# Multivariate Fuzzy-Random and Perturbed Neural Network Approximation

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**Abstract**

In this article we estimate the degree of approximation of multivariate pointwise and uniform convergences in the $p$-mean to the Fuzzy-Random unit operator of multivariate Fuzzy-Random Quasi-Interpolation perturbed activation functions based neural network operators. These multivariate Fuzzy-Random operators arise in a natural way among multivariate Fuzzy-Random neural networks. The rates are given through multivariate Probabilistic-Jackson type inequalities involving the multivariate Fuzzy-Random modulus of continuity of the engaged multivariate Fuzzy-Random function.

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## 1 Fuzzy-Random Functions Background

See also [2], Ch. 22, pp. 497-501.

We start with

**Definition 1.** *(see [8]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:*

*(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R} : \mu(x_0) = 1$.*

*(ii)* $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ *($\mu$ is called a convex fuzzy subset).*

*(iii)* $\mu$ *is upper semicontinuous on* $\mathbb{R}$*, i.e.,* $\forall\ x_0 \in \mathbb{R}$ *and* $\forall\ \varepsilon > 0$, $\exists$ *neighborhood* $V(x_0)$ : $\mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

*(iv) the set* $\overline{supp(\mu)}$ *is compact in* $\mathbb{R}$ *(where* $supp(\mu) := \{x \in \mathbb{R}; \mu(x) > 0\}$*).*

*We call* $\mu$ *a fuzzy real number. Denote the set of all* $\mu$ *with* $\mathbb{R}_{\mathcal{F}}$*.*

*E.g.,* $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$*, for any* $x_0 \in \mathbb{R}$*, where* $\chi_{\{x_0\}}$ *is the characteristic function at* $x_0$*.*

*For* $0 < r \leq 1$ *and* $\mu \in \mathbb{R}_{\mathcal{F}}$ *define* $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ *and* $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \ \forall\ r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [8]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} < u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\ r \in [0,1]$.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\},$$

where $[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that $D$ is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [8], with the properties

$$\begin{aligned} D(u \oplus w, v \oplus w) &= D(u,v), \quad \forall\ u, v, w \in \mathbb{R}_{\mathcal{F}}, \\ D(k \odot u, k \odot v) &= |k|\, D(u,v), \quad \forall\ u, v \in \mathbb{R}_{\mathcal{F}}, \forall\ k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) &\leq D(u,w) + D(v,e), \quad \forall\ u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{aligned} \tag{1}$$

Let $(M, d)$ metric space and $f, g : M \to \mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between $f, g$ is defined by

$$D^*(f,g) := \sup_{x \in M} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall\ r \in [0,1]$.

$\overset{*}{\sum}$ denotes the fuzzy summation, $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ the neutral element with respect to $\oplus$. For more see also [9], [10].

We need

**Definition 2.** *(see also [7], Definition 13.16, p. 654) Let* $(X, \mathcal{B}, P)$ *be a probability space. A fuzzy-random variable is a* $\mathcal{B}$*-measurable mapping* $g : X \to \mathbb{R}_{\mathcal{F}}$ *(i.e., for any open set* $U \subseteq \mathbb{R}_{\mathcal{F}}$*, in the topology of* $\mathbb{R}_{\mathcal{F}}$ *generated by the metric* $D$*, we have*

$$g^{-1}(U) = \{s \in X; g(s) \in U\} \in \mathcal{B}). \tag{2}$$

*The set of all fuzzy-random variables is denoted by* $\mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$. *Let* $g_n, g \in \mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$, $n \in \mathbb{N}$ *and* $0 < p < +\infty$. *We say* $g_n(s) \overset{\text{"p-mean"}}{\underset{n\to+\infty}{\rightarrow}} g(s)$ *if*

$$\lim_{n\to+\infty} \int_X D(g_n(s), g(s))^p P(ds) = 0. \tag{3}$$

**Remark 3.** *(see [7], p. 654) If* $f, g \in \mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$, *let us denote* $F : X \to \mathbb{R}_+ \cup \{0\}$ *by* $F(s) = D(f(s), g(s))$, $s \in X$. *Here,* $F$ *is* $\mathcal{B}$*-measurable, because* $F = G \circ H$, *where* $G(u,v) = D(u,v)$ *is continuous on* $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, *and* $H : X \to \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, $H(s) = (f(s), g(s))$, $s \in X$, *is* $\mathcal{B}$*-measurable. This shows that the above convergence in q-mean makes sense.*

**Definition 4.** *(see [7], p. 654, Definition 13.17) Let* $(T, \mathcal{T})$ *be a topological space. A mapping* $f : T \to \mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$ *will be called fuzzy-random function (or fuzzy-stochastic process) on* $T$. *We denote* $f(t)(s) = f(t,s)$, $t \in T$, $s \in X$.

**Remark 5.** *(see [7], p. 655) Any usual fuzzy real function* $f : T \to \mathbb{R}_{\mathcal{F}}$ *can be identified with the degenerate fuzzy-random function* $f(t,s) = f(t)$, $\forall\ t \in T$, $s \in X$.

