Trigonometric deduced L_p degree of approximation by various smooth singular integral operators

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Abstract

In this article we continue with the study of smooth Gauss-Weierstrass, Poisson-Cauchy and Trigonometric singular integral operators that started in [3], see there chapters 10-14. This time the foundation of our research is a trigonometric Taylor's formula. We establish the univariate L_p convergence of our operators to the unit operator with rates via Jackson type inequalities involving the first L_p modulus of continuity. Of interest here is a residual appearing term. Note that our operators are not positive.

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1 Introduction

We are motivated by [2], [3] chapters 10-14, and [4], [1]. We use a trigonometric new Taylor formula from [4], see also [1]. Here we consider some very general operators, the smooth Gauss-Weierstrass, Poisson-Cauchy and trigonometric singular integral operators over the real line and we study further their L_p, $p \geq 1$, convergencnce properties quantitatively. We establish related inequalities involving the first L_p, $p \geq 1$, modulus of continuity with respect to L_p, $p \geq 1$, norm. We provide detailed proofs.

For recent related works see [11], [14]-[17]. Other important articles on the topic are [6]-[10], [13].

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For the history of the topic we mention about the monograph [5] of 2012, which was the first complete source to deal exclusively with the classic theory of the approximation of singular integrals to the identity-unit operator. The authors there studied quantitatively the basic approximation properties of the general Picard, Gauss-Weierstrass and Poisson-Cauchy singular integral operators over the real line, which are not positive linear operators. In particular they studied the rate of convergence of these operators to the unit operator, as well as the related simultaneous approximation. This is given via inequalities and with the use of higher order modulus of smoothness of the high order derivative of the involved function. Some of these inequalities are proven to be sharp. Also, they studied the global smoothness preservation property of these operators. Furthermore they gave asymptotic expansions of Voronovskaya type for the error of approximation. They continued with the study of related properties of the general fractional Gauss-Weierstrass and Poisson-Cauchy singular integral operators. These properties were studied with respect to L_p norm, $1 \leq p \leq \infty$. The case of Lipschitz type functions approximation was studied separately and in detail. Furthermode they presented the corresponding general approximation theory of general singular integral operators with lots of applications to, the under focused till then, trigonometric singular integral.

2 Basics

By [1], [4], for $f \in C^2(\mathbb{R})$ and $a, x \in \mathbb{R}$, we have by trigonometric Taylor formula

$$f(x) - f(a) = f'(a)\sin(x-a) + 2f''(a)\sin^2\left(\frac{x-a}{2}\right) + \tag{1}$$

$$\int_a^x \left[\left(f''(t) + f(t)\right) - \left(f''(a) + f(a)\right)\right]\sin(x-t)\,dt.$$

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we set

$$\alpha_j := \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, ..., r. \\ 1 - \sum\limits_{j=1}^{r} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{2}$$

that is

$$\sum_{j=0}^{r} \alpha_j = 1. \tag{3}$$

Here we consider both $f, f'' \in L_p(\mathbb{R}) \cap C(\mathbb{R})$, $1 \leq p < \infty$.

Denote by

$$\omega_1(f, h)_p := \sup_{\substack{t \in \mathbb{R} \\ |t| \leq h}} \|f(x+t) - f(x)\|_{p,x}, \quad h > 0. \tag{4}$$

the first L_p modulus of continuity of f, $1 \leq p < \infty$.

I) We define the smooth Gauss-Weierstrass singular integral operators ([5]).

Let $f \in C^2(\mathbb{R})$, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$W_{r,\xi}(f;x) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) e^{-\frac{t^2}{\xi}} dt. \tag{5}$$

We assume that $W_{r,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. We will use also that

$$W_{r,\xi}(f;x) = \frac{1}{\sqrt{\pi\xi}} \sum_{j=0}^{r} \alpha_j \left(\int_{-\infty}^{\infty} f(x+jt) e^{-\frac{t^2}{\xi}} dt \right), \tag{6}$$

notice by $\frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{\xi}} dt = 1$ that $W_{r,\xi}(c;x) = c$, c constant and

$$W_{r,\xi}(f;x) - f(x) = \frac{1}{\sqrt{\pi\xi}} \left(\sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} (f(x+jt) - f(x)) e^{-\frac{t^2}{\xi}} dt \right). \tag{7}$$

We set

$$\Delta_2(x) := W_{r,\xi}(f;x) - f(x) - f''(x) \sum_{j=0}^{r} \alpha_j \left(1 - e^{-\frac{j^2}{4}\xi} \right), \quad x \in \mathbb{R}; \tag{8}$$

$j = 0, 1, ..., r \in \mathbb{N}$.

II) We define the smooth Poisson-Cauchy singular integral operators ([5]).

Let $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$ and $f \in C^2(\mathbb{R})$. We define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$M_{r,\xi}(f;x) = W \int_{-\infty}^{\infty} \frac{\sum_{j=0}^{r} \alpha_j f(x+jt)}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\beta}} dt, \tag{9}$$

where the constant is defined as

$$W = \frac{\Gamma(\beta) \alpha \xi^{2\alpha\beta - 1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)}. \tag{10}$$

We assume that $M_{r,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. We will use also that

$$M_{r,\xi}(f;x) = W \sum_{j=0}^{r} \alpha_j \left(\int_{-\infty}^{\infty} f(x+jt) \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\beta}} dt \right). \tag{11}$$

We notice by $W \int_{-\infty}^{\infty} \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^{\beta}} dt = 1$ that $M_{r,\xi}(c;x) = c$, c constant and

$$M_{r,\xi}(f;x) - f(x) = W \left(\sum_{j=0}^{r} \alpha_j \int_{-\infty}^{\infty} [f(x+jt) - f(x)] \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\beta}} dt \right). \tag{12}$$

We set

$$\Delta_3(x) := M_{r,\xi}(f;x) - f(x) - 4f''(x) \frac{\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)} \sum_{j=0}^{r} \alpha_j \int_0^{\infty} \frac{\sin^2\left(\frac{j\xi}{2}t\right)}{\left(1+t^{2\alpha}\right)^{\beta}} dt, \tag{13}$$

$\xi > 0$, $x \in \mathbb{R}$; $\beta > \frac{1}{2\alpha}$, $\alpha \in \mathbb{N}$; $j = 0, 1, ..., r \in \mathbb{N}$.

III) We define the smooth Trigonometric singular integral operators ([5]) as follows:

Let $\xi > 0$, $f \in C^2(\mathbb{R})$, $x \in \mathbb{R}$, $\beta \in \mathbb{N}$; we set

$$T_{r,\xi}(f;x) := \frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt, \tag{14}$$

where

$$\lambda := \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt = 2\xi^{1-2\beta} \int_0^{\infty} \left(\frac{\sin t}{t} \right)^{2\beta} dt \tag{15}$$

(by [12], p. 210, item 1033)

$$= 2\xi^{1-2\beta}\pi(-1)^{\beta}\beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)!(\beta+k)!}.$$

Denote

$$\lambda_1 := 2\pi(-1)^{\beta}\beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)!(\beta+k)!}; \tag{16}$$

that is

$$\lambda = \lambda_1 \xi^{1-2\beta}. \tag{17}$$

We suppose that $T_{r,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. Clearly, again it is

$$\frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{t}{\xi}\right)}{t} \right)^{2\beta} dt = 1, \tag{18}$$

and $T_{r,\xi}(c;x) = c$, c constant.

We set

$$\Delta_4(x) := T_{r,\xi}(f;x) - f(x) - 4f''(x)\lambda_1^{-1} \sum_{j=0}^{r} \alpha_j \int_0^{\infty} \sin^2\left(\frac{j\xi}{2}t\right) \left(\frac{\sin t}{t}\right)^{2\beta} dt, \tag{19}$$

where $\xi > 0$, $x \in \mathbb{R}$; $\beta \in \mathbb{N} - \{1, 2\}$; $j = 0, 1, ..., r \in \mathbb{N}$.

We need the following set of calculations.

Remark 1 *By (1) we get that*

$$f(x+jt)-f(x)=f'(x)\sin(jt)+2f''(x)\sin^2\left(\frac{jt}{2}\right)+$$

$$\int_x^{x+jt}\left[(f''(s)+f(s))-(f''(x)+f(x))\right]\sin(x+jt-s)\,ds, \tag{20}$$

or better

$$f(x+jt)-f(x)=f'(x)\sin(jt)+2f''(x)\sin^2\left(\frac{jt}{2}\right)+$$

$$\int_0^{jt}\left[(f''(x+z)+f(x+z))-(f''(x)+f(x))\right]\sin(jt-z)\,dz. \tag{21}$$

Furthermore, it holds

$$\sum_{j=0}^{r}\alpha_j\left[f(x+jt)-f(x)\right]=$$

$$f'(x)\sum_{j=0}^{r}\alpha_j\sin(jt)+2f''(x)\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{jt}{2}\right)+$$

$$\sum_{j=0}^{r}\alpha_j\int_0^{jt}\left[(f''(x+z)+f(x+z))-(f''(x)+f(x))\right]\sin(jt-z)\,dz, \tag{22}$$

or better

$$\sum_{j=0}^{r}\alpha_j\left[f(x+jt)-f(x)\right]=$$

$$f'(x)\sum_{j=0}^{r}\alpha_j\sin(jt)+2f''(x)\sum_{j=0}^{r}\alpha_j\sin^2\left(\frac{jt}{2}\right)+$$

$$\sum_{j=0}^{r}\alpha_j j\int_0^{t}\left[(f''(x+jw)+f(x+jw))-(f''(x)+f(x))\right]\sin j(t-w)\,dw. \tag{23}$$

Call

$$R:=R(t):=$$

$$\sum_{j=0}^{r}\alpha_j j\int_0^{t}\left[(f''(x+jw)+f(x+jw))-(f''(x)+f(x))\right]\sin j(t-w)\,dw, \tag{24}$$

$\forall\ t\in\mathbb{R}$.

