, \xi)}{2} \sum_{j=0}^r |\alpha_j| j^2 \frac{\xi^2}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \\
& \quad \left[\Gamma\left(\frac{3}{2\alpha}\right) \Gamma\left(\beta - \frac{3}{2\alpha}\right) + \frac{j}{3} \Gamma\left(\frac{2}{\alpha}\right) \Gamma\left(\beta - \frac{2}{\alpha}\right) \right]. \tag{47}
\end{aligned}$$

We have proved that

$$|\bar{\Delta}_3(x)| \leq \frac{\omega_1(f'' + f, \xi) \xi^2}{2\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)}$$

$$\sum_{j=0}^r |\alpha_j| j^2 \left[\Gamma\left(\frac{3}{2\alpha}\right) \Gamma\left(\beta - \frac{3}{2\alpha}\right) + \frac{j}{3} \Gamma\left(\frac{2}{\alpha}\right) \Gamma\left(\beta - \frac{2}{\alpha}\right) \right]; \quad (48)$$

$\xi > 0$, $x \in \mathbb{R}$; and $\beta > \frac{2}{\alpha}$, $\alpha \in \mathbb{N}$.

Notice that $\bar{\Delta}_3(x) \rightarrow 0$, as $\xi \rightarrow 0$.

Next we simplify the left hand side of (43).

We observe that:

Clearly it is

$$\int_{-\infty}^{\infty} \sin(jt) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt = 0, \quad j = 0, 1, \dots, r. \quad (49)$$

Furthermore, we have that

$$\begin{aligned} W \int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt &= W 2 \int_0^{\infty} \sin^2\left(\frac{jt}{2}\right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt = \\ W 2 \xi^{-2\alpha\beta} \xi \int_0^{\infty} \sin^2\left(\left(\frac{j\xi}{2}\right) \frac{t}{\xi}\right) \frac{1}{\left(\left(\frac{t}{\xi}\right)^{2\alpha} + 1\right)^\beta} d\left(\frac{t}{\xi}\right) &= \end{aligned} \quad (50)$$

($\frac{t}{\xi} =: x$ and $\frac{j\xi}{2} =: \gamma$)

$$\begin{aligned} &= 2W \xi^{1-2\alpha\beta} \int_0^{\infty} \sin^2(\gamma x) \frac{1}{(x^{2\alpha} + 1)^\beta} dx = \\ &2 \frac{\Gamma(\beta) \alpha \xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \xi^{1-2\alpha\beta} \int_0^{\infty} \frac{\sin^2(\gamma x)}{(x^{2\alpha} + 1)^\beta} dx = \\ &\frac{2\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \int_0^{\infty} \frac{\sin^2(\gamma x)}{(x^{2\alpha} + 1)^\beta} dx \leq \quad (51) \\ &\frac{2\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \int_0^{\infty} \frac{dx}{(1 + x^{2\alpha})^\beta} = \end{aligned}$$

(by [7], p. 397, formula 595)

$$\frac{2\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \frac{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)}{2\alpha\Gamma(\beta)} = 1, \quad (52)$$

for $\beta > \frac{1}{2\alpha}$, $\alpha \in \mathbb{N}$.

Therefore it holds ($j = 0, 1, \dots, r$)

$$W \int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt \leq 1, \quad (53)$$

where $\beta > \frac{1}{2\alpha}$, $\alpha \in \mathbb{N}$.

Consequently, we get that

$$\begin{aligned} \overline{\Delta}_3(x) &\stackrel{(43)}{=} M_{r,\xi}(f, x) - f(x) - \\ 4f''(x) &\frac{\alpha\Gamma(\beta)}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta - \frac{1}{2\alpha})} \sum_{j=0}^r \alpha_j \int_0^\infty \frac{\sin^2\left(\frac{j\xi}{2}t\right)}{(t^{2\alpha} + 1)^\beta} dt = \Delta_3(x), \end{aligned} \quad (54)$$

$\xi > 0, x \in \mathbb{R}; \beta > \frac{1}{2\alpha}, \alpha \in \mathbb{N}$.