**Remark 6.** *(see [7], p. 655) Fuzzy-random functions that coincide with probability one for each* $t \in T$ *will be consider equivalent.*

**Remark 7.** *(see [7], p. 655) Let* $f, g : T \to \mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$. *Then* $f \oplus g$ *and* $k \odot f$ *are defined pointwise, i.e.,*

$$\begin{aligned} (f \oplus g)(t,s) &= f(t,s) \oplus g(t,s), \\ (k \odot f)(t,s) &= k \odot f(t,s), \ \ t \in T, s \in X,\ k \in \mathbb{R}. \end{aligned}$$

**Definition 8.** *(see also Definition 13.18, pp. 655-656, [7]) For a fuzzy-random function* $f : W \subseteq \mathbb{R}^N \to \mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$, $N \in \mathbb{N}$, *we define the (first) fuzzy-random modulus of continuity*

$$\Omega_1^{(\mathcal{F})}(f,\delta)_{L^p} =$$

$$\sup\left\{\left(\int_X D^p(f(x,s), f(y,s))\, P(ds)\right)^{\frac{1}{p}} : x, y \in W,\ \ \|x-y\|_\infty \le \delta\right\},$$

$0 < \delta$, $1 \le p < \infty$.

**Definition 9.** *([1]) Here* $1 \le p < +\infty$. *Let* $f : W \subseteq \mathbb{R}^N \to \mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$, $N \in \mathbb{N}$, *be a fuzzy random function. We call* $f$ *a (p-mean) uniformly continuous fuzzy random function over* $W$, *iff* $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ *:whenever* $\|x-y\|_\infty \le \delta$, $x, y \in W$, *implies that*

$$\int_X (D(f(x,s), f(y,s)))^p P(ds) \le \varepsilon.$$

*We denote it as* $f \in C_{FR}^{U_p}(W)$.

**Proposition 10.** *([1]) Let* $f \in C_{FR}^{U_p}(W)$, *where* $W \subseteq \mathbb{R}^N$ *is convex.*

*Then* $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^p} < \infty$, *any* $\delta > 0$.

**Proposition 11.** *([1]) Let $f,g : W \subseteq \mathbb{R}^N \rightarrow \mathcal{L}_{\mathcal{F}}(X,\mathcal{B},P)$, $N \in \mathbb{N}$, be fuzzy random functions. It holds*

*(i)* $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^p}$ *is nonnegative and nondecreasing in* $\delta > 0$.

*(ii)* $\lim_{\delta \downarrow 0} \Omega_1^{(\mathcal{F})}(f,\delta)_{L^p} = \Omega_1^{(\mathcal{F})}(f,0)_{L^p} = 0$, *iff* $f \in C_{FR}^{U_p}(W)$.

We need also

**Proposition 12.** *([1]) Let $f,g$ be fuzzy random variables from $\mathcal{S}$ into $\mathbb{R}_{\mathcal{F}}$. Then*

*(i) Let $c \in \mathbb{R}$, then $c \odot f$ is a fuzzy random variable.*

*(ii) $f \oplus g$ is a fuzzy random variable.*

# 2 About Perturbed Neural Network Background

## 2.1 About $q$-Deformed and $\lambda$-parametrized $A$-generalized logistic function induced real space valued multivariate multi layer neural network approximation

Here we follow [4].

We consider the $q$-deformed and $\lambda$-parametrized function

$$\varphi_{q,\lambda}(x) = \frac{1}{1+qA^{-\lambda x}}, \quad x \in \mathbb{R}, \text{ where } q,\lambda > 0, \ A > 1. \tag{4}$$

which is a sigmoid type function and it is strictly increasing. This is an A-generalized logistic type function. We easily observe that

$$\varphi_{q,\lambda}(+\infty) = 1, \ \ \varphi_{q,\lambda}(-\infty) = 0. \tag{5}$$

Furthermore we have

$$\varphi_{q,\lambda}(x) = 1 - \varphi_{\frac{1}{q},\lambda}(-x). \tag{6}$$

and

$$\varphi_{q,\lambda}(0) = \frac{1}{1+q}. \tag{7}$$

Moreover $\varphi''_{q,\lambda}(x) > 0$, for $x < \frac{\log_A q}{\lambda}$ and there $\varphi_{q,\lambda}$ is concave up. When $x > \frac{\log_A q}{\lambda}$, we have $\varphi''_{q,\lambda}(x) < 0$ and $\varphi_{q,\lambda}$ is concave down. Of course

$$\varphi''_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = 0.$$

So, $\varphi_{q,\lambda}$ is a sigmoid function, see [3].