We notice that

$$\Delta_2^*(x):=W_{r,\xi}(f;x)-f(x)-f'(x)\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin(jt)\,e^{-\frac{t^2}{\xi}}dt\right)$$

$$-2f''(x)\sum_{j=0}^{r}\alpha_j\frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{t^2}{\xi}}dt\right)=$$

$$\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}R(t)\,e^{-\frac{t^2}{\xi}}dt, \tag{25}$$

and

$$\Delta_3^*(x):=M_{r,\xi}(f;x)-f(x)-f'(x)\sum_{j=0}^{r}\alpha_jW\left(\int_{-\infty}^{\infty}\sin(jt)\frac{1}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt\right)$$

$$-2f''(x)\sum_{j=0}^{r}\alpha_jW\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)\frac{1}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt\right)=$$

$$W\int_{-\infty}^{\infty}R(t)\frac{1}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt, \tag{26}$$

and

$$\Delta_4^*(x):=T_{r,\xi}(f;x)-f(x)-f'(x)\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\sin(jt)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)$$

$$-2f''(x)\sum_{j=0}^{r}\alpha_j\frac{1}{\lambda}\left(\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)=$$

$$\frac{1}{\lambda}\int_{-\infty}^{\infty}R(t)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt. \tag{27}$$

Clearly, it holds

$$\int_{-\infty}^{\infty}\sin(jt)\,e^{-\frac{t^2}{\xi}}dt=0, \tag{28}$$

$$\int_{-\infty}^{\infty}\sin(jt)\frac{1}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt=0, \tag{29}$$

and

$$\int_{-\infty}^{\infty}\sin(jt)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt=0, \tag{30}$$

for all $j=0,1,...,r$.

Furthermore, we have that

$$\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{t^2}{\xi}}dt=2\int_{0}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{t^2}{\xi}}dt, \tag{31}$$

$$\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)\frac{1}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt=2\int_0^{\infty}\sin^2\left(\frac{jt}{2}\right)\frac{1}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt, \tag{32}$$

$$\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt=2\int_0^{\infty}\sin^2\left(\frac{jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt, \tag{33}$$

$j=0,1,...,r.$

More specifically we have that

$$\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{t^2}{\xi}}dt=2\int_0^{\infty}\sin^2\left(\frac{jt}{2}\right)e^{-\frac{t^2}{\xi}}dt=$$

$$2\sqrt{\xi}\int_0^{\infty}\sin^2\left(\left(\frac{j\sqrt{\xi}}{2}\right)\frac{t}{\sqrt{\xi}}\right)e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2}d\frac{t}{\sqrt{\xi}}$$

($\frac{t}{\sqrt{\xi}}=:x$ *and* $\frac{j\sqrt{\xi}}{2}=:\beta_1$)

$$=2\sqrt{\xi}\int_0^{\infty}\sin^2\left(\beta_1 x\right)e^{-x^2}dx \tag{34}$$

$$=\sqrt{\xi}\frac{1}{2}\sqrt{\pi}e^{-\beta_1^2}\left(e^{\beta_1^2}-1\right)$$

$$=\frac{\sqrt{\xi}\sqrt{\pi}}{2}e^{-\frac{j^2}{4}\xi}\left(e^{\frac{j^2}{4}\xi}-1\right).$$

Consequently we derive

$$\Delta_2^*(x)\overset{(by\ (8))}{=}W_{r,\xi}(f,x)-f(x)-f''(x)\sum_{j=0}^{r}\alpha_j\left(1-e^{-\frac{j^2}{4}\xi}\right)=\Delta_2(x). \tag{35}$$

Furthermore, we obtain

$$W\int_{-\infty}^{\infty}\sin^2\left(\frac{jt}{2}\right)\frac{1}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt\le W\int_{-\infty}^{\infty}\frac{1}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt=1, \tag{36}$$

$\beta>\frac{1}{2\alpha}$, $\alpha\in\mathbb{N}$; $j=0,1,...,r.$

Hence we get that

$$\Delta_3^*(x)\overset{(13)}{=}M_{r,\xi}(f;x)-f(x)-$$

$$4f''(x)\frac{\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\sum_{j=0}^{r}\alpha_j\int_0^{\infty}\frac{\sin^2\left(\frac{j\xi t}{2}\right)}{\left(1+t^{2\alpha}\right)^{\beta}}dt=\Delta_3(x), \tag{37}$$

$\xi>0,\ x\in\mathbb{R};\ \beta>\frac{1}{2\alpha},\ \alpha\in\mathbb{N};\ j=0,1,...,r\in\mathbb{N}.$

Similarly, we observe that

$$\lambda^{-1}\int_{-\infty}^{\infty}\sin^{2}\left(\frac{jt}{2}\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\leq\lambda^{-1}\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt=1. \quad (38)$$

Consequently it holds

$$\Delta_{4}^{*}(x)=T_{r,\xi}(f;x)-f(x)-$$
$$4f''(x)\lambda_{1}^{-1}\sum_{j=0}^{r}\alpha_{j}\int_{0}^{\infty}\sin^{2}\left(\frac{j\xi}{2}t\right)\left(\frac{\sin t}{t}\right)^{2\beta}dt=\Delta_{4}(x), \quad (39)$$

where $\xi>0$, $x\in\mathbb{R}$; $\beta\in\mathbb{N}-\{1,2\}$; $j=0,1,...,r\in\mathbb{N}$.

Above it is $\lambda_{1}\overset{(17)}{=}\lambda\xi^{2\beta-1}$.

3 Main Results

We present our first main result, about Gauss-Weierstrass operators, L_p approximation, $p>1$.

Theorem 2 *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\xi>0$, *and both* $f,f''\in L_{p}(\mathbb{R})\cap C(\mathbb{R})$, $x\in\mathbb{R}$. *Then*

$$\|\Delta_{2}(x)\|_{p}=\left\|W_{r,\xi}(f)-f-f''\left(\sum_{j=0}^{r}\alpha_{j}\left(1-e^{-\frac{j^{2}}{4}\xi}\right)\right)\right\|_{p}\leq$$
$$\frac{2}{(\sqrt{\pi})^{\frac{1}{p}}}\left(\frac{2}{q}\right)^{\frac{1}{2q}}\left(\frac{r+1}{q+1}\right)^{\frac{1}{q}}\omega_{1}\left(f''+f,\sqrt{\xi}\right)_{p}\xi \quad (40)$$
$$\left\{\sum_{j=0}^{r}|\alpha_{j}|^{p}j^{2p}\left\{\left[\frac{1}{(2p+1)}+\left(\frac{p}{2}\right)^{-(2p+1)}\Gamma(2p+1)\right]+\right.\right.$$
$$\left.\left.\frac{j^{p}}{(p+1)}\left[\frac{1}{(3p+1)}+\left(\frac{p}{2}\right)^{-(3p+1)}\Gamma(3p+1)\right]\right\}\right\}^{\frac{1}{p}}:=\theta_{1}(\xi)\to0,\ \ as\ \xi\to0.$$

Above Γ *stands for the gamma function.*

Proof. From (24) let

$$I:=\int_{0}^{t}\left[\left(f''(x+jw)+f(x+jw)\right)-\left(f''(x)+f(x)\right)\right]\sin j(t-w)\,dw,\ \ \forall\ t\in\mathbb{R}. \quad (41)$$

For $t<0$, we have that

$$|I| = \left| \int_0^t [(f''(x+jw)+f(x+jw)) - (f''(x)+f(x))] \sin j(t-w)\, dw \right| \leq$$

$$\int_0^t |(f''(x+jw)+f(x+jw)) - (f''(x)+f(x))| \, |\sin j(t-w)| \, dw \leq \tag{42}$$

$$j \int_t^0 |(f''(x+jw)+f(x+jw)) - (f''(x)+f(x))| \, (w-t)\, dw =$$

$$-j \int_t^0 |(f''(x-j(-w))+f(x-j(-w))) - (f''(x)+f(x))| \, (-t-(-w))\, d(-w)$$

$(t \leq w \leq 0 \Rightarrow -t \geq -w =: \theta \geq 0)$

$$= -j \int_{-t}^0 |(f''(x-j\theta)+f(x-j\theta)) - (f''(x)+f(x))| \, (-t-\theta)\, d\theta =$$

$$j \int_0^{-t} |(f''(x-j\theta)+f(x-j\theta)) - (f''(x)+f(x))| \, (-t-\theta)\, d\theta = \tag{43}$$

$$j \int_0^{|t|} |(f''(x+sign(t)\, j\theta)+f(x+sign(t)\, j\theta)) - (f''(x)+f(x))| \, (|t|-\theta)\, d\theta.$$

So, we have proved that

$$|I| \leq j \int_0^{|t|} |(f''(x+sign(t)\, j\theta)+f(x+sign(t)\, j\theta)) - (f''(x)+f(x))| \, (|t|-\theta)\, d\theta, \tag{44}$$

$\forall\, t \in \mathbb{R}$,

and by (24),

$$|R(t)| \leq \sum_{j=0}^r |\alpha_j|\, j^2$$

$$\int_0^{|t|} |(f''(x+jsign(t)\,\theta)+f(x+jsign(t)\,\theta)) - (f''(x)+f(x))| \, (|t|-\theta)\, d\theta, \tag{45}$$

$\forall\, t \in \mathbb{R}$.