The theorem is proved. ■

We give the following

Corollary 4 (to Theorem 3) *Additionally assume $f'' \in C_B(\mathbb{R})$. It follows ($\xi > 0, x \in \mathbb{R}$)*

$$\begin{aligned} \|M_{r,\xi}(f) - f\|_\infty &\leq \frac{\omega_1(f'' + f, \xi)\xi^2}{2\Gamma(\frac{1}{2\alpha})\Gamma(\beta - \frac{1}{2\alpha})} \\ &\sum_{j=0}^r |\alpha_j| j^2 \left[\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta - \frac{3}{2\alpha}\right) + \frac{j}{3}\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta - \frac{2}{\alpha}\right) \right] + \\ 4\|f''\|_\infty &\frac{\alpha\Gamma(\beta)}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta - \frac{1}{2\alpha})} \sum_{j=0}^r |\alpha_j| \int_0^\infty \frac{\sin^2\left(\frac{j\xi}{2}t\right)}{(1 + t^{2\alpha})^\beta} dt \rightarrow 0, \text{ as } \xi \rightarrow 0; \end{aligned} \quad (55)$$

$\xi > 0, \beta > \frac{1}{2\alpha}, \alpha \in \mathbb{N}$.

Proof. By (42) and (13). And we notice that

$$\left| \sin\left(\frac{j\xi}{2}t\right) \right| \leq \frac{j\xi}{2}t, \quad \forall t \in \mathbb{R}_+.$$

Hence we get

$$\sin^2\left(\frac{j\xi}{2}t\right) \leq \frac{j^2\xi^2}{4}t^2, \quad \forall t \in \mathbb{R}_+.$$

Consequently, it holds

$$\begin{aligned} \int_0^\infty \frac{\sin^2\left(\frac{j\xi}{2}t\right)}{(1 + t^{2\alpha})^\beta} dt &\leq \frac{j^2\xi^2}{4} \int_0^\infty \frac{t^2}{(1 + t^{2\alpha})^\beta} dt = \\ &\frac{j^2\xi^2}{8} \frac{\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta - \left(\frac{3}{2\alpha}\right)\right)}{\alpha\Gamma(\beta)} \rightarrow 0, \text{ as } \xi \rightarrow 0. \end{aligned}$$

■

The last main result comes next.

Theorem 5 *It holds*

$$|\Delta_4(x)| \leq \|\Delta_4\|_\infty \leq \frac{\omega_1(f'' + f, \xi)}{\lambda_1} \xi^2$$

$$\left\{ \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{\pi (-1)^{\beta-1} (2\beta)!}{8(2\beta-3)!} \left(\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-3}}{(\beta-k)! (\beta+k)!} \right) + \frac{j}{3} \left(\frac{-(2\beta)!}{8(2\beta-4)!} \left(\sum_{k=1}^{\beta} \frac{(-1)^{\beta-k} k^{2\beta-k} \ln(2k)}{(\beta-k)! (\beta+k)!} \right) \right) \right] \right\} =: A_4; \quad (56)$$

$\xi > 0$, $x \in \mathbb{R}$ and $\beta \in \mathbb{N} - \{1, 2\}$.

Notice that $\|\Delta_4\|_\infty \rightarrow 0$, as $\xi \rightarrow 0$. If $f''(x) = 0$, then $|T_{r,\xi}(f, x) - f(x)| \leq A_4$ and $T_{r,\xi}(f, x) \rightarrow f(x)$, as $\xi \rightarrow 0$.

Proof. The proof is the same as the proof of Theorem 1 from start till equation (31).

We call

$$\begin{aligned} \bar{\Delta}_4(x) &:= T_{r,\xi}(f, x) - f(x) - f'(x) \sum_{j=0}^r \alpha_j \lambda^{-1} \left(\int_{-\infty}^{\infty} \sin(jt) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt \right) \\ &\quad - 2f''(x) \sum_{j=0}^r \alpha_j \lambda^{-1} \left(\int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt \right) = \quad (57) \\ &\quad \lambda^{-1} \int_{-\infty}^{\infty} R(t) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt. \end{aligned}$$