We consider the activation function

$${}_1\mathcal{L}_q(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right), \quad x \in \mathbb{R}. \tag{8}$$

Then

$${}_1\mathcal{L}_q(-x) = {}_1\mathcal{L}_{\frac{1}{q},\lambda}(x), \quad \forall \, x \in \mathbb{R}. \tag{9}$$

We have that

$$ {}_1\mathcal{L}_q'(x) = \frac{1}{2}\left(\varphi_{q,\lambda}'(x+1) - \varphi_{q,\lambda}'(x-1)\right) < 0. $$

i.e. ${}_1\mathcal{L}_q$ is strictly decreasing over $\left(\frac{\log_A q}{\lambda}, +\infty\right)$. Furthermore, ${}_1\mathcal{L}_q$ is strictly concave down over $\left(\frac{\log_A q}{\lambda} - 1, \frac{\log_A q}{\lambda} + 1\right)$. Overall ${}_1\mathcal{L}_q$ is a bell-shaped function over $\mathbb{R}$. Of course it holds ${}_1\mathcal{L}q, \lambda''\left(\frac{\log_A q}{\lambda}\right) < 0$. We have that the global maximul of ${}_1\mathcal{L}_q$ is

$$ {}_1\mathcal{L}_q\left(\frac{\log_A q}{\lambda}\right) = \frac{A^\lambda - 1}{2\left(A^\lambda + 1\right)}. \tag{10} $$

Finally we have that

$$ \lim_{x\to+\infty} {}_1\mathcal{L}_q(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(+\infty) - \varphi_{q,\lambda}(+\infty)\right) = 0, \tag{11} $$

and

$$ \lim_{x\to-\infty} {}_1\mathcal{L}_q(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(-\infty) - \varphi_{q,\lambda}(-\infty)\right) = 0. \tag{12} $$

Consequently the $x$-axis is the horizontal asymptote of ${}_1\mathcal{L}_q$. Of course ${}_1\mathcal{L}_q(x) > 0$, $\forall\, x \in \mathbb{R}$. We need

**Theorem 13.** *It holds*

$$ \sum_{i=-\infty}^{\infty} {}_1\mathcal{L}_q(x-i) = 1, \quad \forall x \in \mathbb{R}, \forall\, q, \lambda > 0, A > 1. \tag{13} $$

It follows

**Theorem 14.** *It holds*

$$ \int_{-\infty}^{\infty} {}_1\mathcal{L}_q(x)\, dx = 1, \quad \lambda, q > 0,\ A > 1. \tag{14} $$

So that ${}_1\mathcal{L}_q$ is a density function on $\mathbb{R}$; $\lambda, q > 0,\ A > 1$.

We need the following result

**Theorem 15.** *Let* $0 < \alpha < 1$, *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$. *Then*

$$ \sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} {}_1\mathcal{L}_q(nx-k) < \max\left\{q, \frac{1}{q}\right\} \frac{1}{A^{\lambda(n^{1-\alpha}-2)}} = \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)} =: c_1(n,a), \tag{15} $$

*where* $q, \lambda > 0,\ A > 1$; $\gamma := \max\left\{q, \frac{1}{q}\right\}$.

Let $\lceil\cdot\rceil$ the ceiling of the number, and $\lfloor\cdot\rfloor$ the integral part of the number.

**Theorem 16.** *Let* $x \in [a,b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na\rceil \leq \lfloor nb\rfloor$. *For* $q > 0$, $\lambda > 0$, $A > 1$, *we consider the number* $\lambda_q > z_0 > 0$ *with* ${}_1\mathcal{L}_q(z_0) = {}_1\mathcal{L}_q(0)$ *and* $\lambda_q > 1$. *Then*

$$ \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} {}_1\mathcal{L}_q(nx-k)} < \max\left\{\frac{1}{{}_1\mathcal{L}_q(\lambda_q)}, \frac{1}{{}_1\mathcal{L}_{\frac{1}{q}}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \Psi_1(q). \tag{16} $$

We make

**Remark 17.** *(i) We have that*

$$\lim_{n\to+\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}{}_1\mathcal{L}_q\left(nx-k\right)\neq 1,\quad \textit{for at least some } x\in\left[a,b\right], \tag{17}$$

*where* $\lambda, q > 0$.

*(ii) Let* $[a,b]\subset\mathbb{R}$. *For large* $n$ *we always have* $\lceil na\rceil\leq\lfloor nb\rfloor$. *Also* $a\leq\frac{k}{n}\leq b$, *iff* $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$. *In general it holds*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}{}_1\mathcal{L}_q\left(nx-k\right)\leq 1. \tag{18}$$

We introduce

$${}_1Z_q\left(x_1,...,x_N\right):={}_1Z_q\left(x\right):=\prod_{i=1}^{N}{}_1\mathcal{L}_q\left(x_i\right),\quad x=\left(x_1,...,x_N\right)\in\mathbb{R}^N, \tag{19}$$

$\lambda, q>0$, $A>1$, $N\in\mathbb{N}$.

${}_1Z_q\left(x\right)$ it has the properties:

(i) ${}_1Z_q\left(x\right)>0$, $\forall\ x\in\mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty}{}_1Z_q\left(x-k\right):=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}{}_1Z_q\left(x_1-k_1,...,x_N-k_N\right)=1, \tag{20}$$

where $k:=(k_1,...,k_n)\in\mathbb{Z}^N$, $\forall\ x\in\mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty}{}_1Z_q\left(nx-k\right)=1, \tag{21}$$

$\forall\ x\in\mathbb{R}^N$; $n\in\mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N}{}_1Z_q\left(x\right)dx=1, \tag{22}$$

that is ${}_1Z_q$ is a multivariate density function.