Let now $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

We observe that

$$|R(t)| \leq \sum_{j=0}^r |\alpha_j|\, j^2 \tag{46}$$

$$\left(\int_0^{|t|} |(f''(x+jsign(t)\,\theta)+f(x+jsign(t)\,\theta)) - (f''(x)+f(x))|^p \, d\theta \right)^{\frac{1}{p}}$$

$$\left(\int_0^{|t|} (|t|-\theta)^q \, d\theta\right)^{\frac{1}{q}} = \sum_{j=0}^{r} |\alpha_j| \, j^2$$

$$\left(\int_0^{|t|} |(f''(x+jsign(t)\theta) + f(x+jsign(t)\theta)) - (f''(x)+f(x))|^p \, d\theta\right)^{\frac{1}{p}}$$

$$\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} =$$

$$\left(\int_0^{|t|} |(f''(x+jsign(t)\theta) + f(x+jsign(t)\theta)) - (f''(x)+f(x))|^p \, d\theta\right)^{\frac{1}{p}} \Bigg\}$$

$$\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \leq \Bigg\{ (r+1)^{\frac{1}{q}} \left(\sum_{j=0}^{r} |\alpha_j|^p \, j^{2p}\right. \tag{47}$$

$\left(0 < \frac{1}{p} < 1\right)$

$$\left.\int_0^{|t|} |(f''(x+jsign(t)\theta) + f(x+jsign(t)\theta)) - (f''(x)+f(x))|^p \, d\theta\right)^{\frac{1}{p}} \Bigg\}$$

$$\frac{|t|^{q+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}.$$

Hence, we find that

$$|R(t)|^p \leq \left(\frac{r+1}{q+1}\right)^{\frac{p}{q}} \left[\sum_{j=0}^{r} |\alpha_j|^p \, j^{2p}\right.$$

$$\left.\left(\int_0^{|t|} |(f''(x+jsign(t)\theta) + f(x+jsign(t)\theta)) - (f''(x)+f(x))|^p \, d\theta\right)\right] |t|^{\frac{(q+1)p}{q}}. \tag{48}$$

By (25) we have

$$\Delta_2^*(x) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} R(t) \, e^{-\frac{t^2}{\xi}} \, dt. \tag{49}$$

We observe that

$$\int_{-\infty}^{\infty} |\Delta_2^*(x)|^p \, dx = \frac{1}{\left(\sqrt{\pi\xi}\right)^p} \int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} R(t) \, e^{-\frac{t^2}{\xi}} \, dt\right|^p dx \leq$$

$$\frac{1}{\left(\sqrt{\pi\xi}\right)^p} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)| \, e^{-\frac{t^2}{\xi}} \, dt\right)^p dx =$$

$$\frac{1}{\left(\sqrt{\pi\xi}\right)^p}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|\,e^{-\frac{t^2}{2\xi}}e^{-\frac{t^2}{2\xi}}dt\right)^p dx \leq \tag{50}$$

$$\frac{1}{\left(\sqrt{\pi\xi}\right)^p}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|^p\,e^{-\frac{pt^2}{2\xi}}dt\right)\left(\int_{-\infty}^{\infty}e^{-\frac{qt^2}{2\xi}}dt\right)^{\frac{p}{q}}dx =$$

$$\left(\frac{2\pi\xi}{q}\right)^{\frac{p}{2q}}\frac{1}{(\pi\xi)^{\frac{p}{2}}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|^p\,e^{-\frac{pt^2}{2\xi}}dt\right)dx =$$

$$\left(\frac{2}{q}\right)^{\frac{p}{2q}}\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|^p\,e^{-\frac{pt^2}{2\xi}}dt\right)dx \overset{(48)}{\leq}$$

$$\left(\frac{2}{q}\right)^{\frac{p}{2q}}\frac{1}{\sqrt{\pi\xi}}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\int_{-\infty}^{\infty}\left[\sum_{j=0}^{r}|\alpha_j|^p\,j^{2p}\right.$$

$$\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}|(f''(x+jsign(t)\theta)+f(x+jsign(t)\theta))-(f''(x)+f(x))|^p\,d\theta\right)\right)$$

$$\left.|t|^{\frac{(q+1)p}{q}}\,e^{-\frac{pt^2}{2\xi}}\right]dx$$

(call

$$c_1 := \left(\frac{2}{q}\right)^{\frac{p}{2q}}\frac{1}{\sqrt{\pi\xi}}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\;) \tag{51}$$

$$= c_1\left\{\sum_{j=0}^{r}|\alpha_j|^p\,j^{2p}\int_{-\infty}^{\infty}\left[\int_{0}^{|t|}\right.\right.$$

$$\left(\int_{-\infty}^{\infty}|(f''(x+jsign(t)\theta)+f(x+jsign(t)\theta))-(f''(x)+f(x))|^p\,dx\right)$$

$$\left.\left.d\theta\right]|t|^{\frac{(q+1)p}{q}}\,e^{-\frac{pt^2}{2\xi}}\right\}dt \leq$$

$(\xi > 0)$

$$c_1\left\{\sum_{j=0}^{r}|\alpha_j|^p\,j^{2p}\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\omega_1\left(f''+f,\sqrt{\xi}\frac{j\theta}{\sqrt{\xi}}\right)_p^p d\theta\right)|t|^{\frac{(q+1)p}{q}}\,e^{-\frac{pt^2}{2\xi}}dt\right\} \leq$$

$$\omega_1\left(f''+f,\sqrt{\xi}\right)_p^p c_1$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p\,j^{2p}\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\left(1+\frac{j}{\sqrt{\xi}}\theta\right)^p d\theta\right)|t|^{\frac{(q+1)p}{q}}\,e^{-\frac{pt^2}{2\xi}}dt\right\} \leq \tag{52}$$

$$c_1\omega_1\left(f''+f,\sqrt{\xi}\right)_p^p$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}2^{p-1}\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(1+\frac{j^p}{\sqrt{\xi}^p}\theta^p\right)d\theta\right)|t|^{\frac{(q+1)p}{q}}e^{-\frac{pt^2}{2\xi}}dt\right\}=$$

$$c_1\omega_1\left(f''+f,\sqrt{\xi}\right)_p^p$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}2^{p-1}\int_{-\infty}^{\infty}\left(|t|+\frac{j^p}{\sqrt{\xi}^p}\frac{|t|^{p+1}}{(p+1)}\right)|t|^{\frac{(q+1)p}{q}}e^{-\frac{pt^2}{2\xi}}dt\right\}=$$

$$2^pc_1\omega_1\left(f''+f,\sqrt{\xi}\right)_p^p$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\int_0^{\infty}\left(t+\frac{j^p}{\sqrt{\xi}^p}\frac{t^{p+1}}{(p+1)}\right)t^{(q+1)(p-1)}e^{-\frac{pt^2}{2\xi}}dt\right\}=$$

$$2^pc_1\omega_1\left(f''+f,\sqrt{\xi}\right)_p^p\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\right. \tag{53}$$

$$\left.\left[\int_0^{\infty}t^{(q+1)(p-1)+1}e^{-\frac{pt^2}{2\xi}}dt+\frac{j^p}{\sqrt{\xi}^p(p+1)}\int_0^{\infty}t^{(q+1)(p-1)+(p+1)}e^{-\frac{pt^2}{2\xi}}dt\right]\right\}=$$

$$2^pc_1\omega_1\left(f''+f,\sqrt{\xi}\right)_p^p$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left[\left(\sqrt{\xi}\right)^{(q+1)(p-1)+2}\int_0^{\infty}\left(\frac{t}{\sqrt{\xi}}\right)^{(q+1)(p-1)+1}e^{-\frac{p}{2}\left(\frac{t}{\sqrt{\xi}}\right)^2}d\left(\frac{t}{\sqrt{\xi}}\right)+\right.\right.$$

$$\left.\left.\frac{j^p}{\sqrt{\xi}^p(p+1)}\left(\sqrt{\xi}\right)^{(q+1)(p-1)+(p+2)}\int_0^{\infty}\left(\frac{t}{\sqrt{\xi}}\right)^{(q+1)(p-1)+(p+1)}e^{-\frac{p}{2}\left(\frac{t}{\sqrt{\xi}}\right)^2}d\left(\frac{t}{\sqrt{\xi}}\right)\right]\right\} \tag{54}$$

$$=2^pc_1\omega_1\left(f''+f,\sqrt{\xi}\right)_p^p$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left[\left(\sqrt{\xi}\right)^{(q+1)(p-1)+2}\int_0^{\infty}x^{(q+1)(p-1)+1}e^{-\frac{p}{2}x^2}dx+\right.\right.$$

$$\left.\left.\frac{j^p}{(p+1)}\left(\sqrt{\xi}\right)^{(q+1)(p-1)+2}\int_0^{\infty}x^{(q+1)(p-1)+(p+1)}e^{-\frac{p}{2}x^2}dx\right]\right\} \tag{55}$$

(above it is $(q+1)(p-1)+1=qp-q+p>0$,
$(q+1)(p-1)+(p+1)=qp-q+2p>0$)

$$=2^pc_1\omega_1\left(f''+f,\sqrt{\xi}\right)_p^p$$

$$\left\{\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}\left[\xi^{\left(\frac{qp-q+p+1}{2}\right)}\int_0^\infty x^{(qp-q+p)}e^{-\frac{p}{2}x^2}dx+\right.\right.$$

$$\left.\left.\frac{j^p}{(p+1)}\xi^{\frac{(qp-q+p+1)}{2}}\int_0^\infty x^{(qp-q+2p)}e^{-\frac{p}{2}x^2}dx\right]\right\} =: (*)\,. \tag{56}$$

We estimate the integrals:

$$\int_0^\infty x^{(qp-q+p)}e^{-\frac{p}{2}x^2}dx = \int_0^1 x^{(qp-q+p)}e^{-\frac{p}{2}x^2}dx + \int_1^\infty x^{(qp-q+p)}e^{-\frac{p}{2}x^2}dx \leq$$

$$\int_0^1 x^{(qp-q+p)}dx + \int_1^\infty x^{(qp-q+p)}e^{-\frac{p}{2}x}dx \leq$$

$$\frac{1}{(qp-q+p+1)} + \int_0^\infty x^{(qp-q+p)}e^{-\frac{p}{2}x}dx =$$

$$\frac{1}{(qp-q+p+1)} + \left(\frac{p}{2}\right)^{-qp+q-p-1}\Gamma\left(qp-q+p+1\right), \tag{57}$$

and similarly, it holds

$$\int_0^\infty x^{(qp-q+2p)}e^{-\frac{p}{2}x^2}dx \leq \frac{1}{(qp-q+2p+1)} + \int_0^\infty x^{(qp-q+2p)}e^{-\frac{p}{2}x}dx =$$

$$\frac{1}{(qp-q+2p+1)} + \left(\frac{p}{2}\right)^{-qp+q-2p-1}\Gamma\left(qp-q+2p+1\right), \tag{58}$$

where Γ is the gamma function.