Hence we get

$$|\bar{\Delta}_4(x)| \leq \lambda^{-1} \int_{-\infty}^{\infty} |R(t)| \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt \leq$$

$$\lambda^{-1} \int_{-\infty}^{\infty} \left[\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{t^2}{2} + \frac{j}{6\xi} |t|^3 \right] \right] \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt = \quad (58)$$

$$\omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \lambda^{-1} \int_{-\infty}^{\infty} \left[\frac{t^2}{2} + \frac{j}{6\xi} |t|^3 \right] \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt =$$

$$\begin{aligned}
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{\lambda^{-1}}{2} \int_{-\infty}^{\infty} t^2 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt + \frac{j\lambda^{-1}}{6\xi} \int_{-\infty}^{\infty} |t|^3 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt \right] = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\lambda^{-1} \int_0^{\infty} t^2 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt + \frac{j\lambda^{-1}}{3\xi} \int_0^{\infty} t^3 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt \right] = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\lambda^{-1} \xi^{3-2\beta} \int_0^{\infty} \left(\frac{t}{\xi} \right)^2 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{\left(\frac{t}{\xi}\right)} \right)^{2\beta} d\left(\frac{t}{\xi}\right) \right. \\
& \quad \left. + \frac{j\lambda^{-1}}{3\xi} \xi^{4-2\beta} \int_0^{\infty} \left(\frac{t}{\xi} \right)^3 \left(\frac{\sin\left(\frac{t}{\xi}\right)}{\left(\frac{t}{\xi}\right)} \right)^{2\beta} d\left(\frac{t}{\xi}\right) \right] = \\
& \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\lambda^{-1} \xi^{3-2\beta} \int_0^{\infty} x^2 \left(\frac{\sin x}{x} \right)^{2\beta} dx + \frac{j\lambda^{-1}}{3} \xi^{3-2\beta} \int_0^{\infty} x^3 \left(\frac{\sin x}{x} \right)^{2\beta} dx \right] \\
& \hspace{15em} (60)
\end{aligned}$$

(by [6], p. 210, item 1033)

$$\begin{aligned}
& = \omega_1(f'' + f, \xi) \sum_{j=0}^r |\alpha_j| j^2 \left[\lambda_1^{-1} \xi^2 \left(\frac{\pi (-1)^{\beta-1} (2\beta)!}{8 (2\beta-3)!} \left(\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-3}}{(\beta-k)! (\beta+k)!} \right) \right) \right. \\
& \quad \left. + \frac{j\lambda_1^{-1}}{3} \xi^2 \left(\frac{-(2\beta)!}{8 (2\beta-4)!} \left(\sum_{k=1}^{\beta} \frac{(-1)^{\beta-k} k^{2\beta-k} \ln(2k)}{(\beta-k)! (\beta+k)!} \right) \right) \right] = \\
& \frac{\omega_1(f'' + f, \xi) \xi^2}{\lambda_1} \left\{ \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{\pi (-1)^{\beta-1} (2\beta)!}{8 (2\beta-3)!} \left(\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-3}}{(\beta-k)! (\beta+k)!} \right) \right. \right. \\
& \quad \left. \left. + \frac{j}{3} \left(\frac{-(2\beta)!}{8 (2\beta-4)!} \left(\sum_{k=1}^{\beta} \frac{(-1)^{\beta-k} k^{2\beta-4} \ln(2k)}{(\beta-k)! (\beta+k)!} \right) \right) \right] \right\}. \quad (61)
\end{aligned}$$

We have proved that

$$\begin{aligned}
|\overline{\Delta}_4(x)| & \leq \frac{\omega_1(f'' + f, \xi) \xi^2}{\lambda_1} \left\{ \sum_{j=0}^r |\alpha_j| j^2 \right. \\
& \quad \left[\frac{\pi (-1)^{\beta-1} (2\beta)!}{8 (2\beta-3)!} \left(\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-3}}{(\beta-k)! (\beta+k)!} \right) + \right.
\end{aligned}$$

$$\frac{j}{3} \left(\frac{-(2\beta)!}{8(2\beta-4)!} \left(\sum_{k=1}^{\beta} \frac{(-1)^{\beta-k} k^{2\beta-4} \ln(2k)}{(\beta-k)! (\beta+k)!} \right) \right) \Bigg\}, \quad (62)$$

$\xi > 0$, $x \in \mathbb{R}$; and $\beta \in \mathbb{N} - \{1, 2\}$.

Notice that $\bar{\Delta}_4(x) \rightarrow 0$, as $\xi \rightarrow 0$.

Next we simplify the left hand side of (57).