Here denote $\|x\|_\infty:=\max\left\{|x_1|,...,|x_N|\right\}$, $x\in\mathbb{R}^N$, also set $\infty:=(\infty,...,\infty)$, $-\infty:=(-\infty,...,-\infty)$ upon the multivariate context, and

$$\begin{aligned}\lceil na\rceil&:=\left(\lceil na_1\rceil,...,\lceil na_N\rceil\right),\\ \lfloor nb\rfloor&:=\left(\lfloor nb_1\rfloor,...,\lfloor nb_N\rfloor\right),\end{aligned} \tag{23}$$

where $a:=(a_1,...,a_N)$, $b:=(b_1,...,b_N)$.

We obviously see that for $0 < \beta^* < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_1Z_q(nx-k) =$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^{\beta^*}} \end{cases}}^{\lfloor nb \rfloor} {}_1Z_q(nx-k) + \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^{\beta^*}} \end{cases}}^{\lfloor nb \rfloor} {}_1Z_q(nx-k). \tag{24}$$

(v) We derive that

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^{\beta^*}} \end{cases}}^{\lfloor nb \rfloor} {}_1Z_q(nx-k) < \gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)} = c_1(n, \beta^*), \;\; 0 < \beta^* < 1, \tag{25}$$

with $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $x \in \prod_{i=1}^N [a_i, b_i]$.

(vi) We get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_1Z_q(nx-k)} < (\Psi_1(q))^N, \tag{26}$$

$\forall\ x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $n \in \mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^{\beta^*}} \end{cases}}^{\infty} {}_1Z_q(nx-k) < \gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}, \tag{27}$$

$0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $x \in \mathbb{R}^N$.

Furthermore it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_1Z_q(nx-k) \neq 1, \tag{28}$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

## 2.2 About $q$-deformed and $\lambda-$parametrized hyperbolic tangent function $g_{q,\lambda}$

Here we follow [5]. Let us consider the function

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - qe^{-\lambda x}}{e^{\lambda x} + qe^{-\lambda x}}, \;\; \lambda, q > 0, \; x \in \mathbb{R}. \tag{29}$$

We have that $g_{q,\lambda}$ is striclty increasing. We easily observe that,

$$g_{q,\lambda}(+\infty) = 1, \text{and} \quad g_{q,\lambda}(-\infty) = -1 \tag{30}$$

Furthermore,

$$g_{q,\lambda}(0) = \frac{1-q}{1+q}. \tag{31}$$

and

$$g'_{\frac{1}{q},\lambda}(x) = g'_{q,\lambda}(-x). \tag{32}$$

Moreover, in case of $x < \frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave up, with $g''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = 0$.

And in case of $x > \frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave down.

Clearly, $g_{q,\lambda}$ is a shifted sigmoid function with $g_{q,\lambda}(0) = \frac{1-q}{1+q}$, and $g_{q,\lambda}(-x) = -g_{q^{-1},\lambda}(x)$, (a semi-odd function), see also [3].

By $1 > -1$, $x+1 > x-1$, we consider the activation function

$$_2\mathcal{L}_q(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \tag{33}$$

$\forall x \in \mathbb{R}$; $q, \lambda > 0$. Notice that $_2\mathcal{L}_q(\pm\infty) = 0$, so the $x$-axis is horizontal asymptote. We have that

$$_2\mathcal{L}_q(-x) = {_2\mathcal{L}_{\frac{1}{q}}}(x), \ \ \forall\, x \in \mathbb{R};\ q, \lambda > 0, \tag{34}$$

a deformed symmetry.

Next, we have that

$$_2\mathcal{L}'_q(x) = \frac{1}{4}\left(g'_{q,\lambda}(x+1) - g'_{q,\lambda}(x-1)\right), \ \ \forall\, x \in \mathbb{R}. \tag{35}$$

Moreover, $_2\mathcal{L}_q$ is striclty increasing over $\left(-\infty, \frac{\ln q}{2\lambda} - 1\right)$. and strictly decreasing over $\left(\frac{\ln q}{2\lambda} + 1, +\infty\right)$. Furthermore $_2\mathcal{L}_q$ is concave down over $\left[\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1\right]$, and strictly concave down over $\left(\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1\right)$. Consequently $_2\mathcal{L}_q$ has a bell-type shape over $\mathbb{R}$.

Of course it holds $_2\mathcal{L}''_q\left(\frac{\ln q}{2\lambda}\right) < 0$. We also have that the maximum value of $_2\mathcal{L}_q$ is

$$_2\mathcal{L}_q\left(\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2}, \quad \lambda > 0. \tag{36}$$

We give

**Theorem 18.** *We have that*

$$\sum_{i=-\infty}^{\infty} {_2\mathcal{L}_q}(x-i) = 1, \ \ \forall\, x \in \mathbb{R},\ \forall\, \lambda, q > 0. \tag{37}$$

We need

**Theorem 19.** *It holds*

$$\int_{-\infty}^{\infty} {_2\mathcal{L}_q}(x)\, dx = 1, \quad \lambda, q > 0. \tag{38}$$

So that $_2\mathcal{L}_q$ is a density function on $\mathbb{R}$; $\lambda, q > 0$.