We found that

$$\int_{-\infty}^{\infty}\left|\Delta_2^*(x)\right|^p dx \leq 2^p c_1 \omega_1\left(f''+f,\sqrt{\xi}\right)_p^p \xi^{\left(\frac{qp-q+p+1}{2}\right)}$$

$$\left\{\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}\left\{\left[\frac{1}{(qp-q+p+1)} + \left(\frac{p}{2}\right)^{-qp+q-p-1}\Gamma\left(qp-q+p+1\right)\right]\right.\right. \tag{59}$$

$$\left.\left.+\frac{j^p}{(p+1)}\left[\frac{1}{(qp-q+2p+1)} + \left(\frac{p}{2}\right)^{-qp+q-2p-1}\Gamma\left(qp-q+2p+1\right)\right]\right]\right\} =$$

$$2^p\left(\frac{2}{q}\right)^{\frac{p}{2q}}\frac{1}{\sqrt{\pi}}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\omega_1\left(f''+f,\sqrt{\xi}\right)_p^p \xi^{\frac{(qp-q+p)}{2}}$$

$$\left\{\sum_{j=0}^{r}\left|\alpha_j\right|^p j^{2p}\left\{\left[\frac{1}{(qp-q+p+1)} + \left(\frac{p}{2}\right)^{-qp+q-p-1}\Gamma\left(qp-q+p+1\right)\right]\right.\right.$$

$$\left.\left.+\frac{j^p}{(p+1)}\left[\frac{1}{(qp-q+2p+1)} + \left(\frac{p}{2}\right)^{-qp+q-2p-1}\Gamma\left(qp-q+2p+1\right)\right]\right]\right\} = \tag{60}$$

$$\frac{2^p}{\sqrt{\pi}}\left(\frac{2}{q}\right)^{\frac{p}{2q}}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\omega_1\left(f''+f,\sqrt{\xi}\right)_p^p\xi^p$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left\{\left[\frac{1}{(2p+1)}+\left(\frac{p}{2}\right)^{-(2p+1)}\Gamma(2p+1)\right]\right.\right.$$

$$\left.\left.+\frac{j^p}{(p+1)}\left[\frac{1}{(3p+1)}+\left(\frac{p}{2}\right)^{-(3p+1)}\Gamma(3p+1)\right]\right]\right\}. \qquad (61)$$

The proof of the theorem is now completed. ■

We give the following.

Corollary 3 *(to Theorem 2) It holds ($p>1$)*

$$\|W_{r,\xi}(f)-f\|_p\leq\theta_1(\xi)+$$

$$\|f''\|_p\left(\sum_{j=0}^{r}|\alpha_j|\left(1-e^{-\frac{j^2}{4}\xi}\right)\right)\to 0,\ \text{as } \xi\to 0. \qquad (62)$$

Above $\theta_1(\xi)$ is as in (40).

It follows L_p, $p>1$, approximation by Poisson-Cauchy operators.

Theorem 4 *Let $\alpha\in\mathbb{N}$, $\beta>\frac{4}{\alpha}$, $\xi>0$, $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $2\left(1+\frac{1}{\alpha\beta}\right)\geq q>\frac{3p+1}{\alpha\beta}$. Call*

$$\delta_2:=\frac{\Gamma(\beta)\Gamma\left(\frac{q\beta}{2}-\frac{1}{2\alpha}\right)^{\frac{1}{q}}(r+1)^{\frac{1}{q}}}{\Gamma^{\frac{1}{p}}\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)\Gamma\left(\frac{q\beta}{2}\right)(q+1)^{\frac{1}{q}}}.$$

Here $f,f''\in L_p(\mathbb{R})\cap C(\mathbb{R})$. Then

$$\|\Delta_3\|_p=\left\|M_{r,\xi}(f)-f-4f''\frac{\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\sum_{j=0}^{r}\alpha_j\left(\int_0^\infty\frac{\sin^2\left(\frac{j\xi t}{2}\right)}{(1+t^{2\alpha})^\beta}dt\right)\right\|_p$$

$$\leq 2^{\frac{1}{q}}\delta_2\omega_1(f''+f,\xi)_p\,\xi^{2+\alpha\beta(2-q)} \qquad (63)$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left[\Gamma\left(\frac{2p+1}{2\alpha}\right)\Gamma\left(\frac{q\beta}{2}-\left(\frac{2p+1}{2\alpha}\right)\right)+\right.\right.$$

$$\left.\left.\frac{j^p}{(p+1)}\Gamma\left(\frac{3p+1}{2\alpha}\right)\Gamma\left(\frac{q\beta}{2}-\left(\frac{3p+1}{2\alpha}\right)\right)\right]\right\}^{\frac{1}{p}}:=\theta_2(\xi)\to 0,\ \text{as } \xi\to 0.$$

Proof. Equations / inequalities (41)-(48) are all used here.

By (26) we have that

$$\Delta_3^*(x) = W \int_{-\infty}^{\infty} R(t) \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\beta}} dt, \tag{64}$$

where W as in (10); $\alpha \in \mathbb{N}$, $\beta > \frac{4}{\alpha}$; $\xi > 0$.

We observe that $(p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1)$

$$\int_{-\infty}^{\infty} |\Delta_3^*(x)|^p \, dx = W^p \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} R(t) \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\beta}} dt \right|^p dx \leq$$

$$W^p \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)| \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\beta}} dt \right)^p dx =$$

$$W^p \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)| \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{\beta}{2}}} \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{\beta}{2}}} dt \right)^p dx \leq \tag{65}$$

$$W^p \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)|^p \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{p\beta}{2}}} dt \right) \left(\int_{-\infty}^{\infty} \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{q\beta}{2}}} dt \right)^{\frac{p}{q}} dx =$$

$$\left[\frac{\alpha \Gamma(\beta)^p \, \Gamma\left(\frac{q\beta}{2} - \frac{1}{2\alpha}\right)^{p-1} \xi^{\alpha\beta p - 1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma^p\left(\beta - \frac{1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2}\right)^{p-1}} \right] \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)|^p \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{p\beta}{2}}} dt \right) dx =$$

(call

$$\rho := \frac{\alpha \Gamma(\beta)^p \, \Gamma\left(\frac{q\beta}{2} - \frac{1}{2\alpha}\right)^{p-1} \xi^{\alpha\beta p - 1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma^p\left(\beta - \frac{1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2}\right)^{p-1}})$$

$$\rho \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)|^p \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{p\beta}{2}}} dt \right) dx \leq \tag{66}$$

$$\rho \left(\frac{r+1}{q+1} \right)^{(p-1)} \left[\sum_{j=0}^{r} |\alpha_j|^p \, j^{2p} \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \right.\right.\right.$$

$$\left(\int_0^{|t|} \left| \left(f''(x + j\,sign(t)\,\theta) + f(x + j\,sign(t)\,\theta)\right) - \left(f''(x) + f(x)\right) \right|^p d\theta \right)$$

$$\left.\left.\left. |t|^{\frac{(q+1)p}{q}} \frac{dt}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{p\beta}{2}}} \right) dx \right) \right]$$

(call

$$c_2 := \rho \left(\frac{r+1}{q+1}\right)^{(p-1)})$$

$$= c_2 \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left(\int_{-\infty}^{\infty} \left[\int_0^{|t|} \right.\right.\right. \tag{67}$$

$$\left(\int_{-\infty}^{\infty} |(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))|^p dx \right)$$

$$\left.\left.\left. d\theta \right] |t|^{(q+1)(p-1)} \frac{dt}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{p\beta}{2}}} \right) \right] \leq$$

$$c_2 \left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_1 \left(f'' + f, \xi \frac{j\theta}{\xi} \right)_p^p d\theta \right) |t|^{(q+1)(p-1)} \frac{dt}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{p\beta}{2}}} \right] \leq \tag{68}$$

$$c_2 \omega_1 (f'' + f, \xi)_p^p$$

$$\left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1 + \frac{j}{\xi}\theta\right)^p d\theta \right) |t|^{(q+1)(p-1)} \frac{dt}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{p\beta}{2}}} \right] \leq$$

$$2^{p-1} c_2 \omega_1 (f'' + f, \xi)_p^p$$

$$\left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(1 + \frac{j^p}{\xi^p}\theta^p\right) d\theta \right) |t|^{(q+1)(p-1)} \frac{dt}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{p\beta}{2}}} \right] = \tag{69}$$

$$2^{p-1} c_2 \omega_1 (f'' + f, \xi)_p^p$$

$$\left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \int_{-\infty}^{\infty} \left(|t| + \frac{j^p}{\xi^p} \frac{|t|^{p+1}}{(p+1)} \right) |t|^{(q+1)(p-1)} \frac{dt}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{p\beta}{2}}} \right] =$$

$$2^{p} c_2 \omega_1 (f'' + f, \xi)_p^p$$

$$\left[\sum_{j=0}^{r} |\alpha_j|^p j^{2p} \int_0^{\infty} \left(t + \frac{j^p}{\xi^p} \frac{t^{p+1}}{(p+1)} \right) \frac{t^{(q+1)(p-1)}}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{q\beta}{2}}} dt \right] =$$

$$2^{p} c_2 \omega_1 (f'' + f, \xi)_p^p$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\int_0^{\infty} \frac{t^{(q+1)(p-1)+1}}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{q\beta}{2}}} dt + \frac{j^p}{\xi^p (p+1)} \int_0^{\infty} \frac{t^{(q+1)(p-1)+(p+1)}}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^{\frac{q\beta}{2}}} dt \right] \right\} = \tag{70}$$

$$2^{p} c_2 \omega_1 (f'' + f, \xi)_p^p$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\xi^{(q+1)(p-1)+2-\alpha\beta q} \int_0^\infty \frac{\left(\frac{t}{\xi}\right)^{(q+1)(p-1)+1}}{\left(\left(\frac{t}{\xi}\right)^{2\alpha}+1\right)^{\frac{q\beta}{2}}} d\left(\frac{t}{\xi}\right) + \right.\right.$$

$$\left.\left. \frac{j^p}{\xi^p (p+1)} \xi^{(q+1)(p-1)+(p+2)-\alpha\beta q} \int_0^\infty \frac{\left(\frac{t}{\xi}\right)^{(q+1)(p-1)+(p+1)}}{\left(\left(\frac{t}{\xi}\right)^{2\alpha}+1\right)^{\frac{q\beta}{2}}} d\left(\frac{t}{\xi}\right) \right] \right\} =$$

$$2^p c_2 \omega_1 \left(f'' + f, \xi\right)_p^p \xi^{(q+1)(p-1)+2-\alpha\beta q}$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\int_0^\infty \frac{x^{(q+1)(p-1)+1}}{(1+x^{2\alpha})^{\frac{q\beta}{2}}} dx + \right.\right.$$

$$\left.\left. \frac{j^p}{(p+1)} \int_0^\infty \frac{x^{(q+1)(p-1)+(p+1)}}{(1+x^{2\alpha})^{\frac{q\beta}{2}}} dx \right] \right\} = \tag{71}$$

$$2^p c_2 \omega_1 \left(f'' + f, \xi\right)_p^p \xi^{qp-q+p+1-\alpha\beta q}$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\int_0^\infty \frac{x^{qp-q+p}}{(1+x^{2\alpha})^{\frac{q\beta}{2}}} dx + \frac{j^p}{(p+1)} \int_0^\infty \frac{x^{qp-q+2p}}{(1+x^{2\alpha})^{\frac{q\beta}{2}}} dx \right] \right\} =$$

$$2^p c_2 \omega_1 \left(f'' + f, \xi\right)_p^p \xi^{2p+1-\alpha\beta q}$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\int_0^\infty \frac{x^{2p}}{(1+x^{2\alpha})^{\frac{q\beta}{2}}} dx + \frac{j^p}{(p+1)} \int_0^\infty \frac{x^{3p}}{(1+x^{2\alpha})^{\frac{q\beta}{2}}} dx \right] \right\} = \tag{72}$$