We observe that:

Clearly it is

$$\int_{-\infty}^{\infty} \sin(jt) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt = 0, \quad j = 0, 1, \dots, r. \quad (63)$$

Furthermore, we have that

$$\begin{aligned} & \lambda^{-1} \int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt = \\ & 2\lambda_1^{-1} \xi^{2\beta-1} \xi^{1-2\beta} \int_0^{\infty} \sin^2\left(\left(\frac{j\xi}{2}\right) \frac{t}{\xi}\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{\frac{t}{\xi}} \right)^{2\beta} d\left(\frac{t}{\xi}\right) = \\ & \left(\frac{t}{\xi} =: x \text{ and } \frac{j\xi}{2} =: \gamma\right) \end{aligned} \quad (64)$$

$$\begin{aligned} & = 2\lambda_1^{-1} \int_0^{\infty} \sin^2(\gamma x) \left(\frac{\sin x}{x} \right)^{2\beta} dx \leq \\ & 2\lambda_1^{-1} \int_0^{\infty} \left(\frac{\sin x}{x} \right)^{2\beta} dx \end{aligned} \quad (64)$$

(by [6], p. 210, item 1033)

$$= 2\lambda_1^{-1} \pi (-1)^\beta \beta \left(\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)! (\beta+k)!} \right) = 1, \text{ for } \beta \in \mathbb{N}. \quad (65)$$

Therefore it holds ($j = 0, 1, \dots, r$)

$$\lambda^{-1} \int_{-\infty}^{\infty} \sin^2\left(\frac{jt}{2}\right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt \leq 1, \quad (66)$$

where $\beta \in \mathbb{N}$.

Consequently we get that

$$\bar{\Delta}_4(x) \stackrel{(57)}{=} T_{r,\xi}(f, x) - f(x) - 4f''(x) \lambda_1^{-1} \sum_{j=0}^r \alpha_j \int_0^{\infty} \sin^2\left(\frac{j\xi}{2}t\right) \left(\frac{\sin t}{t} \right)^{2\beta} dt = \Delta_4(x), \quad (67)$$

where $\xi > 0$, $x \in \mathbb{R}$; $\beta \in \mathbb{N} - \{1, 2\}$.

The theorem is proved. ■

We finish with the following.

Corollary 6 (to Theorem 5) *Additionally assume $f'' \in C_B(\mathbb{R})$. It follows ($\xi > 0$, $x \in \mathbb{R}$)*

$$\begin{aligned} \|T_{r,\xi}(f) - f\|_\infty &\leq \frac{\omega_1(f'' + f, \xi)\xi^2}{\lambda_1} \\ &\left\{ \sum_{j=0}^r |\alpha_j| j^2 \left[\frac{\pi (-1)^{\beta-1} (2\beta)!}{8 (2\beta-3)!} \left(\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-3}}{(\beta-k)! (\beta+k)!} \right) + \right. \right. \\ &\quad \left. \left. \frac{j}{3} \left(\frac{-(2\beta)!}{8 (2\beta-4)!} \left(\sum_{k=1}^{\beta} \frac{(-1)^{\beta-k} k^{2\beta-4} \ln(2k)}{(\beta-k)! (\beta+k)!} \right) \right) \right] \right\} + \\ &4 \|f''\|_\infty \lambda_1^{-1} \sum_{j=0}^r |\alpha_j| \int_0^\infty \sin^2 \left(\frac{j\xi}{2} t \right) \left(\frac{\sin t}{t} \right)^{2\beta} dt \rightarrow 0, \text{ as } \xi \rightarrow 0; \end{aligned} \quad (68)$$

where $\xi > 0$, $\beta \in \mathbb{N} - \{1, 2\}$.

Proof. By (56) and (19). And from

$$\sin^2 \left(\frac{j\xi}{2} t \right) \leq \frac{j^2 \xi^2}{4} t^2, \quad \forall t \in \mathbb{R}_+,$$

we obtain that

$$\begin{aligned} \int_0^\infty \sin^2 \left(\frac{j\xi}{2} t \right) \left(\frac{\sin t}{t} \right)^{2\beta} dt &\leq \frac{j^2 \xi^2}{4} \int_0^\infty t^2 \left(\frac{\sin t}{t} \right)^{2\beta} dt = \\ &\frac{j^2 \xi^2}{4} \left(\frac{\pi (-1)^{\beta-1} (2\beta)!}{8 (2\beta-3)!} \left(\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-3}}{(\beta-k)! (\beta+k)!} \right) \right) \rightarrow 0, \text{ as } \xi \rightarrow 0. \end{aligned}$$

■

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