We need the following result

**Theorem 20.** *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$; $q, \lambda > 0$. Then*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} {_2\mathcal{L}_q}(nx-k) < \max\left\{q, \frac{1}{q}\right\} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} = T e^{-2\lambda n^{(1-\alpha)}} =: c_2(n,a), \tag{39}$$

*where* $T := \max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Let $\lceil \cdot \rceil$ the ceiling of the number, and $\lfloor \cdot \rfloor$ the integral part of the number.

**Theorem 21.** *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q > 0$, $\lambda > 0$, we consider the number $\lambda_q > z_0 > 0$ with ${}_2\mathcal{L}_q(z_0) = {}_2\mathcal{L}_q(0)$ and $\lambda_q > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_2\mathcal{L}_q(nx-k)} < max\left\{ \frac{1}{{}_2\mathcal{L}_q(\lambda_q)}, \frac{1}{{}_2\mathcal{L}_{\frac{1}{q}}\left(\lambda_{\frac{1}{q}}\right)} \right\} =: \Psi_2(q). \quad (40)$$

We make

**Remark 22.** *(i) We have that*

$$\lim_{n \to +\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_2\mathcal{L}_q(nx-k) \neq 1, \quad \text{for at least some } x \in [a,b], \quad (41)$$

*where $\lambda, q > 0$.*

*(ii) Let $[a,b] \subset \mathbb{R}$. For large $n$ we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_2\mathcal{L}_q(nx-k) \leq 1. \quad (42)$$

We introduce

$${}_2Z_q(x_1,...,x_N) := {}_2Z_q(x) := \prod_{i=1}^{N} {}_2\mathcal{L}_q(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N, \ \lambda, q > 0, \ N \in \mathbb{N}. \quad (43)$$

${}_2Z_q(x)$ it has the properties:

(i) ${}_2Z_q(x) > 0, \ \forall \ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} {}_2Z_q(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z_q(x_1-k_1,...,x_N-k_N) = 1, \quad (44)$$

where $k := (k_1,...,k_n) \in \mathbb{Z}^N$, $\forall \ x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} {}_2Z_q(nx-k) = 1, \quad (45)$$

$\forall \ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} {}_2Z_q(x)\, dx = 1, \quad (46)$$

that is ${}_2Z_q$ is a multivariate density function.

We obviously see that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} {}_2Z_q\left(nx-k\right) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left(\prod_{i=1}^{N} {}_2\mathcal{L}_q\left(nx_i-k_i\right)\right) =$$

$$\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \dots \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} \left(\prod_{i=1}^{N} {}_2\mathcal{L}_q\left(nx_i-k_i\right)\right) = \prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} {}_2\mathcal{L}_q\left(nx_i-k_i\right)\right). \tag{47}$$

For $0<\beta^*<1$ and $n\in\mathbb{N}$, a fixed $x\in\mathbb{R}^N$, we have that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} {}_2Z_q\left(nx-k\right) =$$

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty \le \frac{1}{n^{\beta^*}}\end{cases}}^{\lfloor nb\rfloor} {}_2Z_q\left(nx-k\right) + \sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^{\beta^*}}\end{cases}}^{\lfloor nb\rfloor} {}_2Z_q\left(nx-k\right). \tag{48}$$

In the last two sums the counting is over disjoint vector sets of $k$'s, because the condition $\left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^{\beta^*}}$ implies that there exists at least one $\left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^{\beta^*}}$, where $r\in\{1,...,N\}$.

(v) We also have that

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^{\beta^*}}\end{cases}}^{\lfloor nb\rfloor} {}_2Z_q\left(nx-k\right) < Te^{-2\lambda n^{(1-\beta^*)}} = c_2(n,\beta^*), \quad 0<\beta^*<1, \tag{49}$$

with $n\in\mathbb{N}: n^{1-\beta^*}>2$, $x\in\prod_{i=1}^{N}\left[a_i,b_i\right]$.

(vi) Moreover

$$0 < \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} {}_2Z_q\left(nx-k\right)} < \left(\Psi_2\left(q\right)\right)^N, \tag{50}$$

$\forall\ x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$, $n\in\mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^{\beta^*}}\end{cases}}^{\infty} {}_2Z_q\left(nx-k\right) < Te^{-2\lambda n^{(1-\beta^*)}}, \tag{51}$$

$0<\beta^*<1$, $n\in\mathbb{N}: n^{1-\beta^*}>2$, $x\in\mathbb{R}^N$.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} {}_2Z_q\left(nx-k\right) \neq 1, \tag{52}$$

for at least some $x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$.

### 2.3 About $q$-deformed and parametrized half hyperbolic tangent function $\vartheta_q$

Here we follow [6]. We introduce the function

$$\vartheta_q(t) := \frac{1 - qe^{-\beta t}}{1 + qe^{-\beta t}}, \quad \forall\, t \in \mathbb{R}, \tag{53}$$

where $q, \beta > 0$. $\vartheta_q$ is striclty increasing. We also observe that

$$\vartheta_q(-\infty) = -1 \quad \text{and} \quad \vartheta_q(+\infty) = 1 \tag{54}$$

Furthermore

$$\vartheta_q(0) = \frac{1-q}{1+q} \tag{55}$$

In case of $t < \frac{\ln q}{\beta}$, we have that $\vartheta_q$ is strictly concave up, with $\vartheta_q''\left(\frac{\ln q}{\beta}\right) = 0$.