(by [18], p. 397, formula 595)

$$2^p c_2 \omega_1 \left(f'' + f, \xi\right)_p^p \xi^{2p+1-\alpha\beta q}$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\frac{\Gamma\left(\frac{2p+1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2} - \left(\frac{2p+1}{2\alpha}\right)\right)}{2\alpha \Gamma\left(\frac{q\beta}{2}\right)} + \frac{j^p}{(p+1)} \frac{\Gamma\left(\frac{3p+1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2} - \left(\frac{3p+1}{2\alpha}\right)\right)}{2\alpha \Gamma\left(\frac{q\beta}{2}\right)} \right] \right\} = \tag{73}$$

$$2^p c_2 \omega_1 \left(f'' + f, \xi\right)_p^p \xi^{2p+1-\alpha\beta q}$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\frac{\Gamma\left(\frac{2p+1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2} - \left(\frac{2p+1}{2\alpha}\right)\right) + \frac{j^p}{(p+1)} \Gamma\left(\frac{3p+1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2} - \left(\frac{3p+1}{2\alpha}\right)\right)}{2\alpha \Gamma\left(\frac{q\beta}{2}\right)} \right] \right\}$$

(under the assumption $q > \frac{3p+1}{\alpha\beta}$)

(it is

$$c_2 = \rho \left(\frac{r+1}{q+1}\right)^{p-1} = \frac{\alpha \Gamma(\beta)^p \Gamma\left(\frac{q\beta}{2} - \frac{1}{2\alpha}\right)^{(p-1)} (r+1)^{(p-1)} \xi^{\alpha\beta p - 1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma^p \left(\beta - \frac{1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2}\right)^{(p-1)} (q+1)^{(p-1)}}$$

and call

$$\psi_2 := \frac{\alpha \Gamma(\beta)^p \Gamma\left(\frac{q\beta}{2} - \frac{1}{2\alpha}\right)^{(p-1)} (r+1)^{(p-1)}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma^p \left(\beta - \frac{1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2}\right)^{(p-1)} (q+1)^{(p-1)}} \;)$$

$$= 2^p \psi_2 \omega_1 \left(f'' + f, \xi\right)_p^p \xi^{2p + \alpha\beta(p-q)}$$

$$\left\{ \sum_{j=0}^{r} |\alpha_j|^p j^{2p} \left[\frac{\Gamma\left(\frac{2p+1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2} - \left(\frac{2p+1}{2\alpha}\right)\right) + \frac{j^p}{(p+1)} \Gamma\left(\frac{3p+1}{2\alpha}\right) \Gamma\left(\frac{q\beta}{2} - \left(\frac{3p+1}{2\alpha}\right)\right)}{2\alpha \Gamma\left(\frac{q\beta}{2}\right)} \right] \right\}. \tag{74}$$

The claim is proved. ■

We give the following.

Corollary 5 *(to Theorem 4) Assume that $\beta > \frac{3}{2\alpha}$. It holds ($p > 1$)*

$$\|M_{r,\xi}(f) - f\|_p \leq \theta_2(\xi) +$$

$$4 \|f''\|_p \frac{\alpha \Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)} \sum_{j=0}^{r} |\alpha_j| \left(\int_0^\infty \frac{\sin^2\left(\frac{j\xi t}{2}\right)}{(1+t^{2\alpha})^\beta} dt \right) \to 0, \;\; as\; \xi \to 0. \tag{75}$$

Above $\theta_2(\xi)$ is as in (63).

Proof. By Theorem 4, and by $|\sin x| \leq |x|$, $x \in \mathbb{R}$:

$$\int_0^\infty \frac{\sin^2\left(\frac{j\xi t}{2}\right)}{(1+t^{2\alpha})^\beta} dt \leq \int_0^\infty \frac{\frac{j^2 \xi^2 t^2}{4}}{(1+t^{2\alpha})^\beta} dt = \left(\frac{j^2}{4}\right) \xi^2 \int_0^\infty \frac{t^2}{(1+t^{2\alpha})^\beta} dt \tag{76}$$

(by [18]; p. 397, formula 595)

$$= \xi^2 \left(\frac{j^2}{4}\right) \frac{\Gamma\left(\frac{3}{2\alpha}\right) \Gamma\left(\beta - \frac{3}{2\alpha}\right)}{2\alpha \Gamma(\beta)} \to 0, \text{ as } \xi \to 0.$$

■

We continue with L_p ($p > 1$) approximation by Trigonometric singular operators.

Theorem 6 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\beta\in\mathbb{N}$ and $\beta>\frac{\lceil 3p\rceil+1}{2}$ ($\lceil\cdot\rceil$ the ceiling of the number), $\xi>0$, λ_1 is as in (16).*

When $\overline{\lambda}\in\mathbb{N}$ is even we define

$$\psi_{1\overline{\lambda}}:=\frac{\pi\left(-1\right)^{\frac{2\beta-\overline{\lambda}}{2}}\left(2\beta\right)!}{2^{\overline{\lambda}+1}\left(2\beta-\overline{\lambda}-1\right)!}\left(\sum_{k=1}^{\beta}\left(-1\right)^{k}\frac{k^{2\beta-\overline{\lambda}-1}}{\left(\beta-k\right)!\left(\beta+k\right)!}\right), \tag{77}$$

and when $\overline{\lambda}$ is odd we define

$$\psi_{2\overline{\lambda}}:=\frac{\left(-1\right)^{\frac{\overline{\lambda}-1}{2}}\left(2\beta\right)!}{2^{\overline{\lambda}}\left(2\beta-\overline{\lambda}-1\right)!}\left(\sum_{k=1}^{\beta}\left(-1\right)^{\beta-k}\frac{k^{2\beta-\overline{\lambda}-1}\ln\left(2k\right)}{\left(\beta-k\right)!\left(\beta+k\right)!}\right), \tag{78}$$

and we set

$$\psi_{\overline{\lambda}}:=\left\{\begin{array}{c}\psi_{1\overline{\lambda}},\ \text{ if } \overline{\lambda} \text{ is even,}\\ \psi_{2\overline{\lambda}},\ \text{ if } \overline{\lambda} \text{ is odd}\end{array}\right\}. \tag{79}$$

Then

$$\left\Vert\Delta_4\right\Vert_p=\left\Vert T_{r,\xi}\left(f\right)-f-4f''\left(\lambda_1^{-1}\sum_{j=0}^{r}\alpha_j\int_0^{\infty}\sin^2\left(\frac{j\xi t}{2}\right)\left(\frac{\sin t}{t}\right)^{2\beta}dt\right)\right\Vert_p$$

$$\leq\frac{2^{1+\frac{1}{p}}}{\lambda_1^{\frac{1}{p}}\left(p+1\right)^{\frac{1}{p}}}\left(\frac{r+1}{q+1}\right)^{\frac{1}{q}}\omega_1\left(f''+f,\xi\right)_p\xi^2\left[\sum_{j=1}^{r}\left|\alpha_j\right|^p j^{3p}\right]^{\frac{1}{p}}$$

$$\left\{\left(\frac{-\left(2\beta\right)!}{8\left(2\beta-4\right)!}\sum_{k=1}^{\beta}\left(-1\right)^{\beta-k}\frac{k^{2\beta-4}\ln\left(2k\right)}{\left(\beta-k\right)!\left(\beta+k\right)!}\right)+\psi_{\lceil 3p\rceil}\right\}^{\frac{1}{p}}=:\theta_3\left(\xi\right)\to 0,\ \text{ as } \xi\to 0. \tag{80}$$

Proof. Equations / inequalities (41)-(48) are all used here.

By (27) we have that

$$\Delta_4^*\left(x\right)=\int_{-\infty}^{\infty}R\left(t\right)\frac{1}{\lambda}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt, \tag{81}$$

where $\lambda>0$ is as in (15); infact $\lambda=\lambda_1\xi^{1-2\beta}$ by (17), where λ_1 is as in (16); $\xi>0$.