And in case of $t > \frac{\ln q}{\beta}$, we have that $\vartheta_q$ is strictly concave down.

Clearly, $\vartheta_q$ is a shifted sigmoid function with $\vartheta_q(0) = \frac{1-q}{1+q}$, and $\vartheta_q(-x) = -\vartheta_{q^{-1}}(x)$, $\forall$ $x \in \mathbb{R}$, (a semi-odd function), see also [3].

By $1 > -1$, $x + 1 > x - 1$, we consider the activation function

$${}_3\mathcal{L}_q(x) := \frac{1}{4}\left(\vartheta_q(x+1) - \vartheta_q(x-1)\right) > 0, \tag{56}$$

$\forall x \in \mathbb{R}$; $\beta, q > 0$. Notice that $\mathcal{L}_q(\pm\infty) = 0$, so the $x$-axis is horizontal asymptote. Also it holds,

$${}_3\mathcal{L}_q(-x) = {}_3\mathcal{L}_{\frac{1}{q}}(x), \quad \forall\, x \in \mathbb{R}, \tag{57}$$

a deformed symmetry.

Next we have that

$${}_3\mathcal{L}_q'(x) = \frac{1}{4}\left(\vartheta_q'(x+1) - \vartheta_q'(x-1)\right), \quad \forall\, x \in \mathbb{R}. \tag{58}$$

Hence, ${}_3\mathcal{L}_q$ is striclty increasing over $\left(-\infty, \frac{\ln q}{\beta} - 1\right)$.

and strictly decreasing over $\left(\frac{\ln q}{\beta} + 1, +\infty\right)$.

Moreover, ${}_3\mathcal{L}_q$ is concave down over $\left[\frac{\ln q}{\beta} - 1, \frac{\ln q}{\beta} + 1\right]$, and strictly concave down over $\left(\frac{\ln q}{\beta} - 1, \frac{\ln q}{\beta} + 1\right)$.

Consequently ${}_3\mathcal{L}_q$ has a bell-type shape over $\mathbb{R}$. Of course it holds ${}_3\mathcal{L}_q''\left(\frac{\ln q}{\beta}\right) < 0$. The maximum value of ${}_3\mathcal{L}_q$ is

$${}_3\mathcal{L}_q\left(\frac{\ln q}{\beta}\right) = \frac{1 - e^{-\beta}}{2\left(1 + e^{-\beta}\right)}. \tag{59}$$

We give

**Theorem 23.** *We have that*

$$\sum_{i=-\infty}^{\infty} {}_3\mathcal{L}_q(x - i) = 1, \quad \forall\, x \in \mathbb{R}, \forall\, q, \beta > 0. \tag{60}$$

It follows

**Theorem 24.** *It holds*

$$\int_{-\infty}^{\infty} {}_3\mathcal{L}_q(x)\,dx = 1, \quad q, \beta > 0. \tag{61}$$

So that ${}_3\mathcal{L}_q$ is a density function on $\mathbb{R}$; $q, \beta > 0$.

We need the following result

**Theorem 25.** *Let* $0 < \alpha < 1$, *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$; $q, \beta > 0$. *Then*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} {}_3\mathcal{L}_q(nx - k) < \max\left\{q, \frac{1}{q}\right\} e^{2\beta} e^{-\beta n^{(1-\alpha)}} = K e^{-\beta n^{(1-\alpha)}} =: c_3(n, a), \tag{62}$$

*where* $K := \max\left\{q, \frac{1}{q}\right\} e^{2\beta}$.

We need,

**Theorem 26.** *Let* $x \in [a, b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$. *For* $q > 0$, *we consider the number* $\lambda_q > z_0 > 0$ *with* ${}_3\mathcal{L}_q(z_0) = {}_3\mathcal{L}\phi_q(0)$ *and* $\beta, \lambda_q > 1$. *Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_3\mathcal{L}_q(nx - k)} < \max\left\{\frac{1}{{}_3\mathcal{L}_q(\lambda_q)}, \frac{1}{{}_3\mathcal{L}_{\frac{1}{q}}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \Psi_3(q). \tag{63}$$

We make

**Remark 27.** *(i) We have that*

$$\lim_{n \to +\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_3\mathcal{L}_q(nx - k) \neq 1, \quad \textit{for at least some } x \in [a, b], \tag{64}$$

*where* $\beta, q > 0$.

*(ii) Let* $[a, b] \subset \mathbb{R}$. *For large* $n$ *we always have* $\lceil na \rceil \leq \lfloor nb \rfloor$. *Also* $a \leq \frac{k}{n} \leq b$, *iff* $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. *In general it holds*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_3\mathcal{L}_q(nx - k) \leq 1. \tag{65}$$

We introduce

$${}_3Z_q(x_1, ..., x_N) := {}_3Z_q(x) := \prod_{i=1}^{N} {}_3\mathcal{L}_q(x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^N, \ \beta, q > 0, \ N \in \mathbb{N}. \tag{66}$$

It has the properties:

(i) ${}_3Z_q(x) > 0, \ \forall \, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} {}_3Z_q(x - k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} {}_3Z_q(x_1 - k_1, ..., x_N - k_N) = 1, \tag{67}$$

where $k := (k_1, ..., k_n) \in \mathbb{Z}^N$, $\forall\ x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} {}_3Z_q\left(nx-k\right) = 1, \tag{68}$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} {}_3Z_q\left(x\right) dx = 1, \tag{69}$$

that is ${}_3Z_q$ is a multivariate density function.

For $0 < \beta^* < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_3Z_q\left(nx-k\right) =$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^{\beta^*}} \end{cases}}^{\lfloor nb \rfloor} {}_3Z_q\left(nx-k\right) + \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^{\beta^*}} \end{cases}}^{\lfloor nb \rfloor} {}_3Z_q\left(nx-k\right). \tag{70}$$

In the last two sums the counting is over disjoint vector sets of $k$'s, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^{\beta^*}}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^{\beta^*}}$, where $r \in \{1, ..., N\}$.

(v) We also have that

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^{\beta^*}} \end{cases}}^{\lfloor nb \rfloor} {}_3Z_q\left(nx-k\right) < Ke^{-\beta n^{\left(1-\beta^*\right)}} = c_3(n, \beta^*),\ \ 0 < \beta^* < 1, \tag{71}$$

with $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $x \in \prod_{i=1}^N \left[a_i, b_i\right]$.

(vi) Moreover

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_3Z_q\left(nx-k\right)} < \left(\Psi_3\left(q\right)\right)^N, \tag{72}$$

$\forall\ x \in \left(\prod_{i=1}^N \left[a_i, b_i\right]\right)$, $\ n \in \mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^{\beta^*}} \end{cases}}^{\infty} {}_3Z_q\left(nx-k\right) < Ke^{-\beta n^{\left(1-\beta^*\right)}}, \tag{73}$$

$0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $x \in \mathbb{R}^N$. Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_3Z_q\left(nx-k\right) \neq 1, \tag{74}$$

for at least some $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$. Let $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We define the multivariate averaged positive linear quasi-interpolation neural network operators ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$); $j = 1, 2, 3$:

$$ {}_jF_n(f, x_1, ..., x_N) := {}_jF_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) {}_jZ_q(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} {}_jZ_q(nx-k)} = \tag{75} $$

$$ \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} {}_j\mathcal{L}_q(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} {}_j\mathcal{L}_q(nx_i-k_i)\right)}. $$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

For the next we need, for $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ the first multivariate modulus of continuity

$$ \omega_1(f,h) := \sup_{\substack{x,y \in \prod_{i=1}^{N}[a_i,b_i] \\ \|x-y\|_\infty \leq h}} |f(x)-f(y)|, \quad h > 0. \tag{76} $$

It holds that

$$ \lim_{h\to 0} \omega_1(f,h) = 0. \tag{77} $$

We mention

**Theorem 28.** *(see [4], [5], [6]) Let $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0 < \beta^* < 1$, $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $N, n \in \mathbb{N}$ with $n^{1-\beta^*} > 2$; $j = 1, 2, 3$. Then*

*1)*

$$ |{}_jF_n(f,x) - f(x)| \leq (\Psi_j(q))^N \left[\omega_1\left(f, \frac{1}{n^\beta}\right) + 2c_j(n,\beta^*)\|f\|_\infty\right] =: \lambda_j, \tag{78} $$

*and*

*2)*

$$ \|{}_jF_n(f) - f\|_\infty \leq \lambda_j. \tag{79} $$

*We notice that $\lim_{n\to\infty} {}_jF_n(f) = f$, pointwise and uniformly.*

In this article we extend Theorem 28 to the fuzzy-random level.

# 3 Main Result

**About $p$-mean Approximation by Fuzzy-Random Perturbed Quasi-Interpolation Neural Network Operators**

All terms and assumptions here as in Sections 1, 2.

Let $f \in C_{\mathcal{FR}}^{U_p}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $1 \leq p < +\infty$, $n, N \in \mathbb{N}$, $0 < \beta < 1$, $\overrightarrow{x} \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $(X, \mathcal{B}, P)$ probability space, $s \in X$; $j = 1, 2, 3$.

We define the following multivariate fuzzy random perturbed quasi-interpolation linear neural network operators

$$\left({}_jF_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s) := \sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot \frac{{}_jZ_q\left(n\overrightarrow{x}-\overrightarrow{k}\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} {}_jZ_q\left(n\overrightarrow{x}-\overrightarrow{k}\right)}, \tag{80}$$

(see also (75)).