By (18) we have

$$\int_{-\infty}^{\infty}\left(\frac{1}{\lambda}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}\right)dt=1. \tag{82}$$

We observe that $(p,q>1:\frac{1}{p}+\frac{1}{q}=1)$

$$\int_{-\infty}^{\infty}\left|\Delta_4^*(x)\right|^p dx=\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}R(t)\frac{1}{\lambda}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right|^p dx\leq$$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|\frac{1}{\lambda}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)^p dx\leq$$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|^p\frac{1}{\lambda}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)dx=$$

$$\lambda^{-1}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|^p\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)dx\overset{(48)}{\leq}$$

$$\lambda^{-1}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\left[\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\right.\right.\right. \tag{83}$$

$$\left(\int_0^{|t|}\left|\left(f''(x+jsign(t)\theta)+f(x+jsign(t)\theta)\right)-\left(f''(x)+f(x)\right)\right|^p d\theta\right)$$

$$\left.\left.\left.|t|^{\frac{(q+1)p}{q}}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)dx\right)\right]=$$

$$\lambda^{-1}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\left[\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\right.\right.\right.$$

$$\left(\int_{-\infty}^{\infty}\left|\left(f''(x+jsign(t)\theta)+f(x+jsign(t)\theta)\right)-\left(f''(x)+f(x)\right)\right|^p dx\right)$$

$$\left.\left.\left.d\theta\right)|t|^{\frac{(q+1)q}{p}}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)\right]\leq\lambda^{-1}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}$$

$$\left[\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_1\left(f''+f,\frac{\xi j\theta}{\xi}\right)_p^p d\theta\right)|t|^{\frac{(q+1)q}{p}}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right]\leq$$

$$\lambda^{-1}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\omega_1\left(f''+f,\xi\right)_p^p$$

$$\left[\sum_{j=0}^{r}|\alpha_j|^p j^{2p}\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(1+\frac{j}{\xi}\theta\right)^p d\theta\right)|t|^{\frac{(q+1)p}{q}}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right]= \tag{84}$$

$$\frac{\lambda^{-1}\xi}{(p+1)}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\omega_1\left(f''+f,\xi\right)_p^p$$

$$\left[\sum_{j=1}^{r}|\alpha_j|^p j^{2p-1}\int_{-\infty}^{\infty}\left(\left(1+\frac{j}{\xi}|t|\right)^{p+1}-1\right)|t|^{\frac{(q+1)p}{q}}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right]\leq$$

$$\frac{2^p\lambda^{-1}\xi}{(p+1)}\left(\frac{r+1}{q+1}\right)^{\frac{p}{q}}\omega_1\left(f''+f,\xi\right)_p^p$$

$$\left[\sum_{j=1}^{r}|\alpha_j|^p j^{2p-1}\left(\int_{-\infty}^{\infty}\left(\frac{j^{(p+1)}}{\xi^{(p+1)}}|t|^{p+1}\right)|t|^{(q+1)(p-1)}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)\right]=$$

$$\frac{2^p\lambda^{-1}\xi^{-p}}{(p+1)}\left(\frac{r+1}{q+1}\right)^{(p-1)}\omega_1\left(f''+f,\xi\right)_p^p\left[\sum_{j=1}^{r}|\alpha_j|^p j^{3p}\right]$$

$$\left(\int_{-\infty}^{\infty}|t|^{(q+1)(p-1)+(p+1)}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)= \tag{85}$$

$$\frac{2^p}{\lambda\xi^p(p+1)}\left(\frac{r+1}{q+1}\right)^{(p-1)}\omega_1\left(f''+f,\xi\right)_p^p\left[\sum_{j=1}^{r}|\alpha_j|^p j^{3p}\right]$$

$$\left(\int_{-\infty}^{\infty}|t|^{3p}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)=$$

$$\frac{2^{p+1}}{\lambda\xi^p(p+1)}\left(\frac{r+1}{q+1}\right)^{(p-1)}\omega_1\left(f''+f,\xi\right)_p^p\left[\sum_{j=1}^{r}|\alpha_j|^p j^{3p}\right]$$

$$\left(\int_0^{\infty}t^{3p}\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)=$$

$$\frac{2^{p+1}\xi^{2p+1-2\beta}}{\lambda(p+1)}\left(\frac{r+1}{q+1}\right)^{(p-1)}\omega_1\left(f''+f,\xi\right)_p^p\left[\sum_{j=1}^{r}|\alpha_j|^p j^{3p}\right]$$

$$\left(\int_0^{\infty} x^{3p}\left(\frac{\sin x}{x}\right)^{2\beta} dx\right) = \tag{86}$$

$$\frac{2^{p+1}\xi^{2p}}{\lambda_1(p+1)}\left(\frac{r+1}{q+1}\right)^{(p-1)}\omega_1\left(f''+f,\xi\right)_p^p\left[\sum_{j=1}^{r}\left|\alpha_j\right|^p j^{3p}\right]$$

$$\left(\int_0^{1} x^{3p}\left(\frac{\sin x}{x}\right)^{2\beta} dx + \int_1^{\infty} x^{3p}\left(\frac{\sin x}{x}\right)^{2\beta} dx\right) \leq$$

$$\frac{2^{p+1}\xi^{2p}}{(p+1)\lambda_1}\left(\frac{r+1}{q+1}\right)^{(p-1)}\omega_1\left(f''+f,\xi\right)_p^p\left[\sum_{j=1}^{r}\left|\alpha_j\right|^p j^{3p}\right]$$

$$\left(\int_0^{1} x^{3}\left(\frac{\sin x}{x}\right)^{2\beta} dx + \int_1^{\infty} x^{\lceil 3p\rceil}\left(\frac{\sin x}{x}\right)^{2\beta} dx\right) \leq$$

$$\frac{2^{p+1}\xi^{2p}}{(p+1)\lambda_1}\left(\frac{r+1}{q+1}\right)^{(p-1)}\omega_1\left(f''+f,\xi\right)_p^p\left[\sum_{j=1}^{r}\left|\alpha_j\right|^p j^{3p}\right] \tag{87}$$

$$\left(\int_0^{\infty} x^{3}\left(\frac{\sin x}{x}\right)^{2\beta} dx + \int_0^{\infty} x^{\lceil 3p\rceil}\left(\frac{\sin x}{x}\right)^{2\beta} dx\right) =$$

(we use [12], p. 210, item 1033)

$$\frac{2^{p+1}}{(p+1)}\frac{\xi^{2p}}{\lambda_1}\left(\frac{r+1}{q+1}\right)^{(p-1)}\omega_1\left(f''+f,\xi\right)_p^p\left[\sum_{j=1}^{r}\left|\alpha_j\right|^p j^{3p}\right]$$

$$\left(\left(\frac{(-1)(2\beta)!}{8(2\beta-4)!}\sum_{k=1}^{\beta}(-1)^{\beta-k}\frac{k^{2\beta-4}\ln(2k)}{(\beta-k)!(\beta+k)!}\right)+\psi_{\lceil 3p\rceil}\right) < \infty.$$

The theorem is proved. ■

We give the following.

Corollary 7 *(to Theorem 6) It holds* $(p > 1)$

$$\left\|T_{r,\xi}(f)-f\right\|_p \leq \theta_3(\xi) +$$

$$4\left\|f''\right\|_p \lambda_1^{-1}\left(\sum_{j=0}^{r}\left|\alpha_j\right|\left(\int_0^{\infty}\sin^2\left(\frac{j\xi t}{2}\right)\left(\frac{\sin t}{t}\right)^{2\beta} dt\right)\right) \to 0, \text{ as } \xi \to 0. \tag{88}$$

Above $\theta_3(\xi)$ *is as in (80).*

Proof. By Theorem 6, and by $|\sin x| \le |x|$, $x \in \mathbb{R}$:

$$\int_0^\infty \sin^2\left(\frac{j\xi t}{2}\right)\left(\frac{\sin t}{t}\right)^{2\beta} dt \le \left(\frac{j^2}{4}\right)\xi^2\left(\int_0^\infty t^2\left(\frac{\sin t}{t}\right)^{2\beta} dt\right)$$

(by [12]; p. 210, item 1033)

$$= \left(\frac{j^2}{4}\right)\left\{\frac{\pi(-1)^{\beta-1}(2\beta)!}{8(2\beta-3)!}\left(\sum_{k=1}^{\beta}(-1)^k\frac{k^{2\beta-3}}{(\beta-k)!(\beta+k)!}\right)\right\}\xi^2 \to 0, \text{ as } \xi \to 0. \tag{89}$$

■

L_1 approximation by Gauss-Weierstrass operators follows.

Theorem 8 *Let $\xi > 0$, both $f, f'' \in L_1(\mathbb{R}) \cap C(\mathbb{R})$. Then*

$$\|\Delta_2\|_1 = \left\|W_{r,\xi}(f) - f - f''\left(\sum_{j=0}^{r}\alpha_j\left(1 - e^{-\frac{j^2}{4}\xi}\right)\right)\right\|_1 \le$$

$$\frac{2}{\sqrt{\pi}}\omega_1\left(f'' + f, \sqrt{\xi}\right)_1 \xi\left\{\sum_{j=1}^{r}|\alpha_j| j^2\left[\frac{7}{3} + j\frac{25}{8}\right]\right\} := E_1(\xi) \to 0, \text{ as } \xi \to 0. \tag{90}$$

Proof. We have that

$$\Delta_2^*(x) = \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty} R(t)\, e^{-\frac{t^2}{\xi}} dt. \tag{91}$$

We observe that

$$\|\Delta_2^*\|_1 = \int_{-\infty}^{\infty}|\Delta_2^*(x)|\,dx = \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} R(t)\, e^{-\frac{t^2}{\xi}} dt\right| dx \le$$

$$\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|R(t)|\, e^{-\frac{t^2}{\xi}} dt\right) dx \overset{(45)}{\le}$$

$$\frac{1}{\sqrt{\pi\xi}}\left[\sum_{j=0}^{r}|\alpha_j| j^2\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\right.\right.\right.$$

$$\left(\int_0^{|t|}|(f''(x + j sign(t)\theta) + f(x + j sign(t)\theta)) - (f''(x) + f(x))|(|t| - \theta)\, d\theta\right) \tag{92}$$

$$\left.\left.\left. e^{-\frac{t^2}{\xi}} dt\right) dx\right]$$

$$\leq \frac{1}{\sqrt{\pi\xi}}\left[\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(|t|\int_{0}^{|t|}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left|\left(f''\left(x+jsign\left(t\right)\theta\right)+f\left(x+jsign\left(t\right)\theta\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right|d\theta\right)e^{-\frac{t^{2}}{\xi}}dt\right)dx\right)\right]=$$

$$\frac{1}{\sqrt{\pi\xi}}\left[\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left|\left(f''\left(x+jsign\left(t\right)\theta\right)+f\left(x+jsign\left(t\right)\theta\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right|d\theta\right)dx\right)|t|\,e^{-\frac{t^{2}}{\xi}}dt\right)\right]=$$

$$\frac{1}{\sqrt{\pi\xi}}\left[\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\left(\int_{-\infty}^{\infty}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left|\left(f''\left(x+jsign\left(t\right)\theta\right)+f\left(x+jsign\left(t\right)\theta\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right|dx\right)d\theta\right)|t|\,e^{-\frac{t^{2}}{\xi}}dt\right)\right]\leq$$

$$\frac{1}{\sqrt{\pi\xi}}\left[\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\omega_{1}\left(f''+f,\frac{\sqrt{\xi}}{\sqrt{\xi}}j\theta\right)_{1}d\theta\right)|t|\,e^{-\frac{t^{2}}{\xi}}dt\right)\right]\leq$$