We present

**Theorem 29.** *Let* $f \in C_{\mathcal{FR}}^{U_p}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta^*<1$, $\overrightarrow{x} \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n, N \in \mathbb{N}$, *with* $n^{1-\beta^*}>2$, $1 \le p < +\infty$. *Assume that* $\int_X \left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^p P(ds) < \infty$; $j=1,2,3$. *Then*

*1)*

$$\left(\int_X D^p\left(\left({}_jF_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{p}} \le \tag{81}$$

$$(\Psi_j(q))^N\left\{\Omega_1\left(f,\frac{1}{n^{\beta^*}}\right)_{L^p} + 2c_j(n,\beta^*)\left(\int_X \left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^p P(ds)\right)^{\frac{1}{p}}\right\} =: \mu_j^{(\mathcal{FR})},$$

*2)*

$$\left\|\left(\int_X D^p\left(\left({}_jFn^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{p}}\right\|_{\infty,\left(\prod_{i=1}^{N}[a_i,b_i]\right)} \le \mu_j^{(\mathcal{FR})}, \tag{82}$$

*where* $(\Psi_j(q))^N$ *as in (26), (50), (72) and* $c_j(n,\beta^*)$ *as in (25), (49), (71).*

**Proof.** We notice that

$$D\left(f\left(\frac{\overrightarrow{k}}{n},s\right), f(\overrightarrow{x},s)\right) \le D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),\widetilde{o}\right) + D\left(f(\overrightarrow{x},s),\widetilde{o}\right) \tag{83}$$

$$\le 2D^*\left(f(\cdot,s),\widetilde{o}\right).$$

Hence

$$D^p\left(f\left(\frac{\overrightarrow{k}}{n},s\right), f(\overrightarrow{x},s)\right) \le 2^p D^{*p}\left(f(\cdot,s),\widetilde{o}\right), \tag{85}$$

and

$$\left(\int_X D^p\left(f\left(\frac{\overrightarrow{k}}{n},s\right), f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{p}} \le 2\left(\int_X \left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^p P(ds)\right)^{\frac{1}{p}}. \tag{86}$$

We observe that

$$D\left(\left({}_jF_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right) = \tag{87}$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{\overrightarrow{k}}{n},s\right) \odot \frac{{}_jZ_q(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} {}_jZ_q(nx-k)}, f(\overrightarrow{x},s)\odot 1\right) =$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{{}_jZ_q(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_jZ_q(nx-k)}, f(\overrightarrow{x},s)\odot\frac{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_jZ_q(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_jZ_q(nx-k)}\right)= \tag{88}$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{{}_jZ_q(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_jZ_q(nx-k)}, \sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *} f(\overrightarrow{x},s)\odot\frac{{}_jZ_q(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_jZ_q(nx-k)}\right)$$

$$\leq\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\frac{{}_jZ_q(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_jZ_q(nx-k)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f(\overrightarrow{x},s)\right). \tag{89}$$

So that

$$D\left(\left({}_jF_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s),f(\overrightarrow{x},s)\right)\leq$$

$$\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\frac{{}_jZ_q(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_jZ_q(nx-k)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f(\overrightarrow{x},s)\right)= \tag{90}$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty\leq\frac{1}{n^{\beta^*}}}}^{\lfloor nb\rfloor}\left(\frac{{}_jZ_q(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_jZ_q(nx-k)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f(\overrightarrow{x},s)\right)+$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty>\frac{1}{n^{\beta^*}}}}^{\lfloor nb\rfloor}\left(\frac{{}_jZ_q(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_jZ_q(nx-k)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f(\overrightarrow{x},s)\right).$$

Hence it holds

$$\left(\int_X D^p\left(\left({}_jF_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s),f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{p}}\leq \tag{91}$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty\leq\frac{1}{n^{\beta^*}}}}^{\lfloor nb\rfloor}\left(\frac{{}_jZ_q(nx-k)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_jZ_q(nx-k)}\right)\left(\int_X D^p\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{p}}+$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta^*}}}}^{\lfloor nb\rfloor}\left(\frac{{}_{j}Z_{q}\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_{j}Z_{q}\left(nx-k\right)}\right)\left(\int_{X}D^{p}\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)P\left(ds\right)\right)^{\frac{1}{p}}\leq$$

$$\left(\frac{1}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}{}_{j}Z_{q}\left(nx-k\right)}\right)\cdot\left\{\Omega_{1}^{(\mathcal{F})}\left(f,\frac{1}{n^{\beta^*}}\right)_{L^{p}}+\right. \tag{92}$$

$$\left.2\left(\int_{X}\left(D^{*}\left(f\left(\cdot,s\right),\widetilde{o}\right)\right)^{p}P\left(ds\right)\right)^{\frac{1}{p}}\left(\sum_{\substack{\overrightarrow{k}=\lceil na\rceil \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta^*}}}}^{\lfloor nb\rfloor}{}_{j}Z_{q}\left(nx-k\right)\right)\right\}$$

(by (25), (26); (49), (50); (71), (72)

$$\leq\left(\Psi_{j}(q)\right)^{N}\left\{\Omega_{1}^{(\mathcal{F})}\left(f,\frac{1}{n^{\beta^*}}\right)_{L^{p}}+2c_{j}\left(n,\beta^{*}\right)\left(\int_{X}\left(D^{*}\left(f\left(\cdot,s\right),\widetilde{o}\right)\right)^{p}P\left(ds\right)\right)^{\frac{1}{p}}\right\}. \tag{93}$$

We have proved claim. □

**Conclusion 30.** *By Theorem 29 we obtain the pointwise and uniform convergences with rates in the $p$-mean and $D$-metric of the operator ${}_{j}F_{n}^{\mathcal{FR}}$ to the unit operator for $f\in C_{\mathcal{FR}}^{U_{p}}\left(\prod_{i=1}^{N}\left[a_{i},b_{i}\right]\right)$, $j=1,2,3$.*

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