$$\frac{1}{\sqrt{\pi\xi}}\omega_{1}\left(f''+f,\sqrt{\xi}\right)_{1}$$

$$\left[\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\left(1+\frac{j}{\sqrt{\xi}}\theta\right)d\theta\right)|t|\,e^{-\frac{t^{2}}{\xi}}dt\right)\right]= \tag{93}$$

$$\frac{1}{\sqrt{\pi\xi}}\omega_{1}\left(f''+f,\sqrt{\xi}\right)_{1}\left[\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{-\infty}^{\infty}\left(|t|+\frac{j}{\sqrt{\xi}}\frac{t^{2}}{2}\right)|t|\,e^{-\frac{t^{2}}{\xi}}dt\right)\right]=$$

$$\frac{2}{\sqrt{\pi\xi}}\omega_{1}\left(f''+f,\sqrt{\xi}\right)_{1}\left[\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left(\int_{0}^{\infty}\left(t+\frac{j}{\sqrt{\xi}}\frac{t^{2}}{2}\right)te^{-\frac{t^{2}}{\xi}}dt\right)\right]=$$

$$\frac{2}{\sqrt{\pi}\sqrt{\xi}}\omega_{1}\left(f''+f,\sqrt{\xi}\right)_{1}$$

$$\left[\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left[\int_{0}^{\infty}t^{2}e^{-\frac{t^{2}}{\xi}}dt+\frac{j}{2\sqrt{\xi}}\int_{0}^{\infty}t^{3}e^{-\frac{t^{2}}{\xi}}dt\right]\right]=$$

$$\frac{2}{\sqrt{\pi}\sqrt{\xi}}\omega_{1}\left(f''+f,\sqrt{\xi}\right)_{1} \tag{94}$$

$$\left[\sum_{j=0}^{r}\left|\alpha_{j}\right|j^{2}\left[\left(\sqrt{\xi}\right)^{3}\int_{0}^{\infty}\left(\frac{t}{\sqrt{\xi}}\right)^{2}e^{-\left(\frac{t}{\sqrt{\xi}}\right)^{2}}d\frac{t}{\sqrt{\xi}}+\right.\right.$$

$$\frac{j}{2\sqrt{\xi}}\left(\sqrt{\xi}\right)^4\int_0^\infty\left(\frac{t}{\sqrt{\xi}}\right)^3 e^{-\left(\frac{t}{\sqrt{\xi}}\right)^2}d\frac{t}{\sqrt{\xi}}\Bigg]\Bigg] =$$

$$\frac{2\xi}{\sqrt{\pi}}\omega_1\left(f''+f,\sqrt{\xi}\right)_1$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\int_0^\infty x^2e^{-x^2}dx+\frac{j}{2}\int_0^\infty x^3e^{-x^2}dx\right]\right\} =$$

$$\frac{2\xi}{\sqrt{\pi}}\omega_1\left(f''+f,\sqrt{\xi}\right)_1\left\{\sum_{j=0}^{r}|\alpha_j|\,j^2\right.$$

$$\left.\left[\left[\int_0^1 x^2e^{-x^2}dx+\int_1^\infty x^2e^{-x^2}dx\right]+\frac{j}{2}\left[\int_0^1 x^3e^{-x^2}dx+\int_1^\infty x^3e^{-x^2}dx\right]\right]\right\} \leq$$

$$\frac{2\xi}{\sqrt{\pi}}\omega_1\left(f''+f,\sqrt{\xi}\right)_1$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\left[\frac{1}{3}+\int_1^\infty x^2e^{-x}dx\right]+\frac{j}{2}\left[\frac{1}{4}+\int_1^\infty x^3e^{-x}dx\right]\right]\right\} \leq$$

$$\frac{2\xi}{\sqrt{\pi}}\omega_1\left(f''+f,\sqrt{\xi}\right)_1$$

$$\left\{\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\left[\frac{1}{3}+\int_0^\infty x^2e^{-x}dx\right]+\frac{j}{2}\left[\frac{1}{4}+\int_0^\infty x^3e^{-x}dx\right]\right]\right\} = \tag{95}$$

$$\frac{2\xi}{\sqrt{\pi}}\omega_1\left(f''+f,\sqrt{\xi}\right)_1\left\{\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\left[\frac{1}{3}+2\right]+\frac{j}{2}\left[\frac{1}{4}+6\right]\right]\right\} =$$

$$\frac{2\xi}{\sqrt{\pi}}\omega_1\left(f''+f,\sqrt{\xi}\right)_1\left\{\sum_{j=0}^{r}|\alpha_j|\,j^2\left[\frac{7}{3}+j\frac{25}{8}\right]\right\}.$$

The theorem is proved. ■

We give.

Corollary 9 *(to Theorem 8) It holds*

$$\|W_{r,\xi}(f)-f\|_1 \leq E_1(\xi)+$$

$$\|f''\|_1\left(\sum_{j=0}^{r}|\alpha_j|\left(1-e^{-\frac{j^2}{4}\xi}\right)\right)\to 0,\ \text{as } \xi\to 0. \tag{96}$$

Above $E_1(\xi)$ is as in (90).

It follows the L_1 approximation by Poisson-Cauchy operators.

Theorem 10 *Let $\xi > 0$, $f, f'' \in L_1(\mathbb{R}) \cap C(\mathbb{R})$ and $\beta > \frac{2}{\alpha}$, $\alpha \in \mathbb{N}$. Then*

$$\|\Delta_3\|_1 =$$

$$\left\| M_{r,\xi}(f) - f - 4f'' \left[\left(\frac{\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)} \right) \left(\sum_{j=0}^{r} \alpha_j \int_0^\infty \frac{\sin^2\left(\frac{j\xi t}{2}\right)}{(1+t^{2\alpha})^\beta} dt \right) \right] \right\|_1$$

$$\leq \frac{\omega_1(f'' + f, \xi)_1 \xi^2}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta - \frac{1}{2\alpha}\right)}$$

$$\left[\sum_{j=0}^{r} |\alpha_j| j^2 \left(\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta - \frac{3}{2\alpha}\right) + \frac{j}{2}\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta - \frac{2}{\alpha}\right) \right) \right] := E_2(\xi) \to 0, \tag{97}$$

as $\xi \to 0$.

Proof. We have that

$$\Delta_3^*(x) = W \int_{-\infty}^{\infty} R(t) \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^\beta} dt. \tag{98}$$

We observe that

$$\|\Delta_3^*\|_1 = \int_{-\infty}^{\infty} |\Delta_3^*(x)| dx = W \int_{-\infty}^{\infty} \left(\left| \int_{-\infty}^{\infty} R(t) \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^\beta} dt \right| \right) dx \leq$$

$$W \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |R(t)| \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^\beta} dt \right) dx \overset{(45)}{\leq}$$

$$W \left[\sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \right.\right.\right.$$

$$\left(\int_0^{|t|} |(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))| (|t| - \theta) d\theta \right)$$

$$\left.\left.\left. \frac{1}{\left(t^{2\alpha} + \xi^{2\alpha}\right)^\beta} dt \right) dx \right) \right] \leq \tag{99}$$

$$W \left[\sum_{j=0}^{r} |\alpha_j| j^2 \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left(|t| \int_0^{|t|} \right.\right.\right.\right.$$

$$|(f''(x + jsign(t)\theta) + f(x + jsign(t)\theta)) - (f''(x) + f(x))| d\theta)$$

$$\frac{1}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt\Bigg)\,dx\Bigg)\Bigg] =$$

$$W\left[\sum_{j=0}^{r}\left|\alpha_j\right| j^2\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(|t|\int_{0}^{|t|}\right.\right.\right.\right.$$

$$\left|\left(f''\left(x+jsign\left(t\right)\theta\right)+f\left(x+jsign\left(t\right)\theta\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right|d\theta)\,dx)$$

$$\frac{1}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt\Bigg)\Bigg] =$$

$$W\left[\sum_{j=0}^{r}\left|\alpha_j\right| j^2\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\left(\int_{-\infty}^{\infty}\right.\right.\right.\right.$$

$$\left|\left(f''\left(x+jsign\left(t\right)\theta\right)+f\left(x+jsign\left(t\right)\theta\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right|dx)\,d\theta)$$

$$\frac{|t|}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt\Bigg)\Bigg] \leq$$

$$W\left[\sum_{j=0}^{r}\left|\alpha_j\right| j^2\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\omega_1\left(f''+f,\frac{\xi}{\xi}j\theta\right)_1 d\theta\right)\frac{|t|}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt\right)\right] \leq \tag{100}$$

$$W\omega_1\left(f''+f,\xi\right)_1$$

$$\left[\sum_{j=0}^{r}\left|\alpha_j\right| j^2\left(\int_{-\infty}^{\infty}\left(\int_{0}^{|t|}\left(1+\frac{j}{\xi}\theta\right)d\theta\right)\frac{|t|}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt\right)\right] =$$

$$W\omega_1\left(f''+f,\xi\right)_1\left[\sum_{j=0}^{r}\left|\alpha_j\right| j^2\left(\int_{-\infty}^{\infty}\left(|t|+\frac{j}{\xi}\frac{t^2}{2}\right)\frac{|t|}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt\right)\right] =$$

$$2W\omega_1\left(f''+f,\xi\right)_1\left[\sum_{j=0}^{r}\left|\alpha_j\right| j^2\left(\int_{0}^{\infty}\left(t+\frac{j}{\xi}\frac{t^2}{2}\right)\frac{t}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt\right)\right] =$$

$$2W\omega_1\left(f''+f,\xi\right)_1$$

$$\left[\sum_{j=0}^{r}\left|\alpha_j\right| j^2\left(\int_{0}^{\infty}\frac{t^2}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt+\frac{j}{2\xi}\int_{0}^{\infty}\frac{t^3}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt\right)\right] =$$

$$2W\omega_1\left(f''+f,\xi\right)_1 \tag{101}$$

$$\left[\sum_{j=0}^{r}\left|\alpha_j\right| j^2\left(\xi^{3-2\alpha\beta}\int_{0}^{\infty}\frac{\left(\frac{t}{\xi}\right)^2}{\left(\left(\frac{t}{\xi}\right)^{2\alpha}+1\right)^{\beta}}d\frac{t}{\xi}+\right.\right.$$

$$\left.\left.\frac{j}{2\xi}\xi^{4-2\alpha\beta}\int_0^\infty \frac{\left(\frac{t}{\xi}\right)^3}{\left(\left(\frac{t}{\xi}\right)^{2\alpha}+1\right)^\beta} d\frac{t}{\xi}\right)\right] =$$

$$2W\xi^{3-2\alpha\beta}\omega_1\left(f''+f,\xi\right)_1$$

$$\left[\sum_{j=0}^r |\alpha_j| j^2\left[\int_0^\infty \frac{x^2}{\left(1+x^{2\alpha}\right)^{2\beta}}dx+\frac{j}{2}\int_0^\infty \frac{x^3}{\left(1+x^{2\alpha}\right)^{2\beta}}dx\right]\right] \overset{\text{(by (10))}}{=}$$

(and by [18], p. 397, formula 595)

$$\frac{2\alpha\Gamma(\beta)\xi^2}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\omega_1\left(f''+f,\xi\right)_1$$

$$\left[\sum_{j=0}^r |\alpha_j| j^2\left[\frac{\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\frac{3}{2\alpha}\right)}{2\alpha\Gamma(\beta)}+\frac{j}{2}\frac{\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta-\frac{2}{\alpha}\right)}{2\alpha\Gamma(\beta)}\right]\right] = \tag{102}$$

$$\frac{\xi^2}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\omega_1\left(f''+f,\xi\right)_1$$

$$\left[\sum_{j=0}^r |\alpha_j| j^2\left(\Gamma\left(\frac{3}{2\alpha}\right)\Gamma\left(\beta-\frac{3}{2\alpha}\right)+\frac{j}{2}\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(\beta-\frac{2}{\alpha}\right)\right)\right],$$

under the assumption $\beta>\frac{2}{\alpha}$.

The proof is completed. ■

We also give.

Corollary 11 *(to Theorem 10) It holds ($\beta>\frac{2}{\alpha}$, $\alpha\in\mathbb{N}$),*

$$\left\|M_{r,\xi}(f)-f\right\|_1 \le E_2(\xi)+4\left\|f''\right\|_1$$

$$\left[\left(\frac{\alpha\Gamma(\beta)}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\right)\left(\sum_{j=0}^r|\alpha_j|\left(\int_0^\infty \frac{\sin^2\left(\frac{j\xi t}{2}\right)}{\left(1+t^{2\alpha}\right)^\beta}dt\right)\right)\right] \to 0, \text{ as } \xi\to 0. \tag{103}$$

Above $E_2(\xi)$ is as in (97).

We continue with L_1 approximation by Trigonometric singular operators.

Theorem 12 *Let $\xi>0$, $f,f''\in L_1(\mathbb{R})\cap C(\mathbb{R})$, and $\beta\in\mathbb{N}-\{1,2\}$. Then*

$$\left\|\Delta_4\right\|_1 = \left\|T_{r,\xi}(f)-f-4f''\left(\lambda_1^{-1}\sum_{j=0}^r\alpha_j\int_0^\infty \sin^2\left(\frac{j\xi t}{2}\right)\left(\frac{\sin t}{t}\right)^{2\beta}dt\right)\right\|_1$$

$$\leq \lambda_1^{-1}\omega_1\left(f''+f,\xi\right)_1\xi^2\left\{\sum_{j=1}^{r}\left|\alpha_j\right|j^2\right. \tag{104}$$

$$\left[\left(\frac{\pi\left(-1\right)^{(\beta-1)}\left(2\beta\right)!}{4\left(2\beta-3\right)!}\sum_{k=1}^{\beta}\left(-1\right)^k\frac{k^{2\beta-3}}{\left(\beta-k\right)!\left(\beta+k\right)!}\right)+\right.$$

$$\left.\frac{j}{2}\left(\frac{-\left(2\beta\right)!}{4\left(2\beta-4\right)!}\sum_{k=1}^{\beta}\left(-1\right)^{\beta-k}\frac{k^{2\beta-4}\ln\left(2k\right)}{\left(\beta-k\right)!\left(\beta+k\right)!}\right)\right]=:E_3\left(\xi\right)\to 0,\ \ as\ \xi\to 0.$$

Proof. We have that

$$\Delta_4^*\left(x\right)=\lambda^{-1}\int_{-\infty}^{\infty}R\left(t\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt. \tag{105}$$

We observe that

$$\left\|\Delta_4^*\right\|=\int_{-\infty}^{\infty}\left|\Delta_4^*\left(x\right)\right|dx=\lambda^{-1}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}R\left(t\right)\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right|dx\leq$$

$$\lambda^{-1}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left|R\left(t\right)\right|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)dx\overset{(45)}{\leq}$$

$$\lambda^{-1}\left[\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\right.\right.\right.$$

$$\left(\int_0^{|t|}\left|\left(f''\left(x+jsign\left(t\right)\theta\right)+f\left(x+jsign\left(t\right)\theta\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right|\left(\left|t\right|-\theta\right)d\theta\right)$$

$$\left.\left.\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)dx)\right]\leq$$

$$\lambda^{-1}\left[\sum_{j=0}^{r}\left|\alpha_j\right|j^2\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(\left|t\right|\int_0^{|t|}\right.\right.\right.\right.$$

$$\left|\left(f''\left(x+jsign\left(t\right)\theta\right)+f\left(x+jsign\left(t\right)\theta\right)\right)-\left(f''\left(x\right)+f\left(x\right)\right)\right|d\theta)$$

$$\left.\left.\left.\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)dx\right)\right]=$$

$$\lambda^{-1}\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\left(|t|\int_0^{|t|}\right.\right.\right.\right.$$

$$|(f''(x+jsign(t)\theta)+f(x+jsign(t)\theta))-(f''(x)+f(x))|\,d\theta)\,dx)$$

$$\left.\left.\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)\right]=$$

$$\lambda^{-1}\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(\int_{-\infty}^{\infty}\right.\right.\right.\right.$$

$$|(f''(x+jsign(t)\theta)+f(x+jsign(t)\theta))-(f''(x)+f(x))|\,dx)\,d\theta)$$

$$\left.\left.|t|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)\right]\le$$

$$\lambda^{-1}\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_1\left(f''+f,\frac{\xi}{\xi}j\theta\right)_1 d\theta\right)|t|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)\right]\le \tag{107}$$

$$\lambda^{-1}\omega_1(f''+f,\xi)_1$$

$$\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(1+\frac{j}{\xi}\theta\right)d\theta\right)|t|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)\right]=$$

$$\lambda^{-1}\omega_1(f''+f,\xi)_1\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\int_{-\infty}^{\infty}\left(|t|+\frac{j}{\xi}\frac{t^2}{2}\right)|t|\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)\right]=$$

$$2\lambda^{-1}\omega_1(f''+f,\xi)_1\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\int_0^{\infty}\left(t+\frac{j}{\xi}\frac{t^2}{2}\right)t\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)\right]=$$

$$2\lambda^{-1}\omega_1(f''+f,\xi)_1$$

$$\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\int_0^{\infty}t^2\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt+\frac{j}{2\xi}\int_0^{\infty}t^3\left(\frac{\sin\left(\frac{t}{\xi}\right)}{t}\right)^{2\beta}dt\right)\right]=$$

$$2\lambda^{-1}\omega_1(f''+f,\xi)_1 \tag{108}$$

$$\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\xi^{3-2\beta}\int_0^{\infty}\left(\frac{t}{\xi}\right)^2\left(\frac{\sin\left(\frac{t}{\xi}\right)}{\frac{t}{\xi}}\right)^{2\beta}d\frac{t}{\xi}+\right.\right.$$

$$\left.\left.\frac{j}{2\xi}\xi^{4-2\beta}\int_0^{\infty}\left(\frac{t}{\xi}\right)^3\left(\frac{\sin\left(\frac{t}{\xi}\right)}{\frac{t}{\xi}}\right)^{2\beta}d\frac{t}{\xi}\right)\right]=$$

$$2\lambda^{-1}\omega_1\left(f''+f,\xi\right)_1\xi^{3-2\beta}$$

$$\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\int_0^{\infty}x^2\left(\frac{\sin x}{x}\right)^{2\beta}dx+\frac{j}{2}\int_0^{\infty}x^3\left(\frac{\sin x}{x}\right)^{2\beta}dx\right)\right]=$$

$$2\lambda_1^{-1}\omega_1\left(f''+f,\xi\right)_1\xi^2$$

$$\left[\sum_{j=0}^{r}|\alpha_j|\,j^2\left(\int_0^{\infty}x^2\left(\frac{\sin x}{x}\right)^{2\beta}dx+\frac{j}{2}\int_0^{\infty}x^3\left(\frac{\sin x}{x}\right)^{2\beta}dx\right)\right]$$

(we use [12], p. 210, item 1033)

$$=2\lambda_1^{-1}\omega_1\left(f''+f,\xi\right)_1\xi^2\left\{\sum_{j=1}^{r}|\alpha_j|\,j^2\right.$$

$$\left[\left(\frac{\pi\,(-1)^{(\beta-1)}\,(2\beta)!}{8\,(2\beta-3)!}\sum_{k=1}^{\beta}(-1)^k\,\frac{k^{2\beta-3}}{(\beta-k)!\,(\beta+k)!}\right)+\right. \tag{109}$$

$$\left.\frac{j}{2}\left(\frac{(-1)\,(2\beta)!}{8\,(2\beta-4)!}\sum_{k=1}^{\beta}(-1)^{\beta-k}\,\frac{k^{2\beta-4}\,(\ln(2k))}{(\beta-k)!\,(\beta+k)!}\right)\right],$$

under $\beta>2$, $\beta\in\mathbb{N}$.

The theorem is proved. ■

We finish with the following.

Corollary 13 *(to Theorem 12) It holds ($\beta\in\mathbb{N}-\{1,2\}$),*

$$\|T_{r,\xi}(f)-f\|_1\le E_3(\xi)+$$

$$4\,\|f''\|_1\,\lambda_1^{-1}\left(\sum_{j=0}^{r}|\alpha_j|\left(\int_0^{\infty}\sin^2\left(\frac{j\xi t}{2}\right)\left(\frac{\sin t}{t}\right)^{2\beta}dt\right)\right)\to 0,\ \ as\ \xi\to 0. \tag{110}$$

Above $E_3(\xi)$ is as in (104).

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